$$\vec{E}(\vec{x},t) = \frac{qR}{(\vec{R}\cdot\vec{w})^3}\left[\vec{w}(c^2-u^2) + \vec{R}\times(\vec{w}\times\vec{a})\right]$$

and

$$\vec{B}(\vec{x},t) = \hat{n}\times\vec{E}$$

Sec. 14.2 Larmor formula for the radiation by an accelerated charge

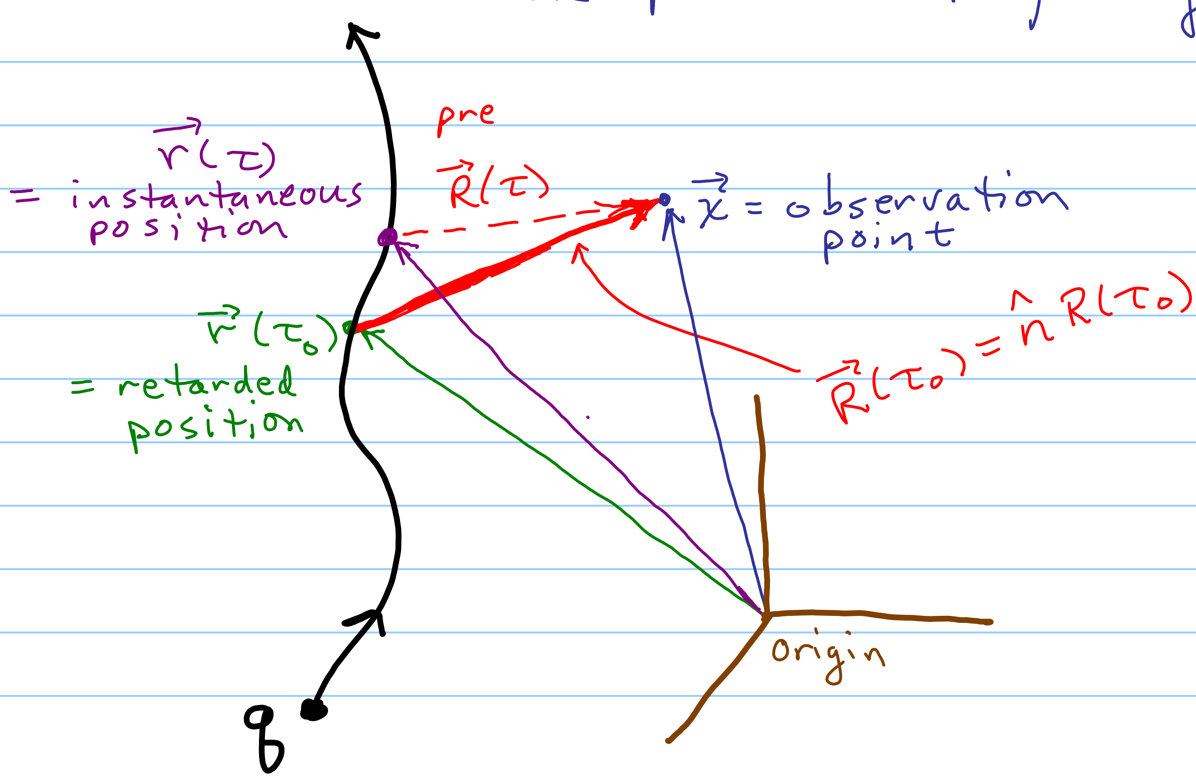
Assume now that $\dot{\vec{\beta}} \neq 0$

$\Rightarrow$ radiation occurs

Also, for now, suppose $\beta \ll 1$

$\Rightarrow$ motion is nonrelativistic in observer's frame

Then visualize the particle's trajectory as



Then in the limit $\beta \to 0$, the dominant term is

$$\vec{E}(\vec{x},t) = \frac{q}{c} \frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{R(\tau_0)}$$

$$\vec{B}(\vec{x},t) = \hat{n}\times\vec{E}(\vec{x},t)$$

Here $\vec{R}(\tau_0) = \vec{x} - \vec{r}(\tau_0) \simeq \vec{R}(\tau)$ at $\beta \ll 1$

