Chapter 14 Radiation by moving charges

Now let $r^\alpha(\tau)$ = spacetime position 4-vector of charge q, as a function of its proper time τ.

In a specific inertial frame K it looks like

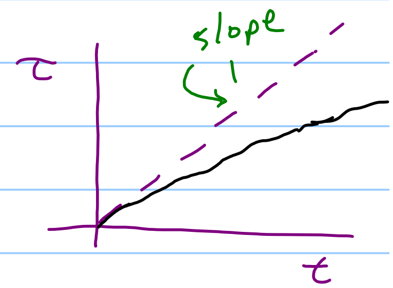
$$r^\alpha(\tau) = \left(ct(\tau), \vec{r}(\tau)\right)$$

and $\tau(t) = \int_0^t \gamma_{v(t)}^{-1}\, dt$, i.e. $d\tau = \gamma_{v(t)}^{-1} dt$

$$\gamma_{v(t)}^{-1} = \sqrt{1 - \frac{v(t)^2}{c^2}}$$

So we can also write

$$\vec{V}(\tau) = \frac{1}{\gamma(\tau)} \frac{d\vec{r}(\tau)}{d\tau}$$



Then the 4-velocity of q is

$$U^\alpha(\tau) = \frac{d}{d\tau} r^\alpha(\tau) = \gamma_{\vec{v}(\tau)}\left(c, \vec{V}(\tau)\right)$$

and the EM potentials generated by this charged particle motion are found using

$$\boxed{A^\alpha(x) = \frac{4\pi}{c} \int d^4x'\, D_r(x-x')\, J^\alpha(x')}$$

and this assumes: NO incident fields
NO boundary surfaces.

Therefore, plugging in $D_r(x-x') = \frac{\theta(x_0 - x_0')}{2\pi} \delta[(x-x')^2]$

