

# Vector Spherical Harmonics, Definitions and Identities

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I will use somewhat interchangeably the notations of Edmonds,  $\vec{Y}_{JLM}(\theta, \phi)$ , and that of Varshalovich et al.,  $\vec{Y}_{JM}^L(\theta, \phi)$ . The basic definition of these is in terms of Clebsch-Gordan coefficients  $\langle Lm, s\sigma | JM \rangle$ , and the spherical tensor components of the Cartesian unit vectors (with spin  $s = 1$ ),  $\hat{e}_{\pm 1} \equiv \mp \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$ ,  $\hat{e}_0 \equiv \hat{z}$ :

$$\vec{Y}_{JLM}(\theta, \phi) \equiv \vec{Y}_{JM}^L(\theta, \phi) \equiv \sum_{m\sigma} Y_{Lm}(\theta, \phi) \hat{e}_\sigma \langle Lm, s\sigma | JM \rangle.$$

These obey nice orthonormality relations,

$$\sum_M \vec{Y}_{JM}^{L'}(\theta, \phi)^* \cdot \vec{Y}_{JM}^L(\theta, \phi) = \delta_{LL'},$$

and

$$\int \vec{Y}_{J'M'}^{L'}(\theta, \phi)^* \cdot \vec{Y}_{JM}^L(\theta, \phi) d\Omega = \delta_{LL'} \delta_{JJ'} \delta_{MM'}.$$

The components transverse and parallel to  $\hat{n} \equiv (\theta, \phi)$ , are given by:

$$\hat{n} \cdot \vec{Y}_{JM}^{J+1}(\theta, \phi) = -\sqrt{\frac{J+1}{2J+1}} Y_{JM}(\theta, \phi), \quad \hat{n} \cdot \vec{Y}_{JM}^J(\theta, \phi) = 0,$$

$$\hat{n} \cdot \vec{Y}_{JM}^{J-1}(\theta, \phi) = \sqrt{\frac{J}{2J+1}} Y_{JM}(\theta, \phi).$$

$$\hat{n} \times \vec{Y}_{JM}^{J+1}(\theta, \phi) = i\sqrt{\frac{J}{2J+1}} \vec{Y}_{JM}^J(\theta, \phi), \quad \hat{n} \times \vec{Y}_{JM}^{J-1}(\theta, \phi) = i\sqrt{\frac{J+1}{2J+1}} \vec{Y}_{JM}^J(\theta, \phi)$$

$$\hat{n} \times \vec{Y}_{JM}^J(\theta, \phi) = i\sqrt{\frac{J+1}{2J+1}} \vec{Y}_{JM}^{J-1}(\theta, \phi) + i\sqrt{\frac{J}{2J+1}} \vec{Y}_{JM}^{J+1}(\theta, \phi).$$

Of particular interest in many applications are the curl and divergence of the vector spherical harmonics, multiplied by an arbitrary solution to the spherical Bessel equation,  $z_L(kr)$ :

$$\frac{1}{k} \nabla \times [z_J(kr) \vec{Y}_{JM}^J(\theta, \phi)] = -i\sqrt{\frac{J}{2J+1}} z_{J+1}(kr) \vec{Y}_{JM}^{J+1}(\theta, \phi) + i\sqrt{\frac{J+1}{2J+1}} z_{J-1}(kr) \vec{Y}_{JM}^{J-1}(\theta, \phi)$$

$$\frac{1}{k} \nabla \times [z_{J+1}(kr) \vec{Y}_{JM}^{J+1}(\theta, \phi)] = i\sqrt{\frac{J}{2J+1}} z_J(kr) \vec{Y}_{JM}^J(\theta, \phi)$$

$$\frac{1}{k} \nabla \times [z_{J-1}(kr) \vec{Y}_{JM}^{J-1}(\theta, \phi)] = -i\sqrt{\frac{J+1}{2J+1}} z_J(kr) \vec{Y}_{JM}^J(\theta, \phi)$$

$$\frac{1}{k} \nabla \cdot [z_{J+1}(kr) \vec{Y}_{JM}^{J+1}(\theta, \phi)] = -\sqrt{\frac{J+1}{2J+1}} z_J(kr) Y_{JM}(\theta, \phi)$$

$$\frac{1}{k} \nabla \cdot [z_J(kr) \vec{Y}_{JM}^J(\theta, \phi)] = 0$$

$$\frac{1}{k} \nabla \cdot [z_{J-1}(kr) \vec{Y}_{JM}^{J-1}(\theta, \phi)] = -\sqrt{\frac{J}{2J+1}} z_J(kr) Y_{JM}(\theta, \phi)$$

Sometimes it is useful to have the components of the curl that are parallel or perpendicular to  $\hat{r}$ . These are:

$$\hat{r} \times \{\nabla \times [z_J(kr) \vec{X}_{JM}(\theta, \phi)]\} = \frac{k}{2J+1} [Jz_{J+1}(kr) - (J+1)z_{J-1}(kr)] \vec{X}_{JM}(\theta, \phi)$$

$$\hat{r} \cdot \{\nabla \times [z_J(kr) \vec{X}_{JM}(\theta, \phi)]\} = ik \frac{\sqrt{J(J+1)}}{2J+1} [z_{J+1}(kr) + z_{J-1}(kr)] Y_{JM}(\theta, \phi)$$

And the expansion of a vector plane wave can be written as:

$$\hat{\varepsilon}(\hat{k}) e^{i\vec{k} \cdot \vec{r}} = 4\pi \sum_{JLM} i^L j_L(kr) [\hat{\varepsilon}(\hat{k}) \cdot \vec{Y}_{JM}^{L*}(\hat{k})] \vec{Y}_{JM}^L(\hat{r})$$

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- [1] A. R. Edmonds, Angular Momentum in Quantum Mechanics, 2nd edition (1960, Princeton University Press, 3rd printing with corrections 1974).  
 [2] D. A. Varshalovich et al., Quantum Theory of Angular Momentum.