Again, τ_0 is found from the equation

$$[x - r(\tau_0)]^2 = 0, \quad x^\alpha = (ct, \vec{x})$$

The instantaneous radiated Poynting vector is

$$\vec{S} = \frac{c}{4\pi} \vec{E}\times\vec{B}$$

since $\vec{E}, \vec{B}$ are <u>real</u> fields at each spacetime point x^α

$$\Rightarrow \vec{S} = \frac{c}{4\pi} |\vec{E}|^2 \hat{n}$$

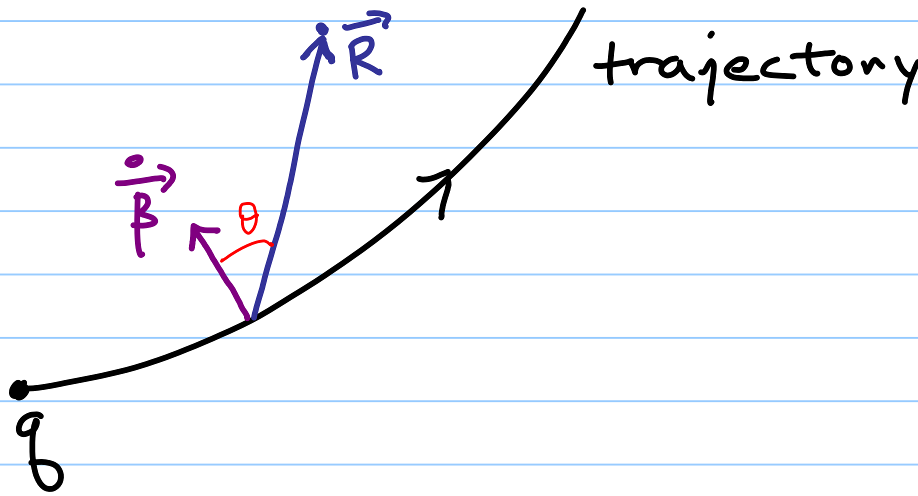
Next work out the <u>angular distribution</u> of radiated power. Owing to retardation, the apparent source is the position $\vec{r}(\tau_0)$.

$$\frac{dP}{d\Omega} = R(\tau_0)^2\, \hat{n}\cdot\vec{S} = \frac{q^2}{4\pi c} |\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})|^2$$

and since $\vec{r}(\tau) \simeq \vec{r}(\tau_0)$, $\vec{R}(\tau) \simeq \vec{R}(\tau_0)$ in this low β limit, i.e. the apparent source of the radiation is approximately the PRESENT position, and so

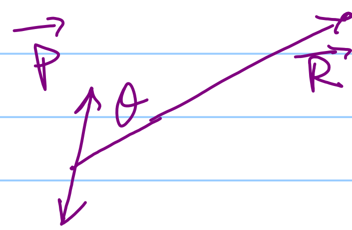
$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3}\sin^2\theta}$$

where $\vec{a} = c\,\dot{\vec{\beta}}$ and $\hat{R}\cdot\hat{a} = \cos\theta$



This result compares with the time-averaged power radiated by a monochromatic electric dipole, which is

$$\frac{dP^{dipole}}{d\Omega} = \frac{ck^4}{8\pi}|\vec{p}|^2\sin^2\theta$$



The total instantaneous power radiated by the accelerated nonrelativistic charge is

$$P = \int \frac{dP}{d\Omega}\,d\Omega = \frac{q^2a^2}{4\pi c^3}\,2\pi\int_{-1}^{1}(1-x^2)\,dx$$

or $\boxed{P = \frac{2}{3}\frac{q^2a^2}{c^3}}$ $\xrightarrow{\text{SI units}}$ $\boxed{\frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}}$ = Larmor Formula

And comparing this with Jackson's Eq. 9.24 for the power radiated by a dipole,

$$P_{E1} = \frac{c^2 Z_0 k^4}{12\pi} |\vec{p}|^2 = \frac{|\vec{p}|^2 \omega^4}{4\pi\epsilon_0 (3c^3)}$$

which resembles our formula for a radiating point charge, with the identifications $p = q\ell$ for the oscillating dipole

$$z = \ell \cos\omega t$$

$$\Rightarrow v = -\omega \ell \sin\omega t$$

$$a = -\omega^2 \ell \cos\omega t$$

$$\Rightarrow \langle a^2 \rangle = \frac{1}{2}\omega^4 \ell^2$$

$$\Rightarrow P_{E1} = \frac{2q^2}{3(4\pi\epsilon_0)} \frac{\langle a^2 \rangle}{c^3}$$

which agrees with the accelerated charge result above.

Total power radiated

Rather than working out the total radiated power from the fields of a relativistic accelerated charge, let's follow the textbook's approach. We generalize the Larmor formula, utilizing covariance arguments to obtain a formula valid even when β is not small.

