

Invasion percolation: a new form of percolation theory

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Abstract. A new kind of percolation problem is described which differs from ordinary percolation theory in that it automatically finds the critical points of the system. The model is motivated by the problem of one fluid displacing another from a porous medium under the action of capillary forces, but in principle it may be applied to any kind of invasion process which proceeds along a path of least resistance. The name invasion percolation is proposed for this new process. Similarities to, and differences from, ordinary percolation theory are discussed.

1. Introduction

The standard theory of percolation (Broadbent and Hammersley 1957, Frisch and Hammersley 1963) has been shown to have application to a broad variety of physical problems. Interestingly, although the above authors motivated their studies of percolation with examples involving transport in random media, their theory is really one of static properties of the medium. The purpose of this paper is to describe a new form of percolation theory which explicitly takes into account the transport process taking place.

The new form of percolation which we discuss was motivated by the study of the flow of two immiscible fluids in porous media (Chandler *et al* 1982), although the emphasis of this paper will be entirely on the percolation theory aspects of the process. We will first describe the physical context of the model in brief. Then we will present results from computer simulations which realise the model. Finally, we shall attempt to set the model in a more general context and raise some interesting questions suggested by the model.

Many porous media may be represented conveniently as a network of pores joined by narrower connecting throats (Lin and Cohen 1982). In an idealised medium the network may be viewed as a regular lattice in which the sites and bonds of the lattice represent pores and throats respectively. For simplicity, in three dimensions we may think of the pores as spheres and the throats as cylinders. Randomness of the medium may be incorporated by assigning random numbers to the sites and bonds to represent the sizes of these pores and throats.

Let us consider the process of a non-wetting fluid, say oil, being displaced from such a medium by a wetting fluid, say water, at constant but infinitesimal flow rate. In this limit, viscous forces are completely dominated by the capillary forces acting at the oil–water interfaces. The capillary forces are such as to make the water spontaneously displace the oil, and in fact in order to keep the flow rate infinitesimal a negative pressure gradient must be placed across the system. The capillary forces

are strongest at the narrowest places in the medium. Thus if all the throats are smaller than all the pores, the water–oil interface moves quickly through the throats, but gets stuck entering the larger pores. It is consistent with both a simple theoretical model and experimental observations to represent this motion as a series of discrete jumps in which at each time step the water displaces oil from the smallest available pore. Simulation of the process in a given realisation of the lattice thus consists of following the motion of the water–oil interface as it advances through the smallest available pores.

An important feature which arises in this displacement process is trapping of the oil. As the water advances it is possible for it to completely surround regions of oil, i.e., disconnect finite clusters from a connected cluster of oil which reaches to the exit face of the sample. This is one origin of the phenomenon of ‘residual oil’, a great economic problem in the oil industry. Since the oil is incompressible, we must introduce a new rule that water cannot invade trapped regions of oil. As we shall see, this rule has a significant effect of the percolation problem described in the next section.

The problem of the displacement of one fluid by another in a porous medium under the action of capillary forces, and its connection with percolation theory, has been studied by a number of authors (de Gennes and Guyon 1978, Lenormand 1980, Larson *et al* 1981). The present work differs from these in that it is a specific dynamical model on the microscopic scale which describes the displacement process at constant flow rate, rather than at constant applied pressure. It is this feature which leads to the dynamical rule of advancing the interface at the point of least resistance, as opposed to advancing all interfaces up to some chosen threshold resistance, and for which we suggest the new name *invasion percolation*. In the presence of the trapping rule, the distinction between constant flow rate and constant pressure becomes very important, at least from the point of view of simulation, because invasion percolation implies a unique time sequence of advances of the interface, and hence a unique way of deciding whether or not a given portion of the displaced fluid becomes trapped. By contrast, at a given applied pressure, the interface can advance in many places, and different time orderings can lead to different trapping configurations.

In the above discussion it was argued that the advance of the interface was determined by the sizes of the pores, which are the sites in our lattice analogue of the medium. There is another version of the model, representing the case of a non-wetting fluid displacing a wetting one, in which the advance is determined by the sizes of the connecting throats, i.e. the bonds of the lattice. In fact this was the version of the model originally considered by Chandler *et al*. In the absence of trapping, this bond version can be reduced to the site version by the usual bond to site transformation, but with trapping a difference arises because it is natural to adopt the convention that a site becomes occupied by the invading fluid, and hence unavailable to the displaced fluid, whenever any of its connecting bonds becomes invaded. The question of trapping thus becomes a correlated site–bond problem, analogous to a correlated percolation problem (Blumberg *et al* 1980). Despite this complication, it is found that the bond version of the problem is qualitatively very similar to the site case discussed in this paper.

