Qualitative picture – Think of someone firing bullets from a moving car. The rate of bullets striking a stationary target is NOT the same as the rate they must be fed to the gun. This is analogous to the Doppler effect.

Next consider examples of linear + circular acceleration

Example 1 Linear motion with $\vec{\beta} \parallel \dot{\vec{\beta}}$

$$\Rightarrow \quad \frac{dP^{emission}}{d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

As expected, this reduces to Larmor's result in the limit $\beta \ll 1$. But as $\beta \to 1^-$ the angular distribution is oriented more + more along $\vec{\beta}$.

Now, the total rate of energy emission by the particle is

$$P^{emission} = \int \frac{dP^{emission}}{d\Omega} d\Omega = \frac{2}{3} \frac{q^2}{c^3} \frac{\dot{u}^2}{(1-\beta^2)^3}$$

or

$$P^{emission} = \frac{2 q^2 \dot{u}^2 \gamma^6}{3 c^3}$$

← agrees with Eq. 14.26

Ultrarelativistic limit: Set $\beta \to 1-\delta$
where $\delta \ll 1$ $\Rightarrow$ small θ dominates

and $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$

and $\gamma = \dfrac{1}{\sqrt{1-(1-\delta)^2}} \simeq \dfrac{1}{\sqrt{2\delta}}$

so $(1-\beta\cos\theta)^{-5} \simeq \left[1-(1-\delta)(1-\frac{1}{2}\theta^2)\right]^{-5}$

$$= \left(\delta + \tfrac{1}{2}\theta^2 - \frac{\delta\theta^2}{2}\right)^{-5} \cong \delta^{-5}\left(1+\frac{\theta^2}{2\delta}\right)^{-5}$$

and of course $\sin^2\theta \simeq \theta^2$ at $\theta \ll 1$

and since $\gamma = \dfrac{1}{\sqrt{2\delta}} \Rightarrow \delta \simeq \dfrac{1}{2\gamma^2}$

Thus we have in this ultrarelativistic limit,

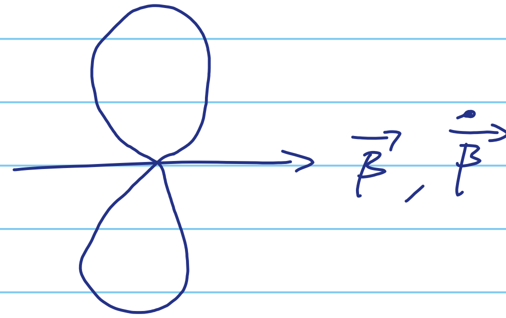
$$\frac{dP^{\text{emission}}}{d\Omega} \cong \frac{q^2\dot{u}^2}{4\pi c^3}\, \frac{2^5\gamma^{10}\,\theta^2}{(1+\gamma^2\theta^2)^5}$$

$\Rightarrow$ This function peaks at $\theta_{max} = \dfrac{1}{2\gamma}$
and the peak intensity approaches

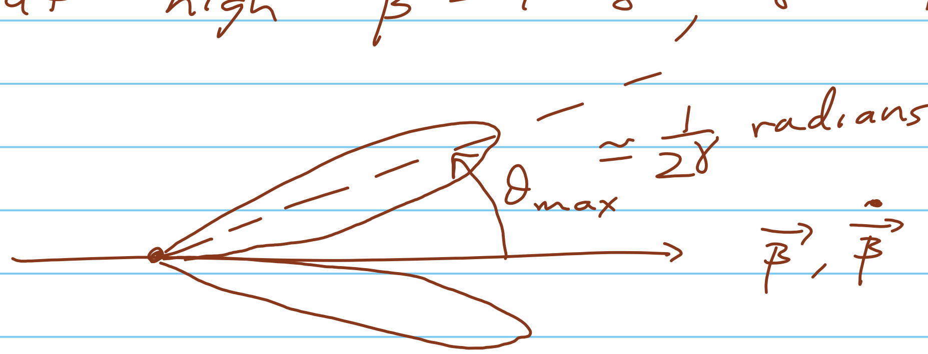
$$\left(\frac{dP^{\text{emission}}}{d\Omega}\right)_{max} \xrightarrow[\gamma \gg 1]{} \frac{2048}{3125}\,\frac{q^2\dot{u}^2}{\pi c^3}\,\gamma^8$$

and note that the integrated result for P^{emiss}
does agree with (14.25)

Contrast in the angular distribution at low $\beta << 1$:

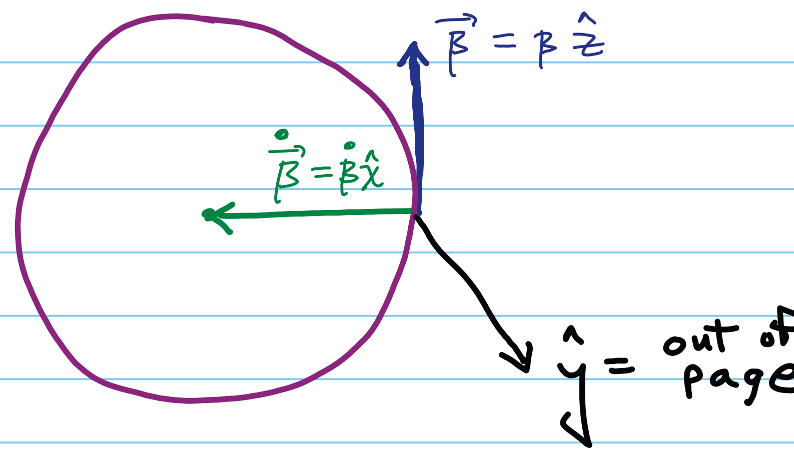


versus at high $\beta \simeq 1 - \delta$, $\gamma >> 1$:



Example 2 Charge q in instantaneously circular motion

In this coordinate system depicted here → we adopt spherical polar coordinates for the observation direction $\hat{n} = (\theta, \phi)$



⇒ Now evaluate

$$\frac{dP^{\text{emission}}}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}|^2}{(1 - \vec{\beta} \cdot \hat{n})^5}$$

$$= \frac{q^2}{4\pi c} \frac{|\dot{\beta}[\hat{n}\times(\hat{n}\times\hat{x}) - \beta\hat{n}\times(\hat{z}\times\hat{x})]|^2}{(1-\beta\cos\theta)^5}$$

Now evaluate the explicit angle dependence, setting $\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$

$$\Rightarrow \hat{n}\times\hat{x} = -\sin\theta\sin\phi\,\hat{z} + \cos\theta\,\hat{y}$$

and $\hat{n}\times(\hat{n}\times\hat{x}) = (-\cos^2\theta - \sin^2\theta\sin^2\phi)\,\hat{x}$

$$+ \cos\phi\sin\phi\sin^2\theta\,\hat{y} + \cos\phi\sin\theta\cos\theta\,\hat{z}$$

and $\hat{n}\times(\hat{z}\times\hat{x}) = \hat{n}\times\hat{y} = -\cos\theta\,\hat{x} + \sin\theta\cos\phi\,\hat{z}$

So our expression simplifies to

$$\frac{dP^{\text{emission}}}{d\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\frac{1}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

Concentrate now on the limit $\gamma \gg 1$

$$\Rightarrow 1 - \beta\cos\theta \simeq \frac{1}{2\gamma^2}(1+\gamma^2\theta^2)$$

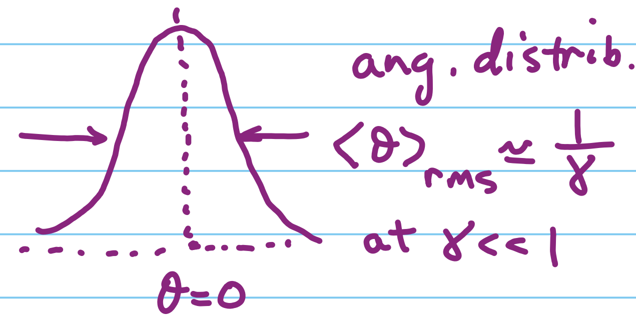
$$\Rightarrow \frac{dP^{\text{emission}}}{d\Omega} \longrightarrow \frac{q^2a^2}{4\pi c^3}\frac{8\gamma^6}{(1+\gamma^2\theta^2)^3}\left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right]$$

This peaks at $\theta \simeq 0$ with $\langle\theta\rangle_{rms} \simeq \frac{1}{\gamma}$ for $\gamma \gg 1$

