Sec 12.7 Lagrangian for the $\vec{E}, \vec{B}$-fields

Now let's find a Lagrangian for the fields, acknowledging that the dynamical variables $\vec{E}(\vec{x}, t)$, $\vec{B}(\vec{x}, t)$ are continuously infinite. Here $\vec{x}, t$ are the independent variables which the dependent variables $\vec{E}, \vec{B}$ depend on.

The following correspondences can be established with the case of particle motion:

Particles, $q_i(t)$		Fields or Potentials, $\phi_k(x)$
index i	$\longrightarrow$	index k
independent variable, t	$\longrightarrow$	variables, x^α
$q_i(t)$	$\longrightarrow$	$\phi_k(x)$
$\dot{q}_i(t)$	$\longrightarrow$	$\partial^\alpha \phi_k(x)$
$L = \sum_i L(q_i, \dot{q}_i)$	$\longrightarrow$	$\int \mathcal{L}(\phi_k, \partial^\alpha \phi_k)\, d^3x$ ($\mathcal{L}$ = Lagrangian density)
action $A = \int L\, dt$	$\longrightarrow$	$A = \iint \mathcal{L}\, d^3x\, dt = \int \mathcal{L}\, d^4x$

i.e. For the EM field, the "positions" are the fields or potentials A^α, while the role of "velocities" are $\partial^\beta A^\alpha$

The Euler - Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \longrightarrow \partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta \phi_k)} = \frac{\partial \mathcal{L}}{\partial \phi_k}$$

What are the desired properties of the action, A?

$\Rightarrow$ As before, we desire that A should be a Lorentz invariant

$\Rightarrow$ $\mathcal{L}$ = Lorentz scalar, because d^4x = invariant

How to construct $\mathcal{L}_{EM}$?

$\Rightarrow$ By analogy with the case for nonrelativistic particles, we expect that $\mathcal{L}_{EM}$ might be quadratic in the "velocities", i.e. in $\partial^\beta A^\alpha$ or $F^{\alpha\beta}$

Recalling that $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$
and $\mathcal{F}^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$

Therefore, two plausible candidates are

$F_{\alpha\beta}F^{\alpha\beta}$ or $F_{\alpha\beta}\mathcal{F}^{\alpha\beta}$

invariant under inversion

changes sign under inversion $\Rightarrow$ pseudoscalar

This suggests that we should try

$$\mathcal{L}_{EM}^{\text{Free}} \propto F_{\alpha\beta} F^{\alpha\beta}$$

For interactions with charges and currents, recall that we found earlier

$$\gamma L_{int} = -\frac{q}{c} U_\alpha A^\alpha$$

and this suggests that for continuous charges + currents,

$$\mathcal{L}_{int} \propto -\frac{J_\alpha A^\alpha}{c}$$

In fact, when this is "tried out", the Lagrangian density that gives the correct "equations of motion" = Maxwell's equations are found to be

$$\boxed{\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha}$$

Next let's verify that this works:

(1) $\dfrac{\partial \mathcal{L}}{\partial A^\beta} = -\dfrac{J_\beta}{c}$ is one partial derivative needed

(2) To simplify the algebra, call $K \equiv -\frac{1}{16\pi}$.
The next derivative we need is

$$\partial^\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial^\alpha A^\beta)}\right] = K \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial(\partial^\alpha A^\beta)} \left(F_{\gamma\delta} F^{\gamma\delta}\right)$$

$$= K \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial(\partial^\alpha A^\beta)} \left\{ (\partial^\gamma A^\delta - \partial^\delta A^\gamma) g_{\gamma\eta} g_{\delta\mu} (\partial^\eta A^\mu - \partial^\mu A^\eta) \right\}$$

... some details follow... Observe that

$$\frac{\partial}{\partial(\partial^\alpha A^\beta)} (\partial^\gamma A^\delta - \partial^\delta A^\gamma) = \delta^\gamma{}_\alpha \delta^\delta{}_\beta - \delta^\delta{}_\alpha \delta^\gamma{}_\beta$$