2. Percolation model

In terms of our lattice model of the porous medium, we may construct a computer simulation of the displacement process as follows. Since the process in question need

not necessarily be fluid flow, we shall adopt a different nomenclature in which water is termed the invader and oil the defender. Let us assign a random number, drawn from a uniform distribution on the unit interval $[0, 1]$, to each site in the lattice. In the initial configuration, the defender forms a spanning cluster, while the invader resides entirely in some compact region. Starting from such an initial configuration, the invader grows at each time step by displacing the defender from that site on the interface which has the smallest random number. It is assumed that the displaced defender has an 'escape route' to some boundary (or even local) sink or sinks.

The choice of initial configuration depends on the problem one has in mind. One natural choice is to have the invader grow from a single point into an infinite medium; detailed Monte Carlo results for this geometry will be presented elsewhere. Here we will consider a geometry which is more appropriate to a laboratory two-fluid flow experiment: a finite domain with the invader initially occupying one side or face of the lattice. To be definite, we will consider a lattice of size $L \times L$ in two dimensions or $L \times L \times L$ in three dimensions, with periodic boundary conditions on the sides and the invader initially occupying one edge or face at the end. The defender must then escape from the opposite edge or face of the lattice. The main quantities of interest will be the fraction of sites which become occupied by the invader, and the distribution of random numbers of those sites. In order to eliminate end effects as much as possible, these quantities will be measured over the central $L \times L$ or $L \times L \times L$ region. All our simulations will be done for four lattices: honeycomb, square and triangular in two dimensions, and simple cubic in three dimensions. For reference, the accepted percolation thresholds (in ordinary percolation) for these lattices are shown in table 1.

Table 1. Percolation thresholds in ordinary percolation for the four lattices considered in this paper. The value $p_c = 0.5$ for the triangular lattice is exact (Sykes and Essam 1963). For the other values see for example Djordjevic *et al* (1982), Sur *et al* (1976).

Lattice	p_c
Honeycomb	0.6962
Square	0.5923
Triangular	0.5000
Simple Cubic	0.3115

Many of the results and observations of this paper are based on our Monte Carlo simulations of the process. These simulations are performed on comparatively small lattices: $L_{\max} = 100$ in two dimensions and $L_{\max} = 30$ in three dimensions, and therefore cannot be expected to give accurate values for critical exponents. Their main purpose is rather to indicate the general features of the phenomenon, and in particular to show the effect of the trapping rule mentioned in the introduction. While some improvement in the Monte Carlo work is doubtless possible, there are two features of invasion percolation which make it much more time consuming to simulate than ordinary percolation:

(a) The move selection algorithm involves searching a list of random numbers. Thus the time taken to generate an invasion percolation cluster of n sites grows at least as fast as $n \ln n$.

(b) In the presence of the trapping rule, it is necessary to check after each move whether a trap has taken place. This cannot be determined locally, but rather involves a global search of the system.

It is interesting to compare the growth of the invader with the growth which would occur in ordinary percolation theory. Let us first do this in the absence of the trapping mechanism discussed earlier, as would be appropriate for the displacement of an infinitely compressible fluid by an incompressible one (Larson and Morrow 1981, Chandler and Willemsen 1981). In the percolation case one starts, in a single realisation, with the same lattice, i.e. the same set of random numbers assigned to each site. Then, for a given choice of occupation probability p , $0 \leq p \leq 1$, the cluster grows by occupying all available sites with random number r such that $r \leq p$. The process stops when no more such numbers are left on the boundary of the cluster. By contrast, in our modified percolation, the cluster never terminates, but always grows into the smallest available random number, no matter how large. However, once a given number r_0 has been chosen, it is not necessarily true that subsequently every number $r \leq r_0$ will be chosen—smaller numbers will in general become available at the interface, and will thus be chosen in preference to numbers close to r_0 .