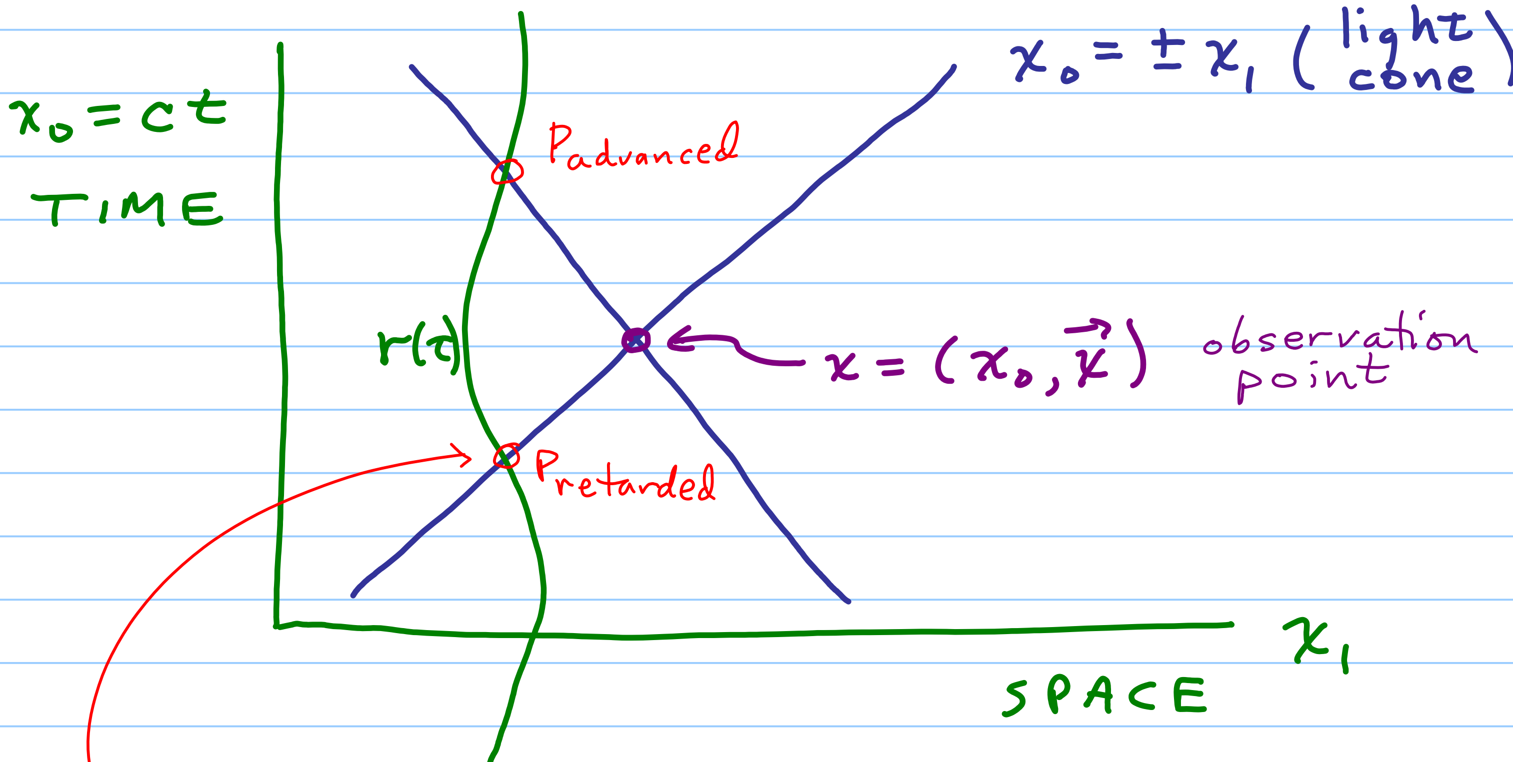
and

$$J^\alpha(x') = qc \int d\tau\, U^\alpha(\tau)\, \delta^{(4)}(x - r(\tau))$$

$$\Rightarrow A^\alpha(x) = 2q \int d\tau\, U^\alpha(\tau) \underbrace{\int d^4x'\, \theta(x_0 - x_0')\, \delta[(x-x')^2]\, \delta^{(4)}[x' - r(\tau)]}_{\text{Let's evaluate this integral first}}$$

$$\Rightarrow A^{\alpha}(x) = 2q \int d\tau \, U^{\alpha}(\tau)\, \theta(x_0 - r_0(\tau))\, \delta[(x-r(\tau))^2]$$

To see the big picture, consider the particle trajectory as viewed with respect to the observation point:



Call the physically-relevant intersection with the light cone, $r(\tau_0)$, i.e. where the light cone intersects the particle's trajectory in the PAST.

Mathematical solution for the root:

$$[x - r(\tau)]^2 = (x_0 - r_0(\tau))^2 - |\vec{x} - \vec{r}(\tau)|^2$$

$$= 0 \quad \text{when} \quad x_0 - r_0(\tau_0) = \pm |\vec{x} - \vec{r}(\tau_0)|$$

only the '+' sign contributes because of the causality θ-function

Next use $\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{\left|\frac{df}{dx}\right|_{x=x_i}}$ ← summed over all points where $f(x_i) = 0$

Hence $\delta[(x - r(\tau))^2] = \frac{\delta(\tau - \tau_0)}{\left|\frac{d}{d\tau}[x - r(\tau)]^2\right|_{\tau=\tau_0}}$

where

$$\frac{d}{d\tau}[x - r(\tau)]^2 = \frac{d}{d\tau}\left\{(x^\alpha - r^\alpha(\tau))\, g_{\alpha\beta}\,(x^\beta - r^\beta(\tau))\right\}_{\tau_0}$$

$$= -2\,\frac{dr^\alpha(\tau)}{d\tau}\,(x_\alpha - r_\alpha(\tau))\Big|_{\tau_0}$$

So putting this all together into

$$A^\alpha(x) = 2q\int d\tau\, U^\alpha(\tau)\,\theta(x_0 - r_0(\tau))\,\delta[(x - r(\tau))^2]$$

gives

$$\boxed{A^\alpha(x) = \frac{q\,U^\alpha(\tau)}{|U(\tau)\cdot(x - r(\tau))|}\Bigg|_{\tau=\tau_0}}$$

where τ_0 is the causality-appropriate root of $[x - r(\tau_0)]^2 = 0$, i.e. obeying $x_0 > r_0(\tau_0)$

Note that this denominator will be positive definite even without absolute value signs. (Convince yourself why!)

This formula for $A^\alpha(x)$ is called the

Lienard-Wiechert potential.

