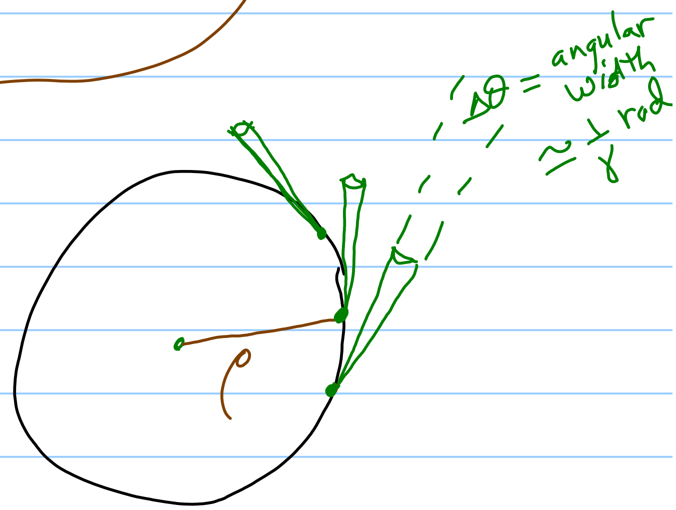
Sec. 14.4 Frequency distribution of radiated light – qualitative analysis

Let's still consider circular motion

$$\Rightarrow \quad \boxed{\begin{aligned} a_\perp &= \omega^2 \rho = \omega u \simeq \omega c \\ u &= \omega \rho \simeq c \end{aligned}}$$

Recall that the angular width of the radiation cone is $\Delta\theta \simeq \dfrac{1}{\gamma}$



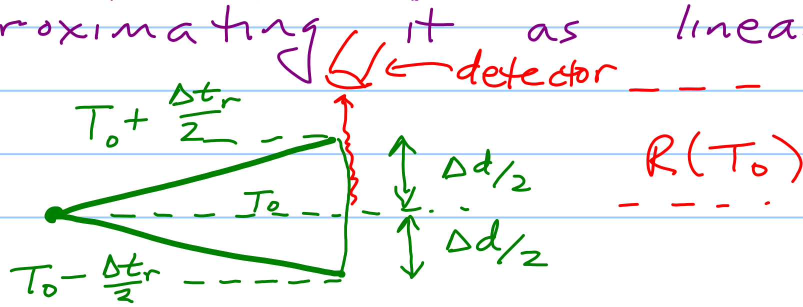
So in a time $\Delta t_{rad} \simeq \dfrac{\Delta\theta}{\omega} \simeq \dfrac{\rho}{\gamma u}$,

the distance the particle moves is approximately

$$\Delta d \simeq \rho \Delta\theta = \frac{\rho}{\gamma}$$

$\Rightarrow \quad \Delta t_{rad} \simeq \dfrac{\rho}{\gamma u}$ is the time duration when the radiated "searchlight" illuminates an observer

Let's expand this stretch of the particle's motion, approximating it as linear



To the observer, radiation begins to arrive at $t_1 = T_0 - \frac{\Delta t_r}{2} + \frac{R + d/2}{c}$

and it ends at

$$t_2 = T_0 + \frac{\Delta t_r}{2} + \frac{R - \frac{d}{2}}{c}$$

$$\Rightarrow \Delta t = t_2 - t_1 = \Delta t_r - \frac{d}{c} = \frac{\rho}{\gamma u} - \frac{\rho}{\gamma c}$$

Or since $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = (1-\gamma^{-2})^{1/2}$
or $\beta \simeq 1 - \frac{1}{2\gamma^2}$

$$\Rightarrow \boxed{\Delta t = \frac{\rho}{\gamma c}\left(\frac{1}{\beta} - 1\right) \simeq \frac{\rho}{2c\gamma^3}}$$

Now, from Fourier analysis, we expect that $\Delta\omega\,\Delta t \gtrsim \frac{1}{2}$, whereby

$\omega_{cutoff} \simeq \Delta\omega \simeq \frac{1}{2\Delta t}$ is an estimate of the highest usable light frequency, i.e., having appreciable intensity.

$$\Rightarrow \omega_{cutoff} = \frac{c}{\rho}\gamma^3 = \frac{a_\perp \gamma^3}{c}$$

e.g. at the 7 GeV APS synchrotron, $\gamma \simeq 14{,}000$; and the bending radius is $\rho \simeq 39\,m$

$$\Rightarrow \omega_{cutoff} \simeq \frac{3\times10^8 \frac{m}{s}}{39\,m}(1.4\times10^4)^3 = 2.1\times10^{19}\,\frac{rad}{s}$$

$$\Rightarrow \hbar\omega_{cutoff} \simeq 14 \text{ keV photons}$$

In fact, from the APS website, the photon energy at peak intensity is $\hbar\omega_{peak} \simeq 20$ keV although usable intensity is delivered up to even 100 keV photons, and somewhat beyond.

Sec. 14.5 Quantitative treatment of frequency and angle distributions.