Naturally, in the finite geometry we are considering, the invader will gradually fill the entire lattice. But an interesting configuration to study occurs at the point in time when the invader first percolates, i.e. first forms a connected path across the lattice. Let us denote the volume fraction of the invader by S_I . In figure 1, the value of this quantity when the invader first percolates is plotted as a function of the lattice size L . We see that our Monte Carlo data is well fitted by the relation

$$S_I = AL^{-\alpha}, \quad (2.1)$$

where the exponent α is tabulated in the first column of table 2. The results are consistent with a universal value $\alpha \sim 0.11$ in two dimensions, and $\alpha \sim 0.48$ in three dimensions. In terms of the number of sites N_I occupied by the invader this result

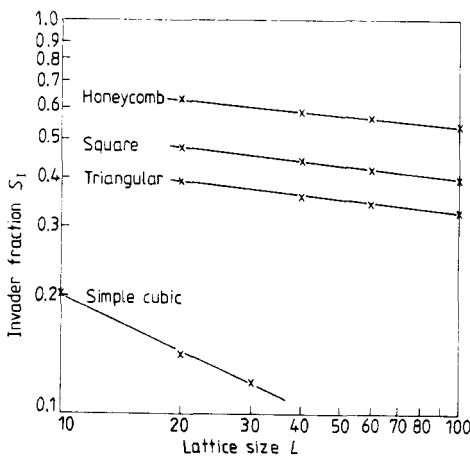


Figure 1. Invader fraction S_I at the percolation point without the trapping rule, plotted as a function of lattice size L . In figures 1 and 2, the number of realisations in two dimensions varies from about 2000 for $L = 20$ to 500 for $L = 100$, and in three dimensions from 1000 for $L = 10$ to 200 for $L = 30$.

may be expressed as

$$N_1 = AL^\phi, \tag{2.2}$$

with $\phi \sim 1.89$ in two dimensions and $\phi \sim 2.52$ in three dimensions. It is natural to identify ϕ as the fractal dimension of the cluster; we note that the values we have obtained are consistent with the fractal dimensions of ordinary percolation theory (see e.g. Stauffer 1979, Essam 1980). The natural identification with ordinary percolation exponents is

$$\alpha = \beta/\nu, \quad \phi = \Delta/\nu, \tag{2.3}$$

where β , Δ and ν are the usual order parameter, gap, and correlation length exponents respectively. The relation $\alpha + \phi = d$, where d is the space dimension, corresponds to the hyperscaling relation $\Delta + \beta = d\nu$.

Although the fractal dimension of our clusters is consistent with that of ordinary percolation theory, the clusters are not precisely the same. One simple way to express this is to consider the acceptance profile $a(r)$, which is the number of random numbers in the interval $[r, r + dr]$ which were accepted into the cluster, expressed as a fraction of the number of random numbers in that range which became available. This quantity, averaged over many realisations, is plotted in figure 2 for two different values of L .

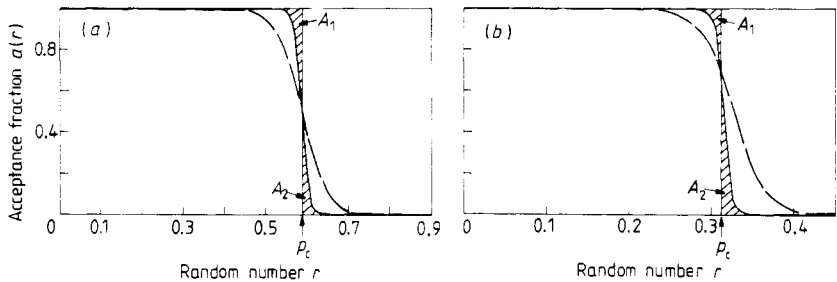


Figure 2. Acceptance profile $a(r)$ for the (a) square and (b) simple cubic lattices at the percolation point without the trapping rule. Two different values of lattice size L are shown. The shaded areas represent the quantities A_1 and A_2 , defined in (2.6), when $r = p_c$. It is conjectured that both A_1 and A_2 vanish as $L \rightarrow \infty$. Figures 2, 3 and 5 were obtained by dividing the r axis into 100 divisions and counting the number of sites chosen in each interval. The data points have been suppressed. (a) --- $L = 20$, — $L = 100$. (b) --- $L = 10$, — $L = 30$.

