Sec. 16.2 Radiative reaction force

Since acceleration of a charge q by an external force $\vec{F}_{ext}$ causes it to radiate, we know that the process of radiation must remove kinetic energy from q.

i.e. the radiation must exert a recoil force on q, which we denote by $\vec{F}_{rad}$.

To keep things simple, consider first the nonrelativistic limit where

$$m\dot{\vec{u}} = \vec{F}_{ext}$$

But since the power radiated is

$$P(t) = \frac{2}{3}\frac{q^2}{c^3}\dot{\vec{u}}^2,$$

this suggests that there must be some force opposing the motion, which therefore reduces $|\vec{u}|$.

An energy conservation might seem to suggest that this force should satisfy

$$\vec{F}_{rad} \cdot \vec{u} = -\frac{2}{3}\frac{q^2}{c^3}\dot{\vec{u}}^2,$$

but in fact this is not true in general.

Why?

⇒ One reason is that EM fields associated with a charge moving at uniform velocity $\vec{u}$ ALSO carry energy.

⇒ It is not ONLY in the radiated fields.

⇒ A more correct statement of energy conservation would be:

(Energy lost by particle during $t_1 \to t_2$)

= (Energy carried away by radiation)

+ (change in energy stored in the velocity fields)

But, observe that if the particle motion is periodic (a special case), then the only net energy loss is due to radiation, i.e.

$$\int_{t_1}^{t_2} \vec{F}_{rad} \cdot \vec{u}\, dt = -\int_{t_1}^{t_2} \frac{2}{3}\frac{q^2}{c^3} \dot{\vec{u}}^2 dt$$

But $\int_{t_1}^{t_2} a^2 dt = \int_{t_1}^{t_2} \frac{d\vec{u}}{dt} \cdot \frac{d\vec{u}}{dt} dt$

$$= \cancel{\vec{u} \cdot \frac{d\vec{u}}{dt}\Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2} \frac{d^2\vec{u}}{dt^2} \cdot \vec{u}\, dt$$

0

One can also consider this valid in a perturbative sense, if the change in velocity during this time interval is "small".

$$\Rightarrow \int_{t_1}^{t_2} \left[\vec{F}_{rad} - \frac{2}{3}\frac{q^2}{c^3}\ddot{\vec{u}}\right] \cdot \vec{u}\, dt = 0$$

$$\Rightarrow \boxed{\vec{F}_{rad} = \frac{2}{3}\frac{q^2}{c^3}\dot{\vec{a}}}$$

← This is called the Abraham-Lorentz equation.

Or defining $\tau \equiv \frac{2q^2}{3mc^3}$ which has units of time,

$$\Rightarrow \boxed{\vec{F}_{rad} = m\tau\ddot{\vec{u}}}$$

Note that for an e^-,

$\tau = 6.3 \times 10^{-24}$ s

or $c\tau = 10^{-15}$ m

(Of course, this derivation will not determine any component of $\vec{F}_{rad}$ that is $\perp$ to $\vec{u}$.)

And it is a plausible conjecture that the modified equation of motion might be

$$m\dot{\vec{u}} - m\tau\ddot{\vec{u}} = \vec{F}_{ext}$$

This is called the Abraham–Lorentz equation of motion.

One interesting (though seemingly trivial) limit is when $\vec{F}_{ext} = 0$.

$$\Rightarrow \vec{a} - \tau\dot{\vec{a}} = 0$$

This simple 1^{st}-order equation has the solution

$$\vec{a}(t) = \vec{a}(0)\, e^{t/\tau}$$

This solution exhibits exponentially divergent self-acceleration, which is unphysical! This is unacceptable *except* in the special case,

$$\vec{a}(t) = \vec{a}(0) = 0.$$

Accordingly, one usually considers an alternative approach that does not suffer from this pathological runaway behavior:

$\Rightarrow$ use $\frac{d}{dt}\left(m\dot{\vec{u}} = \vec{F}_{ext}\right)$ to eliminate $m\ddot{\vec{u}}$.

i.e. $\quad m\ddot{\vec{u}} = \dfrac{d\vec{F}_{ext}}{dt}$ which gives instead

$$\boxed{m\dot{\vec{u}} = \vec{F}_{ext} + \tau \frac{d\vec{F}_{ext}}{dt}}$$

or since $\quad \vec{F}_{ext} = \vec{F}_{ext}(\vec{x}(t), t)$

$$\Rightarrow \frac{d}{dt}\vec{F}_{ext} = \frac{\partial \vec{F}_{ext}}{\partial t} + (\vec{u}\cdot\nabla)\vec{F}_{ext}$$

$$\Rightarrow \boxed{m\dot{\vec{u}} = \vec{F}_{ext} + \tau\left[\frac{\partial \vec{F}_{ext}}{\partial t} + (\vec{u}\cdot\nabla)\vec{F}_{ext}\right]}$$

This is Eq. 16.10, and it has no runaway solutions.