Next we investigate these potentials and fields. First of all, notice that

$$U(\tau_0)\cdot\left(x-r(\tau_0)\right)=\overbrace{U_0(\tau_0)}^{\gamma c}\overbrace{\left(x_0-r_0(\tau_0)\right)}^{c(t_{obs}-t_{ret})=R}-\vec{U}(\tau_0)\cdot\left(\vec{x}-\vec{r}(\tau_0)\right)$$

and the light cone constraint reads:

$$x_0-r_0(\tau_0)=|\vec{x}-\vec{r}(\tau_0)|\equiv R$$

source - observer distance at the retarded time

The following notation is helpful:

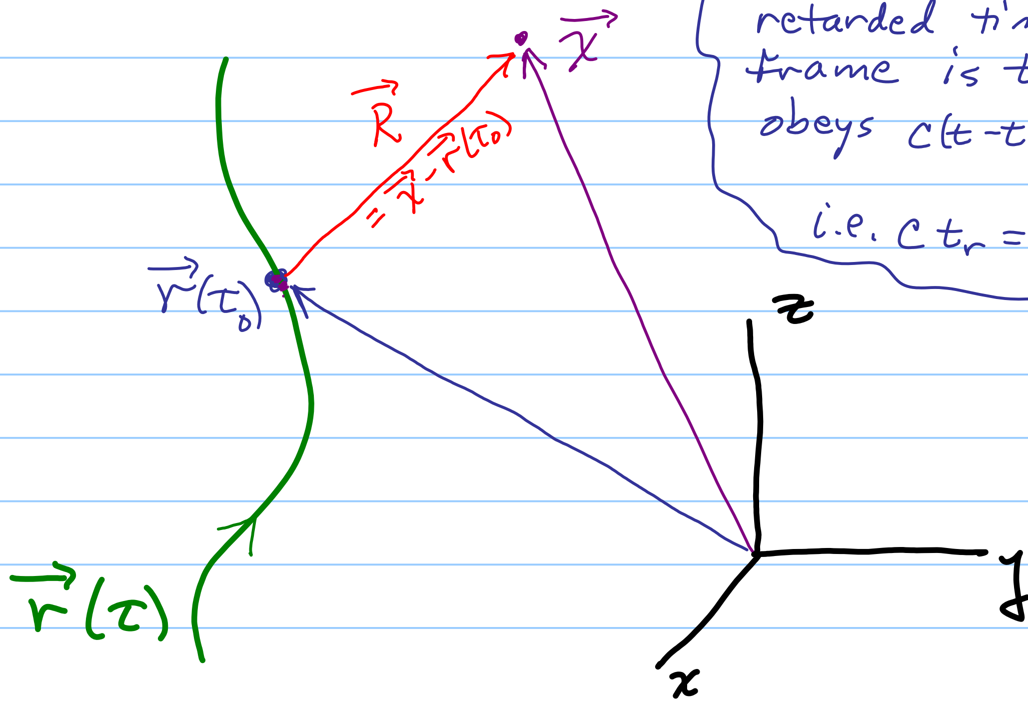
$$\hat{n}=\frac{\vec{R}}{R}=\frac{\vec{x}-\vec{r}(\tau_0)}{|\vec{x}-\vec{r}(\tau_0)|}$$

$$\Rightarrow U\cdot(x-r(\tau))=\gamma cR-\gamma\vec{u}(\tau_0)\cdot\hat{n}R$$
$$=\gamma cR\left(1-\vec{\beta}\cdot\hat{n}\right)$$

where $\vec{\beta}\equiv\dfrac{\vec{u}(\tau_0)}{c}$

To reiterate, the retarded time in OUR frame is t_r, which obeys $c(t-t_r)=|\vec{x}-\vec{r}(\tau_0)|=R$

i.e. $ct_r=r_0(\tau_0)$



And from the above derivation we have

$$A^\alpha(x) = q\,\frac{\gamma(c,\vec{u})}{\gamma c R(1-\vec{\beta}\cdot\hat{n})}$$ RHS evaluated at the retarded time

or

$$\boxed{\Phi(\vec{x},t) = \left[\frac{q}{(1-\vec{\beta}\cdot\hat{n})R}\right]_{ret}, \quad \vec{A}(\vec{x},t) = \left[\frac{q\vec{\beta}}{(1-\vec{\beta}\cdot\hat{n})R}\right]_{ret}}$$