$\Rightarrow$ First recall that $\text{Power} = \dfrac{\text{energy}}{\text{time}}$

and since both dE and dt are timelike components of 4-vectors, we expect that

Power = Lorentz invariant

$\Rightarrow$ So it is plausible that if we are able to find a Lorentz invariant expression that reduces to the Larmor formula for $\beta \ll 1$, it is likely to be the correct relativistic formula for radiated power.

Therefore, let's start from the correct nonrelativistic formula in the form

$$P^{\text{nonrel}} = \frac{2}{3}\frac{q^2}{m^2c^3}\, m\dot{\vec{u}} \cdot m\dot{\vec{u}}$$

and one can guess that a relativistic expression for the power might then likely be

$$\boxed{P = -\frac{2}{3}\frac{q^2}{m^2c^3}\frac{dp^\alpha}{d\tau}\frac{dp_\alpha}{d\tau}}\ {}^{*},$$ where $d\tau = \dfrac{dt}{\gamma_u}$ as usual

this has the correct nonrelativistic limit because $\dfrac{dp^0}{d\tau} \to \dfrac{d}{d\tau}(\gamma mc) \underset{\substack{\gamma \to 1 \\ \beta \to 0}}{\longrightarrow} O(\beta\dot{\beta})$

$\left(\dfrac{d}{d\tau}(1-\beta^2)^{-1/2} \simeq \beta\dot{\beta}\gamma \text{ at } \beta \simeq 0\right)$

and so $-\frac{dp^\alpha}{d\tau}\frac{dp_\alpha}{d\tau} \simeq \left(\frac{d\vec{p}}{d\tau}\right)^2 - \beta^2\dot{\beta}^2\gamma^2$ at low β

whereas $\frac{d\vec{p}}{d\tau} = \gamma\frac{d}{dt}(\gamma m\vec{\beta}c) = \gamma^2 mc\dot{\vec{\beta}} + \gamma\frac{d\gamma}{dt}$

$$\simeq \gamma^2 mc\dot{\vec{\beta}} + O(\beta)$$

and so in the limit $\beta\to 0$
the leading term is $-\frac{dp^\alpha}{d\tau}\frac{dp_\alpha}{d\tau} \xrightarrow[\beta\to 0]{} m^2c^2\dot{\beta}^2$

and plugging this into (*) above does indeed give the Larmor formula at $\beta\to 0$

While the result (*) is elegant, it is more useful to work out a noncovariant expression in one specific inertial frame, starting from

$$\frac{dU^\alpha}{d\tau} = \frac{d}{d\tau} = (\gamma c, \gamma\vec{u}) = c\gamma\frac{d}{dt}(\gamma, \gamma\vec{\beta})$$

and now $\frac{d\gamma}{dt} = \gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}$ where $\dot{\vec{\beta}} = \frac{1}{c}\frac{d\vec{u}}{dt}$

$$\Rightarrow \frac{dU^\alpha}{d\tau} = c\gamma^2\left(\gamma^2\vec{\beta}\cdot\dot{\vec{\beta}},\ \dot{\vec{\beta}} + \gamma^2\vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})\right)$$

and so

$$\frac{dU^\alpha}{d\tau}\frac{dU_\alpha}{d\tau} = c^2\gamma^4\left\{\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - \dot{\vec{\beta}}^2 - \gamma^4\beta^2(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - 2\gamma^2(\vec{\beta}\cdot\dot{\vec{\beta}})^2\right\}$$

$$= c^2\gamma^4\left\{\gamma^2(\gamma^2 - \beta^2\gamma^2 - 2)(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - \dot{\vec{\beta}}^2\right\}$$

$$= -c^2\gamma^4\left[\dot{\vec{\beta}}^2 + \gamma^2(\vec{\beta}\cdot\dot{\vec{\beta}})^2\right]$$

(using $\gamma^2 - \beta^2\gamma^2 = 1$)

Now, observing that

$$(\vec{\beta}\times\dot{\vec{\beta}})^2 = (\vec{\beta}\times\dot{\vec{\beta}})\cdot(\vec{\beta}\times\dot{\vec{\beta}}) = \vec{\beta}\cdot[\dot{\vec{\beta}}\times(\vec{\beta}\times\dot{\vec{\beta}})]$$

$$= \beta^2\dot{\beta}^2 - (\vec{\beta}\cdot\dot{\vec{\beta}})^2$$

$1+\gamma^2\beta^2 = \gamma^2$

we obtain:

$$-\frac{dU^\alpha}{d\tau}\frac{dU_\alpha}{d\tau} = c^2\gamma^4\left[(\gamma^2\beta^2+1)\dot{\beta}^2 - \gamma^2(\vec{\beta}\times\dot{\vec{\beta}})^2\right]$$

So finally we have both the covariant form

$$P = -\frac{2}{3}\frac{q^2}{m^2c^3}\frac{dp^\alpha}{d\tau}\frac{dp_\alpha}{d\tau}$$

and the form in a specific chosen inertial frame,

$$\boxed{P = \frac{2}{3}q^2\gamma^6\left[\dot{\beta}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right]}$$

This is the Lienard result (1898), and Jackson Eq. 14.26

IMPORTANT – Note the γ^6 dependence!