The total power radiated can be calculated by integrating this last expression, or else from the relativistic formula,

$$P^{emission} = \frac{2}{3}\frac{q^2\gamma^6}{c}\left\{\dot{\beta}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right\} = \frac{2q^2a^2}{3c^2}\gamma^6(1-\beta^2)$$

$$= \frac{2q^2a^2}{3c^3}\gamma^4$$



Note also that for circular motion, the rate of momentum change is

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{u}) = \gamma m\dot{\vec{u}} + \underbrace{\cancel{m\vec{u}\frac{d\gamma}{dt}}}_{\simeq 0}$$

$$\Rightarrow P^{emission}_{circ} = \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2\left(\frac{d\vec{p}}{dt}\right)^2$$

as compared to the formula for linear (parallel) velocity & acceleration, Eq. 14.27

$$P^{emission}_{linear} = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2$$

$$\Rightarrow P^{emiss}_{circular} = \gamma^2 P^{emiss}_{linear}$$

$\Rightarrow$ Circular motion produces radiated power γ^2 higher than linear acceleration, for the same applied force, $\vec{F} = \frac{d\vec{p}}{dt}$

Real-life example

The Advanced Photon Source at Argonne National Laboratory

http://www.aps.anl.gov/

Relevant parameters of the machine:

- positrons circulate at 7 GeV ($mc^2 = 511$ keV)

$$\Rightarrow \gamma - 1 = \frac{7 \text{ GeV}}{511 \times 10^3} \Rightarrow \gamma \simeq 14{,}000$$

$$\Rightarrow \theta_{rms} \simeq \frac{1}{\gamma} \simeq 70 \; \mu\text{radians}$$

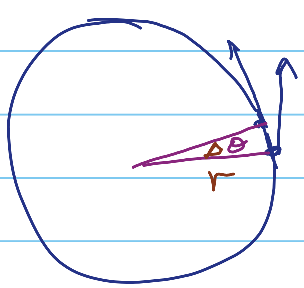
(away from insertion devices)

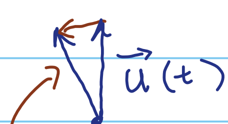
- beam circumference $= 768$ m $= 2\pi R$

$\Rightarrow$ circulation frequency is $\simeq \dfrac{c}{2\pi R} = 390$ kHz $= \nu$

- average energy lost to radiation by 1 e^+ per second is

$$P^{\text{emission}}_{\text{circ}} = \frac{2e^2 a^2}{3c^3} \gamma^4$$





$\vec{u} + d\vec{u} = \vec{u}(t+dt)$

$$e = (4.8 \times 10^{-10} \text{ esu})$$

$$c = 3 \times 10^{10} \text{ cm/s}$$

$$a = \left| \frac{d\vec{u}}{dt} \right| = \frac{u^2}{r} \simeq \frac{c^2}{r}$$

since $\dfrac{|\Delta\vec{u}|}{u} = \dfrac{|\Delta\vec{r}|}{r} \simeq \Delta\theta$

$\ell \simeq \dfrac{2\pi r}{N}$,

and $|\Delta\vec{r}| = \vec{u}\,\Delta t$

$$\Rightarrow \frac{\Delta u}{\Delta t} = a = \frac{u^2}{r}$$

$$a = \frac{\left(3\times10^{10}\,\frac{cm}{s}\right)^2}{(76800\,cm/2\pi)} = 7.4\times10^{16}\,\frac{cm}{s^2}$$

$$\Rightarrow P^{emission}_{circ} = \frac{2\,(4.8\times10^{-10})^2\,(7.4\times10^{16}\,\frac{cm}{s^2})^2}{3\,(3\times10^{10})^2}\,\gamma^4 = 1.2\,\frac{ergs}{sec}$$

(1 positron)

$1. = 6.24\times10^{5}\,\frac{MeV}{erg}$, $1\,J = 10^{7}\,ergs$

$\Rightarrow$ 1 positron radiates 1.9 GeV per revolution

The typical APS beam current is $300\,mA = 1.9\times10^{18}\,\frac{e^+}{sec}$

$$\Rightarrow N_{e^+} = \frac{1.9\times10^{18}}{sec}\times\frac{1}{\#\,revs/sec} = \frac{1.9\times10^{18}}{390{,}000} = 4.9\times10^{12}\,e^+$$

So when the APS is running, just to replenish the energy radiated away as photons requires

$$P^{emiss} = (4.9\times10^{12}\,e^+)\times1.2\,\frac{ergs}{sec}\times10^{-7}\,\frac{J}{erg} = 580{,}000\,W$$
$$= 0.58\,MW$$

Note: there are $\simeq$ 439 nuclear power plants worldwide
- average power output = 0.85 GW per plant.

$\Rightarrow$ it only takes 0.1% of the power output of a typical nuclear power plant to run the APS!

And in 24 hours of running, APS uses about 14000 kW-hours of energy @ \$0.10/kW-hr = \$1400 worth of photons