Consider $\dfrac{dP(t)}{d\Omega} = |\vec{A}(t)|^2$ with $\vec{A}(t) = \sqrt{\frac{c}{4\pi}}\,[R\vec{E}]_{ret}$

and we write this in the OBSERVER's time since we are interested in the frequencies detected.

$$\Rightarrow \frac{\text{Total energy}}{\text{unit solid angle}} = \int_{-\infty}^{\infty} |\vec{A}(t)|^2 dt \equiv \frac{dW}{d\Omega}$$

The following Fourier representation is useful,

$$\vec{A}(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \vec{A}(t)\, e^{i\omega t}\, dt$$

and $$\vec{A}(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \vec{A}(\omega)\, e^{-i\omega t}\, d\omega$$

So, $$\frac{dW}{d\Omega} = \frac{1}{2\pi}\int dt \int d\omega \int d\omega' \vec{A}^*(\omega')\cdot\vec{A}(\omega)\, e^{i(\omega'-\omega)t}$$

t-integral gives $\delta(\omega'-\omega)$

$$\Rightarrow \boxed{\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\vec{A}(\omega)|^2\, d\omega}$$

this result is just Parseval's theorem

Since $\vec{A}(t) = \text{real}$, $\Rightarrow \vec{A}(-\omega) = \vec{A}^*(\omega)$, this can be written as an integral over just POSITIVE frequencies, i.e. as

$$\frac{dW}{d\Omega} = \int_0^{\infty} \frac{d^2 I(\omega,\hat{n})}{d\omega\, d\Omega}\, d\omega$$

where $$\frac{d^2 I}{d\omega\, d\Omega} \equiv |\vec{A}(\omega)|^2 + |\vec{A}(-\omega)|^2$$
$$= 2|\vec{A}(\omega)|^2$$

And in our problem,

$$\vec{A}(\omega) = \left(\frac{q^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega t}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3}\right]_{ret} dt$$

and as usual $t = t_r + \frac{R(t_r)}{c}$

Then we can change variables to t_r using $\frac{dt}{dt_r} = 1 - \vec{\beta}\cdot\hat{n}$, giving

$$\vec{A}(\omega) = \left(\frac{q^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega\left(t_r + \frac{R(t_r)}{c}\right)} \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2}\, dt_r$$

and at large distances as we saw back in Chap. 9,

$$R(t_r) \simeq x - \hat{n}\cdot\vec{r}(t_r)$$

And the text points out that this integral will be simpler if we use the fact that:

$$\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2} = \frac{d}{dt_r}\left(\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right)$$

This suggests that we integrate by parts, giving

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{n}\times(\hat{n}\times\vec{\beta})\, e^{i\omega\left[t_r - \frac{\hat{n}\cdot\vec{r}(t_r)}{c}\right]}\right|^2$$

This expression generalizes to <u>many</u> charges by replacing:

$$q\vec{\beta}\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}(t_r)} \longrightarrow \sum_j q_j \vec{\beta}_j\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}_j(t_r)}$$

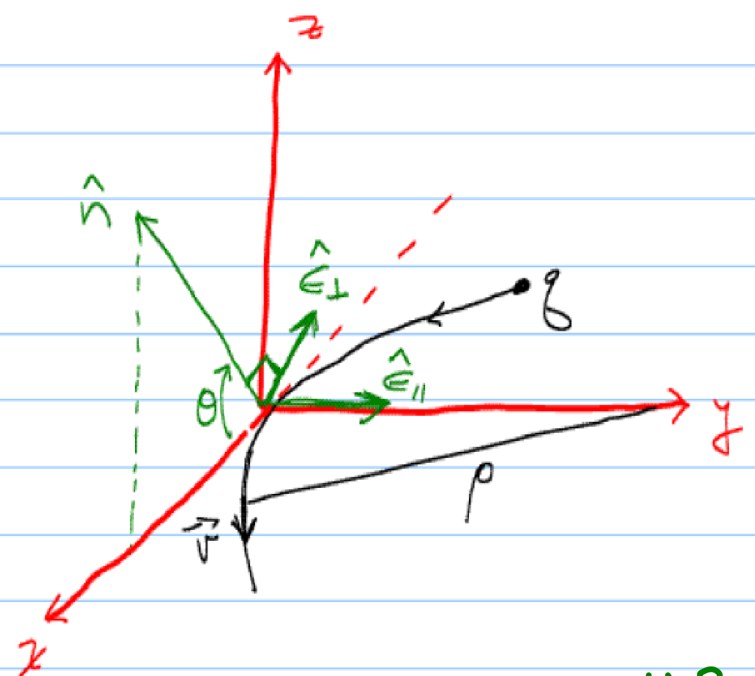
or in the case of a macroscopic current density, to

$$\longrightarrow \frac{1}{c}\int d^3x'\, \vec{J}(\vec{x}', t_r)\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{x}'}$$

Hence we have

$$\boxed{\frac{d^2I}{d\omega\, d\Omega} = \frac{\omega^2}{4\pi^2 c}\left|\int dt' \int d^3x'\, \hat{n}\times\left[\hat{n}\times\vec{J}(\vec{x}', t')\right] e^{i\omega\left(t' - \frac{\hat{n}\cdot\vec{x}'}{c}\right)}\right|^2}$$

Sec. 14.6 Radiation spectrum for $\gamma \gg 1$, from a particle in instantaneously circular motion



Assumptions

(1) Only a very short part of the trajectory contributes, which we choose to define the xy-plane, with the x-axis tangent to the trajectory at $y=0$, $t=0$.