<u>Example</u> Apply this to a simple harmonic oscillator in 1D, with

$$F_{ext} = -kx = -m\omega_0^2 x$$

$$\Rightarrow \quad m\ddot{x} = -m\omega_0^2 x + \tau\left(\cancelto{0}{\frac{\partial F_{ext}}{\partial t}} + \dot{x}\frac{\partial F_{ext}}{\partial x}\right)$$

$$\Rightarrow \quad m\ddot{x} = -m\omega_0^2 x - \tau m\omega_0^2 \dot{x}$$

$$\Rightarrow \ddot{x} + \tau\omega_0^2\dot{x} + \omega_0^2 x = 0$$

This equation has solutions of the form (the real part of):

$$x(t) = x_0 e^{-i\omega t}$$

$$\Rightarrow \left(-\omega^2 - i\tau\omega_0^2\,\omega + \omega_0^2\right) x_0 = 0$$

$$\Rightarrow \omega = \pm\left[\omega_0^2 - \left(\frac{\omega_0^2\tau}{2}\right)^2\right]^{1/2} - \frac{i\omega_0^2\tau}{2}$$

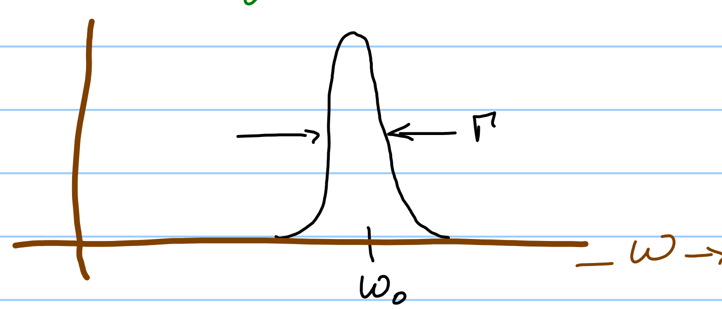
Thus, energy decays at the rate

$$e^{2(\mathrm{Im}\,\omega)t} = e^{-(\omega_0^2\tau)t}$$

i.e. $e^{-\Gamma t}$ where $\Gamma = \omega_0^2\tau$

And if this damped oscillator is driven, monochromatically, its frequency response looks like

See Sec. 16.7



But problems remain in attempting to formulate a fully consistent theory of radiation reaction. One finds intriguing comments (e.g. Dirac, Proc. R. Soc. Lond. A 167, 148 (1938)) "... the interior of the electron being a region ... allowing signals to propagate at v>c" 251

$\Rightarrow$ Dirac proposes to use a modified field $f^{\mu\nu}$ in the equations of motion of q, namely

$$f^{\mu\nu} = F^{\mu\nu}_{actual} - \tfrac{1}{2}\left(F^{\mu\nu}_{ret} + F^{\mu\nu}_{adv}\right)$$

Wheeler + Feynman
1945 Rev. Mod. Phys.
follow + extend some of Dirac's ideas

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Interaction with the Absorber as the Mechanism of Radiation†*

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"We must, therefore, be prepared to find that further advance into this region will require a still more extensive renunciation of features which we are accustomed to demand of the space time mode of description."—Niels Bohr[1]

To carry the analysis further requires us to find a new idea. We go back to a suggestion once made by Tetrode.[10] He proposed to abandon the conception of electromagnetic radiation as an elementary process and to interpret it as a consequence of an interaction between a source and an absorber. In his words,

Tetrode quote 1922

"The sun would not radiate if it were alone in space and no other bodies could absorb its radiation. . . . If for example I observed through my telescope yesterday evening that star which let us say is 100 light years away, then not only did I know that the light which it allowed to reach my eye was emitted 100 years ago, but also the star or individual atoms of it knew already 100 years ago that I, who then did not even exist, would view it yesterday evening at such and such a time. . . . One might accordingly adopt the opinion that the amount of material in the universe determines the rate of emission. Still this is not necessarily so, for two competing absorption centers

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[10] H. Tetrode, Zeits. f. Physik **10**, 317 (1922). When we gave a preliminary account of the considerations which appear in this paper (Cambridge meeting of the American Physical Society, February 21, 1941, Phys. Rev. **59**, 683 (1941)) we had not seen Tetrode's paper. We are indebted to Professor Einstein for bringing to our attention the ideas of Tetrode and also of Ritz, who is cited in this

252

article. An idea similar to that of Tetrode was subsequently proposed by G. N. Lewis, Nat. Acad. Sci. Proc. **12**, 22 (1926): "I am going to make the . . . assumption that an atom never emits light except to another atom, and to claim that it is as absurd to think of light emitted by one atom regardless of the existence of a receiving atom as it would be to think of an atom absorbing light without the existence of light to be absorbed. I propose to eliminate the idea of mere emission of light and substitute the idea of *transmission*, or a process of exchange of energy between two definite atoms or molecules." Lewis went nearly as far as it is possible to go without explicitly recognizing the importance of other absorbing matter in the system, a point touched upon by Tetrode, and shown below to be essential for the existence of the normal radiative mechanism.

Wheeler - Feynman

Our picture of the mechanism of radiation is seen to be self-consistent. Any particle on being accelerated generates a field which is half-advanced and half-retarded. From the source a disturbance travels outward into the surrounding absorbing medium and sets into motion all the constituent particles. They generate a field which is equal to half the retarded minus half the advanced field of the source. In this field we have the explanation of the radiation field assumed by Dirac. The radiation field combines with the field of the source itself to produce the usual retarded effects which we expect from observation, and such retarded effects only. The radiation field also acts on the source itself to produce the force of radiative reaction. What we have said of one particle holds for every particle in a completely absorbing medium. All advanced fields are concealed by interference. Their effects show up directly only in the force of radiative reaction. Otherwise we appear to have a system of particles acting on each other via purely retarded forces.

Greene 2012, wacky idea

A conjecture – using Monte Carlo one could sample the time and direction and energy of photon emission as discrete events.

Then each time a photon is emitted with frequency ω, wavevector $\vec{k}$, the instantaneous 4-momentum of the particle q should be reduced by $\left(\frac{\hbar\omega}{c}, \hbar\vec{k}\right)$

$$\text{i.e.} \quad p^\alpha \longrightarrow p^\alpha - \left(\frac{\hbar\omega}{c}, \hbar\vec{k}\right)$$

Then continue solving the equations of motion in the external field neglecting radiation emission until Monte Carlo sampling says that the next photon is emitted, etc.