As one would expect, all small random numbers are accepted and all large numbers are rejected. However, there is also a transition region in which some numbers are accepted and some rejected, in contrast to the sharp acceptance profile which defines ordinary percolation theory. The figures also suggest that as $L \rightarrow \infty$ the profile has the limiting form of a step function at $r = p_c$, where p_c is the ordinary percolation threshold for the lattice in question. To test this idea it is useful to define the total acceptance fraction p , which is the area under the curve $a(r)$:

$$p = \int_0^1 a(r) dr. \tag{2.4}$$

If the above hypothesis is correct then $p \rightarrow p_c$ as $L \rightarrow \infty$, and it is natural to try a power

law approach to this limit:

$$|p - p_c| \sim L^{-\mu}. \tag{2.5}$$

(In this geometry we find for finite L that $p < p_c$ in two dimensions and $p > p_c$ in three dimensions.) Actually a more precise, and also more sensitive, check is to consider the two quantities

$$A_1(r) = \int_0^r [1 - a(r')] dr', \quad A_2(r) = \int_r^1 a(r') dr', \tag{2.6}$$

which are related respectively to the probability that a random number below r is not chosen, and the probability that a number above r is chosen. Let us further define

$$r_{\min} = \sup\{r; A_1(r) \rightarrow 0 \text{ as } L \rightarrow \infty\}, \quad r_{\max} = \inf\{r; A_2(r) \rightarrow 0 \text{ as } L \rightarrow \infty\}. \tag{2.7}$$

Clearly r_{\min} and r_{\max} exist and satisfy $0 \leq r_{\min} \leq r_{\max} \leq 1$. The above hypothesis is that $r_{\min} = r_{\max} = p_c$. In this case we should expect that at $r = p_c$

$$A_1 \sim L^{-\mu_1}, \quad A_2 \sim L^{-\mu_2}. \tag{2.8a, b}$$

In principle it should be possible to determine separate values of p_c which give the best fit to (2.8a) and (2.8b), and then compare these p_c values and the corresponding exponents μ_1 and μ_2 , but unfortunately our Monte Carlo data does not seem of sufficient quality to do this. However, it is reasonable to suppose the exponents μ_1 and μ_2 to be equal, and we shall use this assumption to determine the value of p_c by demanding that a least squares fit to (2.8a) and (2.8b) yield the same exponent μ . In this way we find the p_c values and exponents $\mu_1 = \mu_2$ shown in columns 2 and 3 of table 2. The values obtained for p_c are very close to the accepted percolation thresholds for these four lattices shown in table 1, thus providing compelling evidence for the correctness of equations (2.5) and (2.8). We also note that the exponent μ has a universal value around 0.7 for the three two-dimensional planar lattices. The identification of this exponent in terms of ordinary percolation is more tricky. Since p in (2.4) is the total acceptance fraction, analogous perhaps to the occupation fraction of ordinary percolation, the natural identification is $\mu = 1/\nu$, where ν is the correlation

Table 2. Results of Monte Carlo simulations of invasion percolation for four different lattices. ‘No trapping’ refers to the invader threshold without trapping, ‘Trapping 1’ to the invader threshold with trapping, and ‘Trapping 2’ to the final point where the defender becomes completely disconnected. The various quantities are defined in and around (2.1), (2.5), (2.8), (2.9), (2.10) and (2.13). It is hard to estimate the accuracy of the given exponents, but we expect that for α and α' the errors are of order 0.02 and for μ , μ_2 and τ , which are harder to measure, the errors are somewhat larger.

Lattice	No Trapping			Trapping 1		Trapping 2		
	α	p_c	μ	α'	μ_2	α'	μ_2	τ
Honeycomb	0.10	0.6983	0.71	0.18	1.27	0.18	1.27	1.80
Square	0.12	0.5931	0.70	0.18	1.19	0.19	1.21	1.84
Triangular	0.11	0.5013	0.69	0.12	0.91	0.12	0.89	1.80
Simple cubic	0.48	0.3116	1.07	0.48	1.26		1.33 [†]	2.07

[†] At $r_{\max} = 0.6884$.

length exponent. A more detailed hypothesis leading to the same conclusion is that as $r \rightarrow p_c$ and $L \rightarrow \infty$, the acceptance profile $a(r)$ depends only on the scaled variable $|r - p_c|^\nu L$. Our obtained values for μ are probably consistent with the accepted values for ν of 1.33 in two dimensions and 0.88 in three dimensions.