So the radiated power increases RAPIDLY as $u\to c$!

Sec. 14.3 Angular distribution of radiation from an accelerated charge

To work this out, begin by generalizing the nonrelativistic expression

$$\frac{dP^{\text{nonrel}}}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})\right|^2, \text{ for } \beta \ll 1$$

In general we can use the field expressions derived earlier, i.e.

falls off as $\frac{1}{R^2}$ and is negligible in the radiation zone

$$\vec{E}(\vec{x},t) = \left[\frac{q(\hat{n}-\vec{\beta})}{\gamma^2 R^2 (1-\vec{\beta}\cdot\hat{n})}\right]_{ret} + \frac{q}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{R(1-\vec{\beta}\cdot\hat{n})^3}\right]_{ret}$$

and

$$\vec{B}(\vec{x},t) = \left[\hat{n}\times\vec{E}\right]_{ret}$$

only this is radiation falling off like $\frac{1}{R}$ at $R\to\infty$

relativistically valid

We start from $\frac{dP}{d\Omega_0} = R^2(\tau_0)\,\hat{n}\cdot\vec{S}$

where $\vec{R}(\tau_0) = \vec{x} - \vec{r}(\tau_0) \leftarrow$ displacement vector to the observation point $\vec{x}$ from the retarded source position of q.

$$\Rightarrow \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}\hat{n}|\vec{E}|^2$$

which we evaluate in the radiation zone where $\vec{B} = \hat{n}_{ret}\times\vec{E}$, with $\vec{E} = \frac{q}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3 R}\right]_{ret}$

Important point This $\vec{S}$ gives the energy flux that is observed at a fixed detector in the observer's rest frame, as a function of the time of detection, t_{obs}.

i.e. $$\frac{d^2W}{dA\,dt_{obs}} = (\vec{S}\cdot\hat{n})^{detected}$$

But we are often interested in the rate of energy EMISSION of the particle rather than the rate of energy detection.

The detection or observation time t is related to the emission or retarded time t_r by

$$t_r = t - \frac{R(t_r)}{c}$$

i.e. $t = t_r + \frac{R(t_r)}{c} \Rightarrow dt = dt_r + \frac{1}{c}\frac{d}{dt_r}|\vec{x} - \vec{r}(t_r)|\, dt_r$

and $\frac{d}{dt_r}|\vec{x} - \vec{r}(t_r)| = \frac{d}{dt_r}\left[(\vec{x} - \vec{r}(t_r))^2\right]^{1/2} = \frac{-\frac{2}{2}\frac{d\vec{r}(t_r)}{dt_r}\cdot(\vec{x} - \vec{r}(t_r))}{\left[(\vec{x} - \vec{r}(t_r))^2\right]^{1/2}}$

$$= -c\vec{\beta}\cdot\hat{n}$$

$$\Rightarrow \quad dt = (1 - \vec{\beta}\cdot\hat{n})\, dt_r$$

or $$(\vec{S}\cdot\hat{n})^{\text{emission}} = (\vec{S}\cdot\hat{n})^{\text{detection}} \frac{dt}{dt_r}$$

i.e. $$\frac{dP^{\text{detector}}}{d\Omega} = R^2\, \hat{n}\cdot\vec{S}^{\,\text{det}} = \frac{q^2}{4\pi c}\frac{|\hat{n}\times[(\hat{n} - \vec{\beta})\times\dot{\vec{\beta}}]|^2}{(1 - \vec{\beta}\cdot\hat{n})^6}$$

whereas

$$\boxed{\frac{dP^{\text{emission}}}{d\Omega} = R^2\hat{n}\cdot\vec{S}^{\,\text{emission}} = \frac{q^2}{4\pi c}\frac{|\hat{n}\times[(\hat{n} - \vec{\beta})\times\dot{\vec{\beta}}]|^2}{(1 - \vec{\beta}\cdot\hat{n})^5}}$$