Now, the electromagnetic fields can be directly calculated from these $\Phi, \vec{A}$, but it turns out to be simpler to use the integral expression (14.3).

$$\Rightarrow A^\beta(x) = 2q\int d\tau\, U^\beta(\tau)\,\theta(x_0 - r_0(\tau))\,\delta[(x-r(\tau))^2]$$

and of course we need $\partial^\alpha A^\beta$ to determine $F^{\alpha\beta}$

$$\Rightarrow \partial^\alpha A^\beta = 2q\int d\tau\, U^\beta(\tau)\,\partial^\alpha\delta[(x-r(\tau))^2]\,\theta(x_0 - r_0(\tau))$$

$$+\left(\begin{array}{l}\text{terms involving } \partial^\alpha\theta, \text{ involving } \delta(R)\\ \text{which do not survive at } R\neq 0\end{array}\right)$$

Recall also that ∂^α acts only on x, and abbreviate $\partial^\alpha\delta[(x-r(\tau))^2] \equiv \partial^\alpha\delta(f)$

where $f = (x-r(\tau))^2$

$$\Rightarrow \partial^\alpha\delta(f) = (\partial^\alpha f)\,\frac{d}{df}\delta(f) = (\partial^\alpha f)\,\frac{d\tau}{df}\,\frac{d}{d\tau}\delta(f)$$

And, moreover,

$$(i)\ \partial^\alpha f = \partial^\alpha (x-r(\tau))^2 = 2\,(x^\alpha - r^\alpha(\tau))$$

$$(ii)\ \frac{df}{d\tau} = -2\,(x-r(\tau))\cdot U(\tau)$$

So

$$\partial^\alpha \delta[(x-r(\tau))^2] = \frac{-(x^\alpha - r^\alpha(\tau))}{U(\tau)\cdot(x-r(\tau))}\frac{d}{d\tau}\delta[(x-r(\tau))^2]$$

So if we plug this into the above integral for $\partial^\alpha A^\beta$ and integrate by parts, this gives

$$\partial^\alpha A^\beta = 2q\int d\tau\, \theta(x_0 - r_0(\tau))\,\delta[(x-r(\tau))^2] \times \frac{d}{d\tau}\left[\frac{(x^\alpha - r^\alpha(\tau))\,U^\beta(\tau)}{(x-r(\tau))\cdot U(\tau)}\right]$$

+ surface terms that vanish because they are not on the light cone at $\tau \to \pm\infty$

Now, let's again use $\delta[(x-r(\tau))^2] = \dfrac{\delta(\tau-\tau_0)}{2U(\tau)\cdot(x-r(\tau))}$

Then,

$$F^{\alpha\beta}(x) = \frac{q}{(x-r(\tau_0))\cdot U(\tau_0)}\frac{d}{d\tau_0}\left\{\frac{(x-r(\tau_0))^\alpha U^\beta(\tau_0) - (x-r(\tau_0))^\beta U^\alpha(\tau_0)}{(x-r(\tau_0))\cdot U(\tau_0)}\right\}$$

and continue to adhere to the conditions

$(x-r(\tau_0))^2 = 0$ (light cone)

and $x_0 > r_0(\tau_0)$ (causality)

Next, determine the fields:

1) $(x - r(\tau_0))^\alpha = (R, \vec{R})^\alpha$; $\vec{R} = \vec{x} - \vec{r}(\tau_0)$; $r_0(\tau_0) = c\,t_r$, $t - t_r = \dfrac{R}{c}$

2) $U^\alpha(\tau_0) = \gamma(\tau_0)\,(c, \vec{u}(\tau_0))$

3) $$\frac{d}{d\tau_0}(x - r(\tau_0))^\alpha = -\left.\frac{dr^\alpha}{d\tau}\right|_{\tau=\tau_0} = -U^\alpha(\tau_0)$$