Let us now introduce the trapping phenomenon described in the introduction. In this mode we must add the rule that once a cluster of the defender has become isolated, it can no longer be invaded. Let us again stop the process at the point in time when the invader first percolates. In two dimensions, our Monte Carlo data indicate that the invader fraction S_I again has a power-law dependence on lattice size:

$$S_I = A' L^{-\alpha'}, \tag{2.9}$$

where the exponent α' is shown in the fourth column of table 2. For the triangular lattice the exponent is close to its no-trapping value, but for the honeycomb and square lattices the exponent now takes the larger value $\alpha' \sim 0.18$, corresponding to a fractal dimension 1.82. (In the bond version of the problem mentioned in the introduction, all planar two-dimensional lattices exhibit this modified fractal dimension around 1.82 (Chandler *et al* 1982).) Despite the relatively small lattices used in these simulations ($L_{\max} = 100$) we believe that this change in fractal dimension is significant, and that the phenomenon of trapping has changed the universality class from that of ordinary percolation. Intuitively we see that the change of fractal dimension is in the right direction, since the trapping rule prevents the invader from occupying some sites which it would otherwise have occupied, thereby conceivably lowering its fractal dimension. A possible reason for the different behaviour of the triangular lattice will be discussed below.

The acceptance profile shows even more difference from the no-trapping case. The results for the square lattice in figure 3(a) suggest that as $L \rightarrow \infty$ the profile does not approach a step function, but rather that $A_1(r)$ has a finite limit as $L \rightarrow \infty$ for any choice of r . In terms of the definitions (2.7) this means that $r_{\min} = 0$. This is intuitively correct, since with the trapping rule in effect there is a finite probability that any random number which appears on the interface will be surrounded by the invader before it is itself invaded. We expect, however, that r_{\max} should still equal p_c , since on the one hand the trapping rule cannot force the invader to occupy larger random

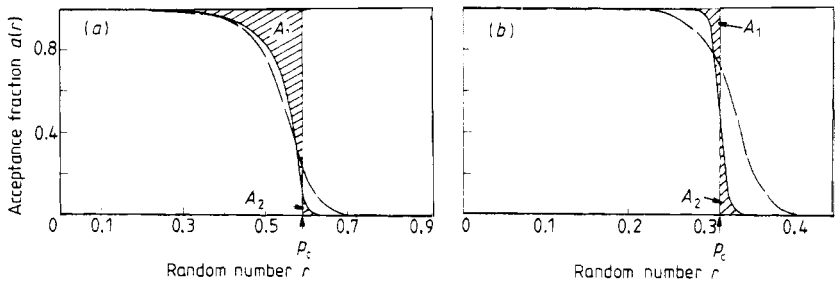


Figure 3. Acceptance profile $a(r)$ for the (a) square and (b) simple cubic lattices at the percolation point in the presence of the trapping rule. It is conjectured that as $L \rightarrow \infty$ the area A_2 vanishes but A_1 approaches a finite value. This is more apparent in two dimensions, but we believe that it is also the case in three dimensions. In figures 3, 4 and 5 the number of realisations in two dimensions varies from about 1600 for $L = 20$ to 400 for $L = 100$, and in three dimensions from 200 for $L = 10$ to 50 for $L = 30$. (a) --- $L = 20$, — $L = 100$; (b) ---, $L = 10$, — $L = 30$.

numbers than it would otherwise, while on the other hand the invader still has to pick numbers right up to p_c in order to percolate. A fit to the power law (2.8b) with the same values of p_c as before gives an exponent μ_2 shown in column five of table 2. We see that the values are considerably larger than in the no-trapping case, around 1.2 for the honeycomb and square lattices, and 0.9 for the triangular lattice. This is consistent with the observation that random numbers above p_c become more likely to be trapped, and hence not chosen, as the lattice size grows, since it becomes possible for them to be trapped inside larger and larger clusters.