Kronecker δ-functions

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial^\alpha A^\beta)} = K \Big[(\delta^\gamma{}_\alpha \delta^\delta{}_\beta - \delta^\delta{}_\alpha \delta^\gamma{}_\beta) g_{\gamma\eta} g_{\delta\mu} (\partial^\eta A^\mu - \partial^\mu A^\eta)$$
$$+ (\partial^\gamma A^\delta - \partial^\delta A^\gamma) g_{\gamma\eta} g_{\delta\mu} (\delta^\eta{}_\alpha \delta^\mu{}_\beta - \delta^\mu{}_\alpha \delta^\eta{}_\beta) \Big]$$

$$= K \Big[(g_{\alpha\eta} g_{\beta\mu} - g_{\alpha\mu} g_{\beta\eta}) (\partial^\eta A^\mu - \partial^\mu A^\eta)$$
$$+ (\partial^\gamma A^\delta - \partial^\delta A^\gamma)(g_{\gamma\alpha} g_{\delta\beta} - g_{\delta\alpha} g_{\gamma\beta}) \Big]$$

and since $\quad g_{\alpha\eta} = g_{\alpha\alpha} \delta_{\alpha\eta}$ (no sum on α here)

we have $\quad g_{\alpha\eta} \partial^\eta = g_{\alpha\alpha} \partial^\alpha$ "

$$\text{So } \frac{\partial \mathcal{L}}{\partial(\partial^\alpha A^\beta)} = K g_{\alpha\alpha} g_{\beta\beta} \Big[(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - (\partial^\beta A^\alpha - \partial^\alpha A^\beta)$$
$$+ (\partial^\alpha A^\beta - \partial^\beta A^\alpha) - (\partial^\beta A^\alpha - \partial^\alpha A^\beta) \Big]$$

no sum on α, β

$$= 4K g_{\alpha\alpha} g_{\beta\beta} F^{\alpha\beta} \quad \text{(no sum on } \alpha, \beta)$$

$$= 4K g_{\alpha\gamma} g_{\beta\delta} F^{\gamma\delta} = 4K F_{\alpha\beta}$$

Accordingly, the Euler-Lagrange equations are

$$4K\,\partial^\alpha F_{\alpha\beta} = -J_\beta/c\,, \quad \text{with } K = \frac{-1}{16\pi}$$

or finally,

$$\boxed{\partial^\alpha F_{\alpha\beta} = \frac{4\pi}{c} J_\beta}$$

← agrees with Eq. 11.141

These give only the INHOMOGENEOUS Maxwell's equations. However, the 2 HOMOGENEOUS equations have been automatically satisfied, by construction, i.e. one can verify that

$$\partial^\alpha \mathcal{F}_{\alpha\beta} = 0$$

This is enforced, in effect, because the field tensor was originally defined using potentials, and those were initially created to satisfy

$$\vec{B} = \nabla \times \vec{A} \Rightarrow \nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0 \quad \checkmark$$

and since $\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

$$\Rightarrow \nabla \times \vec{E} = \cancelto{0}{\nabla \times (-\nabla\Phi)} - \frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{A}$$

$$= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\,, \text{ automatically} \quad \checkmark$$

Note also that change conservation or continuity holds here, since $\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c} J_\alpha$

used antisymmetry of $F_{\alpha\beta}$

$$\Rightarrow \partial^\alpha \partial^\beta F_{\beta\alpha} = \frac{4\pi}{c}\partial^\alpha J_\alpha = -\partial^\alpha\partial^\beta F_{\alpha\beta} = -\partial^\beta\partial^\alpha F_{\beta\alpha}$$

or $\boxed{\partial^\alpha J_\alpha = 0}$ $\checkmark$

Sec. 12.11 Invariant Green Functions

Recall that (the inhomogeneous) Maxwell's equations are

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

and in terms of the 4-potential A^α,

$$\partial_\alpha\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) = \Box A^\beta - \partial^\beta\left(\partial_\alpha A^\alpha\right)$$

so if we adopt the Lorentz gauge, $\partial_\alpha A^\alpha = 0$,

$$\Rightarrow \boxed{\Box A^\beta = \frac{4\pi}{c} J^\beta}$$

This inhomogeneous wave equation can be solved formally using a Green's function satisfying

