Chap. 12 Relativistic dynamics

By now we know the equations of motion of a charged particle e through $\vec{E}, \vec{B}$ - fields, i.e.

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{\vec{u}}{c}\times\vec{B}\right), \quad \frac{dE}{dt} = e\,\vec{u}\cdot\vec{E}$$

or their equivalent manifestly covariant form,

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta$$

While these equations are classically "complete", it is desirable to recast them into the language of Lagrangian or Hamiltonian dynamics, anticipating eventual quantum treatments.

$\Rightarrow$ Start with the Principle of Least Action:

If a particle moves from one phase space point $a = (q_i^a, \dot{q}_i^a)$ at time t_a to another point $b = (q_i^b, \dot{q}_i^b)$ at time t_b, it will follow the trajectory that minimizes the ACTION:

$$A = \int_{t_a}^{t_b} L\left[q_i(t), \dot{q}_i(t)\right] dt$$

From this, one derives the equations of motion (the Euler-Lagrange eqns) by demanding the stationary minimum equation:

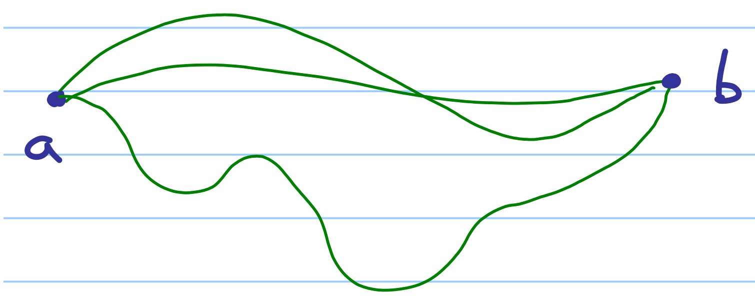
$$\delta\left(\frac{dq_i}{dt}\right) = \frac{d}{dt}\delta q_i$$

$$0 = \delta A = \sum_i \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right) dt$$

integrate this term by parts

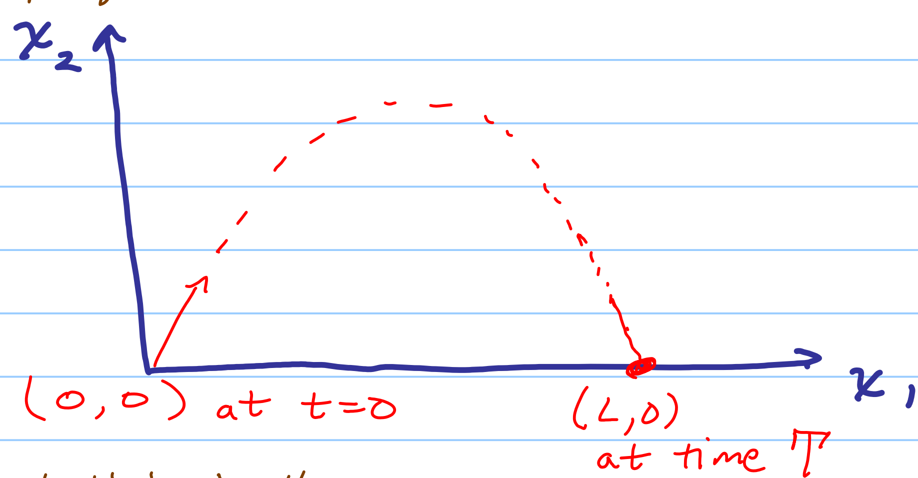
$$0 = \delta A = \sum_i \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \, dt + \cancel{\frac{\partial L}{\partial \dot{q}_i}\delta q_i\Big|_{t_a}^{t_b}}$$

vanishes if $\delta q_i = 0$ at the endpoints, i.e. if we only consider those paths that start at q_i^a at time t_a and end at q_i^b at time t_b