(2) The path has an instantaneous radius of curvature ρ at $t=0$.

(3) The detector is taken to lie in the xz-plane, i.e. $\hat{n} = \hat{x}\cos\theta + \hat{z}\sin\theta$

(4) We further define 2 polarization unit vectors,

$\hat{\epsilon}_{\parallel} = \hat{y}$ lying IN the orbit plane,

$\hat{\epsilon}_{\perp} = \hat{n} \times \hat{\epsilon}_{\parallel}$ representing polarization $\perp$ (perp.) to the orbit plane

Next write the circular approximation to the trajectory at $t \simeq 0$:

$$\vec{r}(t) = \rho\,\hat{x}\sin\frac{ut}{\rho} + \rho\,\hat{y}\left(1 - \cos\frac{ut}{\rho}\right)$$

$$\vec{u}(t) = u\,\hat{x}\cos\frac{ut}{\rho} + u\,\hat{y}\sin\frac{ut}{\rho}$$

and
$$\hat{n}\times(\hat{n}\times\vec{\beta}) = -\hat{x}\,\beta\cos\frac{ut}{\rho}\sin^2\theta - \hat{y}\,\beta\sin\frac{ut}{\rho} + \hat{z}\,\beta\cos\frac{ut}{\rho}\sin\theta\cos\theta$$

$$= \hat{\epsilon}_{\perp}\,\beta\cos\frac{ut}{\rho} - \underbrace{\hat{\epsilon}_{\parallel}}_{\hat{y}}\sin\frac{ut}{\rho}$$

and the argument of the exponential in the integral is:
$$\omega\left(t - \frac{\hat{n}\cdot\vec{r}(t)}{c}\right) = \omega t - \frac{\omega\rho}{c}\sin\frac{ut}{\rho}\cos\theta$$

$$\simeq \omega t - \frac{\omega\rho}{c}\left(1 - \frac{1}{2}\theta^2\right)\left(\frac{ut}{\rho} - \frac{1}{6}\frac{u^3t^3}{\rho^3}\right)$$

$\theta \ll 1$

$t \ll \frac{\rho}{u}$

$$= \omega\left(1 - \frac{u}{c}\right)t + \frac{\theta^2}{2}\frac{u}{c}\omega t + \frac{t^3}{6}\frac{\omega}{c}\frac{u^3}{\rho^2}$$

244

Moreover, $\frac{u}{c} = (1-\gamma^{-2})^{1/2} \simeq 1 - \frac{1}{2}\gamma^{-2} + \ldots$

$$\simeq \frac{\omega}{2}\left[(\gamma^{-2}+\theta^2)t + \frac{c^2}{3\rho^2}t^3\right]$$ + ... terms of order γ^{-2} smaller

and to leading order,

$$\hat{n}\times(\hat{n}\times\vec{\beta}) \simeq \hat{\epsilon}_\perp \theta - \hat{\epsilon}_\parallel \frac{ct}{\rho}$$

So, plugging in these expressions gives our desired integral as

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\hat{\epsilon}_\perp A_\perp(\omega) - \hat{\epsilon}_\parallel A_\parallel(\omega)\right|^2$$

where $A_\perp(\omega) \simeq \theta \int_{-\infty}^{\infty} dt\, e^{i\frac{\omega}{2}\left[(\gamma^{-2}+\theta^2)t + \frac{c^2t^3}{3\rho^2}\right]}$

and $A_\parallel(\omega) = \frac{c}{\rho}\int_{-\infty}^{\infty} dt\, t\, e^{\frac{i\omega}{2}\left[(\gamma^{-2}+\theta^2)t + \frac{c^2t^3}{3\rho^2}\right]}$

After making a variable change to call

$$\xi \equiv \frac{\omega\rho}{3c}(\gamma^{-2}+\theta^2)^{3/2}$$

these are evaluated as

$$\Rightarrow \boxed{\begin{aligned} A_\perp(\omega) &= \frac{\rho}{c\sqrt{3}}\theta(\gamma^{-2}+\theta^2)^{1/2} K_{\frac{1}{3}}(\xi) \\ A_\parallel(\omega) &= \frac{\rho}{c\sqrt{3}}(\gamma^{-2}+\theta^2) K_{\frac{2}{3}}(\xi) \end{aligned}}$$

Kewl! This answer is so simple it can be expressed as Bessel functions/Airy fcns!