$$\Box_x D(x,x') = \delta^{(4)}(x-x') = \delta(x_0 - x_0')\,\delta^{(3)}(\vec{x}-\vec{x}\,')$$

and if there are no boundary surfaces, D can <u>only</u> depend on $z^\alpha = x^\alpha - x'^\alpha$

$$\Rightarrow \Box_z D(z) = \delta^{(4)}(z)$$

In fact we already solved this for the retarded (+) and advanced (−) Green's function last semester (refer to pp 190 − 198, Fall 2011 notes). The solution obtained there, in the present notation, and with $R \equiv |\vec{x}-\vec{x}\,'|$, is:

$$D^{(\pm)}(z) = \frac{1}{4\pi R}\,\delta(x_0 - x_0' \mp R) \qquad \left(\begin{matrix}\text{Jackson}\\ \text{Eq. } 12.132\end{matrix}\right)$$

and if we like, we can add a factor $\theta(x_0 - x_0')$ to select the forward light cone with observation times AFTER$^{(+)}$ the source time or $\theta(x_0' - x_0)$ to select the backward light cone for the advanced GF which has observation times BEFORE$^{(-)}$ the source time

The text shows on p. 614 how to put these GFs into covariant form, e.g. using the identity $\delta[(x-x')^2] = \delta[(x_0-x_0')^2 - |\vec{x}-\vec{x}'|^2]$

$$= \delta[(x_0 - x_0' - R)(x_0 - x_0' + R)]$$

$$= \frac{1}{2R}\delta(x_0 - x_0' - R) + \frac{1}{2R}\delta(x_0 - x_0' + R)$$

and the covariant forms of these two GFs can then be expressed as

$$D_r(x-x') = \frac{1}{2\pi}\theta(x_0 - x_0')\,\delta[(x-x')^2]$$

and

$$D_a(x-x') = \frac{1}{2\pi}\theta(x_0' - x_0)\,\delta[(x-x')^2]$$

This θ-function appears to be noninvariant, but it is in fact when constrained by the δ-function i.e. θ here just selects the forward (backward) light cone for retarded (advanced) waves.

Then, e.g. as usual, the solution is

$$A^\alpha(x) = A^\alpha_{inc}(x) + \frac{4\pi}{c}\int d^4x'\, D_r(x-x')\, J_\alpha(x')$$

Aside: If it is desirable to make D_r manifestly invariant, we can rewrite $\theta(z_0)$, e.g. by introducing a 4-vector

$$\eta^\alpha = (1,0,0,0) = (1,\vec{0}) \text{ in our frame}$$

$$\Rightarrow z^0 = z^\alpha \eta_\alpha \text{ is an invariant in any frame}$$

$$\Rightarrow \boxed{D_r(z) = \frac{\theta(z^\alpha \eta_\alpha)\,\delta(z^\beta z_\beta)}{2\pi}}$$

Later we will need the **covariant current** of a moving point charge. In an inertial frame K, call $\vec{r}(t)$ = position of the point charge q

$$\Rightarrow \rho(\vec{x},t) = q\,\delta[\vec{x}-\vec{r}(t)]$$

and $\vec{J}(\vec{x},t) = q\,\vec{v}(t)\,\delta[\vec{x}-\vec{r}(t)]$

where $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$ = charge's velocity at time t as measured in K

The appropriate contravariant 4-vector is

$$J^\alpha = qc \int d\tau\, U^\alpha(\tau)\,\delta^{(4)}[x-r(\tau)]$$

where $r(\tau) \equiv (ct, \vec{r}(t))$ in frame K

and $U^\alpha(\tau) = \gamma(c, \vec{v}(t))$;

The source proper time τ is connected to frame K's clock through $d\tau = \gamma^{-1}_{v(t)}\,dt$, i.e. $\tau = \int dt\, \gamma^{-1}_{v(t)}$

and $J^\alpha(x) = qc \int dt'\, \gamma_v^{-1}\, \gamma_v (c, \vec{v}(t'))\,\delta(ct-ct')\,\delta(\vec{x}-\vec{r}(t'))$ which checks!