In three dimensions the trapping rule has much less effect, since the invader must form closed surfaces in order to disconnect the defender. Indeed it is found that all the trapped regions are very small, and larger ones do not appear as the lattice size is increased. The invader fraction at the percolation point is indistinguishable from the no-trapping case, obeying the same law (2.1) with the same exponent $\alpha \sim 0.48$. The acceptance profile, shown in figure 3(b), is also much closer to the no-trapping case than in two dimensions. However, it is still true that r_{\min} must be zero; the limiting profile is presumably close to, but not exactly, a step function. Assuming r_{\max} to be the same $p_c \sim 0.3116$ we obtain an exponent μ_2 around 1.26, again larger than in the no-trapping case, though the change is not as marked as in two dimensions.

With the trapping mechanism activated it becomes interesting to continue the invasion process beyond the percolation point, since the invader is now prevented from filling the entire lattice. Eventually a second percolation threshold is reached at which the defender consists only of isolated clusters, and the process must stop. In our porous medium model we say that the system has reached residual oil saturation; no more oil can be displaced from the sample without increasing the flow rate and bringing viscous forces into play. The two percolation thresholds in some sense correspond to the two thresholds of ordinary percolation theory, but the situation is not symmetrical, since the invader always consists of a single cluster whereas the defender gets broken up into many clusters.

The invader fraction at this second percolation threshold is shown in figure 4. Here we see an even greater difference between two and three dimensions. In two dimensions we again have the power-law behaviour

$$S_1 \sim A'' L^{-\alpha'}, \quad (2.10)$$

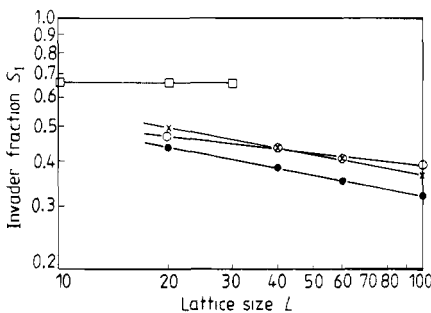


Figure 4. Invader fraction S_1 at the point where the defender consists only of disconnected clusters, plotted as a function of lattice size L . Note the completely different behaviour in two and three dimensions, and also that the triangular lattice is somewhat different from the other two-dimensional lattices. \square , simple cubic; \times , honeycomb; \bullet , square; \circ , triangular.

with $\alpha' \sim 0.18$ for the honeycomb and square lattice and $\alpha' \sim 0.12$ for the triangular lattice, but in three dimensions we see a completely different behaviour suggesting a constant value

$$S_1 \sim 0.66 \tag{2.11}$$

as $L \rightarrow \infty$. This corresponds to a defender fraction of 0.34, which is close to but significantly different from the percolation threshold 0.31 for a simple cubic lattice. This closeness is not surprising since the defender is clearly at a percolation threshold of some kind; the reason for the difference will be discussed in detail below.

Let us now consider the acceptance profile at this second percolation threshold. In two dimensions, the results are almost indistinguishable from those at the invader percolation threshold (figure 3(a)). The reason for this is that in two dimensions the two thresholds are almost identical, since the invader and defender cannot both percolate at the same time; as soon as the invader percolates from end to end of the sample, it also percolates from side to side, thus disconnecting the defender. In three dimensions, however, this is not true, and the invader must occupy many more sites before it forms the closed surface necessary to disconnect the defender. The acceptance profile in three dimensions is shown in figure 5. Again we must have $r_{\min} = 0$, and the data suggest a limiting profile for large L which cuts off at

$$r_{\max} = 0.6884. \tag{2.12}$$

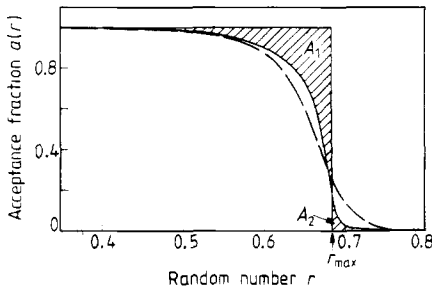


Figure 5. Acceptance profile $a(r)$ for the simple cubic lattice at the point where the defender consists only of isolated clusters. It is conjectured that as $L \rightarrow \infty$ the profile cuts off at $r_{\max} = 1 - p_c \sim 0.6884$. --- $L = 10$, — $L = 30$.

(In this case the value of r_{\max} was determined by requiring the best possible fit to the power law (2.8b); the corresponding exponent is shown in table 2.) Note that, in contrast to S_1 , the value of r_{\max} is precisely that of $1 - p_c$ for the simple cubic lattice. The reasons why the invader fraction S_1 is less than r_{\max} are twofold. Firstly there are some random numbers below r_{\max} which never become available to the invader, since they are contained inside trapped defender clusters. Secondly some random numbers below r_{\max} appear on the boundary, but are surrounded before they can be invaded. The latter is the more important effect, since the defender clusters are very ramified and have very little interior.