4) $$\frac{d}{d\tau_0}U^\alpha(\tau_0) = \left(c\,\frac{d\gamma(\tau_0)}{d\tau_0},\ \frac{d\gamma(\tau_0)}{d\tau_0}\vec{u}(\tau_0) + \gamma^2(\tau_0)\,\vec{a}(\tau_0) \right)$$

since $\dfrac{d\vec{u}}{d\tau} = \gamma\dfrac{d\vec{u}}{dt} = \gamma\vec{a}$

where $\vec{a}$ = ordinary acceleration, $\dot{\vec{u}}$

and note as well that

$$\frac{d\gamma}{d\tau} = \gamma\,\frac{d\gamma}{dt} = \gamma\left[-\frac{1}{2}\gamma^3\left(-\frac{2\,\vec{u}\cdot\vec{a}}{c^2}\right)\right] = \gamma^4\,\vec{\beta}\cdot\dot{\vec{\beta}}$$

so $$\frac{dU^\alpha(\tau_0)}{d\tau_0} = \left(c\gamma^4\,\dot{\vec{\beta}}\cdot\vec{\beta},\ c\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})\vec{\beta} + c\gamma^2\dot{\vec{\beta}} \right)$$

5) $$\frac{d}{d\tau_0}\{U(\tau_0)\cdot(x - r(\tau_0))\} = \frac{dU(\tau_0)}{d\tau_0}\cdot(x - r(\tau_0)) - \underbrace{U^2(\tau_0)}_{c^2}$$

again using $U^2 = \gamma^2(c^2 - u^2) = c^2$

6) $U(\tau_0)\cdot(x - r(\tau)) = \gamma c R\,(1 - \vec{\beta}\cdot\hat{n})$ (derived on p. 261 above)

7) $$\frac{dU}{d\tau_0}\cdot(x - r(\tau_0)) = (c\gamma^4\,\dot{\vec{\beta}}\cdot\vec{\beta})R - [c\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})\vec{\beta} + \gamma^2 c\dot{\vec{\beta}}]\cdot R\hat{n}$$

And now, somewhat tediously, put all of these pieces together:

$$F^{\alpha\beta}_{(x)} = \left(\frac{q}{\gamma c R(1-\vec{\beta}\cdot\hat{n})} \right) \times \qquad (c\gamma^4 \dot{\vec{\beta}}\cdot\vec{\beta})R - [c\gamma^4(\dot{\vec{\beta}}\cdot\vec{\beta})\vec{\beta} + c\gamma^2\dot{\vec{\beta}}]\cdot R\hat{n} - c^2$$

$$\left\{ \frac{-1}{[\gamma c R(1-\vec{\beta}\cdot\hat{n})]^2} \frac{d}{d\tau_0}\left[(x-r(\tau_0))\cdot U(\tau_0)\right] \left[(R,\vec{R})^\alpha (1,\vec{\beta})^\beta - (R,\vec{R})^\beta (1,\vec{\beta})^\alpha \right] \gamma c \right.$$

$$+ \frac{1}{\gamma c R(1-\vec{\beta}\cdot\hat{n})} \left[\left[(1,\vec{\beta})^\alpha (1,\vec{\beta})^\beta - (1,\vec{\beta})^\beta (1,\vec{\beta})^\alpha \right] \gamma^2 c^2 \right]^{=0}$$

$$\left. + \frac{c\gamma^4}{\gamma c R(1-\vec{\beta}\cdot\hat{n})} \left[(R,\vec{R})^\alpha \left(\vec{\beta}\cdot\dot{\vec{\beta}},\ (\vec{\beta}\cdot\dot{\vec{\beta}})\vec{\beta} + \dot{\vec{\beta}}\gamma^{-2}\right)^\beta - (R,\vec{R})^\beta \left(\vec{\beta}\cdot\dot{\vec{\beta}},\ (\vec{\beta}\cdot\dot{\vec{\beta}})\vec{\beta} + \dot{\vec{\beta}}\gamma^{-2}\right)^\alpha \right] \right\}$$

Next we can read off the field components, e.g. pull off the elements $E_i = F^{i,0}$.

After the final steps of algebraic simplification we get the compact form (14.14):

$$\boxed{\vec{E}(\vec{x},t) = \left[\frac{q(\hat{n}-\vec{\beta})}{\gamma^2 R^2 (1-\vec{\beta}\cdot\hat{n})} \right]_{ret} + \frac{q}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{R(1-\vec{\beta}\cdot\hat{n})^3} \right]_{ret}}$$

and

$$\vec{B}(\vec{x},t) = \left[\hat{n}\times\vec{E}\right]_{ret}$$

only this is radiation falling off like $\frac{1}{R}$ at $R \to \infty$

The Griffiths text gives an alternative derivation that is direct but "brute force", in terms of an auxiliary vector $\vec{w} = c\hat{R} - \vec{u}$, namely