We see that the phenomenon of defender trapping is completely different in two and three dimensions. In two dimensions, the trapping occurs as soon as the invader percolates, when it has occupied random numbers only up to p_c , and is still a fractal set. In three dimensions, the trapping does not occur until the invader has consumed

random numbers up to $1 - p_c$, by which time it occupies a finite fraction of the lattice. This suggests a reason why the triangular lattice behaves somewhat differently from the other two-dimensional lattices: p_c is exactly 0.5 so that p_c and $1 - p_c$ coincide, in contrast to the honeycomb and square lattices where $p_c > 0.5$. In some sense the triangular lattice, though more like two dimensions than three dimensions, is an intermediate case. This comes out more clearly if one considers a non-planar two-dimensional lattice such as that obtained by inserting all the diagonals on a square lattice. It is found that this lattice behaves qualitatively like the simple cubic lattice (but with two-dimensional exponents), so that in fact the correct distinction is not between two and three dimensions, but rather between planar and non-planar lattices.

The notion that one is indeed near a critical threshold when the defender becomes disconnected may be sharpened by examining the size distribution of defender clusters at this threshold. By analogy with static percolation, scale invariance at the critical point leads one to predict that the number of occurrences $n(s)$ of blobs containing s sites should scale with s according to the formula (Stauffer 1979)

$$n(s) = n_0 s^{-\tau}. \quad (2.13)$$

Our Monte Carlo data for the largest lattices considered ($L_{\max} = 100$ in two dimensions and $L_{\max} = 30$ in three dimensions) yield a good fit to this power law over a wide range of s with an exponent τ as shown in the last column of table 2. In the case of static percolation, the exponent τ can be expressed in terms of the fractal dimension ϕ and space dimension d through the formula

$$\tau = (d + \phi)/\phi. \quad (2.14)$$

Recall that the fractal dimension of the invader cluster at breakthrough was $\phi = 2.52$ in three dimensions. Inserting this value into the equation above leads to $\tau = 2.19$. This is slightly larger than the value 2.07 observed on a $30 \times 30 \times 60$ lattice. Assuming that (2.14) defines the fractal dimension of the finite clusters of defenders, and that the observed value of τ is considered correct, we see these have a slightly larger fractal dimension than that of the invader at breakthrough. This seems to reflect again that the difference in mechanisms through which the invader spanning cluster and the defender almost-spanning-clusters are generated can have a significant effect on the critical exponents, though the quality of our data is probably not sufficient to make a definitive statement.

In two dimensions we have a completely different situation, since the measured values of τ are less than 2. In ordinary percolation we see from (2.14) that this is impossible since ϕ must be less than d . Another way to see this is to observe that if $n(s)$ is normalised to be the number of s -site clusters per site, then the sum $\sum sn(s)$ must converge, since in principle the process of laying down occupied sites at random can be performed on an infinite lattice. In our modified percolation with trapping, however, the finite size of the system is an essential feature. We have observed that, although the exponent τ is approximately independent of L , the prefactor n_0 in (2.13) decreases with increasing lattice size. Although we have not investigated this in detail, presumably this decrease is just what is required to keep $\sum sn(s)$ finite as $L \rightarrow \infty$.

3. Summary and discussion

We first list what we believe are the essential features of the invasion percolation process illustrated above. Clearly, there must exist local variations in the medium

which have a strong effect on the invasion process. In our specific model, these local variations are random, but there is no difficulty in principle in introducing spatial correlations if desired. In addition, the randomness should be sufficiently broad to warrant the 'one move at a time' dynamic we have used in our simulations. Smoother advance of the interface would occur if, say, all random numbers in the intervals $[0.0-0.1]$, $[0.1-0.2]$, etc, could be invaded simultaneously. Thus, neither the medium nor the process have a built-in length scale (other than the lattice spacing), and the only length which can enter into the simulation of the invasion is the lattice size measured in units of the lattice spacing.

In the absence of the trapping rule we believe that invasion percolation is closely related to ordinary static percolation. This may be understood intuitively as follows. Imagine painting red those lattice sites on a given realisation whose random numbers are less than or equal to p^* , that being the smallest random number required for the red cluster to span. Clearly invasion percolation on this same realisation of the lattice takes place on a subset of points of the red cluster. On average over realisations we have $p^* \sim p_c$. This suggests that the invader fraction S_I , analogous to the order parameter $P(p)$ of static percolation, exhibits finite size scaling, and it appears from our simulations that the exponent which governs this behaviour is the same as that of ordinary percolation theory. By contrast, when the trapping rule is in effect, there are many differences from ordinary percolation, and it is even possible for lattices which exhibit the same behaviour in ordinary percolation to have non-universal behaviour in invasion percolation.

In addition to the quantities discussed so far, it may be of interest to examine a more general set of observables. Let us consider fixed point in 'time' in the simulation, say at the breakthrough point. In the k th realisation at fixed lattice size L , we can define (suppressing the k and L labels) functions

$$\pi(x) = \text{value of random number at lattice site } x, \tag{3.1a}$$

$$\rho(x) = 1 \text{ if } x \text{ has been invaded, } 0 \text{ otherwise.} \tag{3.1b}$$

(Of course, $\pi(x)$ is 'time' independent in the model as defined.) Every other 'observable' of the system is derivable from these two quantities, for example,

$$S_I = \frac{1}{L^d} \sum_{i=1}^{L^d} \rho(x_i). \tag{3.2}$$

The number of sites on the boundary of the cluster is

$$B = \sum_{i=1}^{L^d} \Theta \left((1 - \rho(x_i)) \sum_{\epsilon} \rho(x_i + \epsilon) \right), \tag{3.3}$$

where ϵ is a lattice unit vector, and Θ is a function which equals zero when its argument vanishes, and unity otherwise.

We can also introduce spatial correlation functions:

$$C_2(y) = L^{-d} \sum_x \rho(x) \rho(x + y). \tag{3.4}$$

Finally, every such observable quantity Q_k defined in the k th realisation of an ensemble of R realisations can be ensemble averaged:

$$\langle Q \rangle = \frac{1}{R} \sum_{k=1}^R Q_k. \tag{3.5}$$

Let $[\]$ denote the scaling dimension of a quantity with respect to the lattice size L . For example, we see from (2.1), (2.2) that when the invader first percolates $[\langle S_1 \rangle]$ is $(\phi - d)$. It follows that $[\langle \rho \rangle]$ is also $(\phi - d)$. From this one might predict that the correlation function $\langle C_2(y) \rangle$ scales as $L^{-2(\phi-d)} F(|y|/L)$. Scaling relationships of this form hold for conventional thermodynamic systems (Stanley 1971), and for ordinary percolation. It will be very interesting to investigate whether this hypothesis is true in invasion percolation.

Another question which arises is whether the correlation function in (3.4) is isotropic. In this paper it has been tacitly assumed that invasion percolation should be compared to ordinary isotropic percolation, rather than, say, directed percolation, despite the fact that in the geometry we have considered there is a preferred 'flow' direction. This seems reasonable, since the rules by which the invader grows are isotropic, and one might expect that the effects of the boundary conditions would diminish as $L \rightarrow \infty$, but this remains to be substantiated.

We believe it will be worthwhile to pursue the above considerations not simply because they emerge naturally from the invasion process we are considering, but also because questions of this kind might naturally arise in other more complex transport scenarios. We must ask: Which are the observables of the system? What functionals of these are physically meaningful in the ensemble average? What scaling hypotheses are appropriate for the physically meaning functional near critical points?

Many interesting question such as the above arise in this new form of percolation theory and it would be of interest to study them in a more formal manner, as contrasted to the essentially descriptive approach of the present paper. Unfortunately this is very difficult since at present there exists no framework analogous to, say, the mapping of ordinary percolation onto the Potts model (Kasteleyn and Fortuin 1969). Even the simplest problem of growing a cluster from a point into an infinite lattice without the trapping rule appears intractable, since it is very difficult to write down the weight which should be given to a given configuration. In ordinary percolation this is trivial; the difficulty there resides solely in summing over all configurations. Our model has been motivated by a straightforward real world problem, which seems to disguise a rich and interesting theoretical structure. The development of a mathematical framework for discussing this structure poses a very interesting problem.

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