<u>Nonrelativistic example</u> — how Lagrangian dynamics works

$\Rightarrow$ consider a projectile near the Earth's surface



Recall that nonrelativistically,

$$L = T - V = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - mgx_2$$

For these initial and final points, the action is

$$A = \int_0^T \left\{ \tfrac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - m g x_2(t) \right\} dt$$

where $x_1(t)$, $x_2(t)$ are to be determined

$$\Rightarrow \delta A = 0 = \sum_{i=1}^{2} \int_0^T \left\{ \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} \right\} \delta x_i \, dt$$

+ surface terms that vanish since $\delta x_i = 0$ there

$$\Rightarrow (i) \quad \cancelto{0}{\frac{\partial L}{\partial x_1}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = 0 \Rightarrow \frac{d}{dt}(m\dot{x}_1) = 0$$

or $\dot{x}_1 =$ constant

$$\Rightarrow x_1(t) = \frac{L}{T} t$$

and

$$(ii) \quad \frac{\partial L}{\partial x_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = 0 \Rightarrow -mg - \frac{d}{dt}(m\dot{x}_2) = 0$$

or $\ddot{x}_2 = -g \Rightarrow x_2(t) = \underbrace{V_2}_{\text{integration const.}} t - \frac{1}{2} g t^2$

and V_2 is fixed by demanding that $x_2(T) = 0$

$$\Rightarrow V_2 T - \frac{1}{2} g T^2 = 0 \Rightarrow V_2 = \frac{1}{2} g T$$

and finally the full solution is

$$x_1(t) = \frac{L}{T} t$$

$$x_2(t) = \frac{1}{2} g \, (tT - t^2)$$

Next, let's find the Lagrangian that gives the correct relativistic equations of motion, i.e. find L appropriate to

$$A = \int_{t_1}^{t_2} L\,dt = \int_{\tau_1}^{\tau_2} \gamma L\,d\tau$$

using $d\tau = \frac{dt}{\gamma}$ and $\gamma = \left(1 - \frac{u(t)^2}{c^2}\right)^{-1/2}$

Now, we seek extrema of A, and they must be extrema in ALL inertial frames. Therefore, we anticipate that A must be a Lorentz invariant (scalar)

$\Rightarrow$ inspecting the above A, this means that

$$\gamma L = \text{invariant}$$

Further considerations

(a) For a free particle, we expect that L should depend on speed only, not on position nor the direction of velocity.

$\Rightarrow$ The only plausible invariant constructed from velocity, recalling that $U^\alpha = \gamma(c, \vec{u})$, is

$$U_\alpha U^\alpha = \gamma^2 c^2 - \gamma^2 u^2 = c^2$$

This invariant has dimensions of energy if we multiply by mass m of the particle.

This suggests that L for a free particle might be chosen to be

$$L = -\frac{mc^2}{\gamma_u} = -mc^2\sqrt{1-\frac{u^2}{c^2}}$$

We must check to see whether the Euler-Lagrange equations make sense:

$$\cancelto{0}{\frac{\partial L}{\partial x_i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial u_i} = 0$$

and
$$\frac{\partial L}{\partial u_i} = -\frac{mc^2\left(\frac{1}{2}\right)\left(-\frac{2u_i}{c^2}\right)}{\left(1-\frac{u^2}{c^2}\right)^{1/2}} = \underbrace{\gamma_u m u_i}$$

i^{th} component of the relativistic momentum

$$\Rightarrow \frac{d}{dt}\vec{p} = 0$$, or $\vec{p} =$ constant is the solution, as expected

So we conclude that an appropriate relativistic free-particle Lagrangian is

$$\boxed{L_{free} = -mc^2\sqrt{1-\frac{u^2}{c^2}}}$$

which does satisfy $\gamma L_{free} =$ Lorentz invariant.

(b) Now consider a particle interacting with external $\vec{E}, \vec{B}$ - fields

We know that in the low velocity limit the interaction energy is mainly electrostatic, i.e.

$$V_{int} = q\Phi \longrightarrow \frac{1}{c}\int c\rho\, \Phi\, d^3x$$

Then, since the nonrelativistic Lagrangian is $L = T - V$, we expect $L_{int} \xrightarrow[u\to 0]{} -q\Phi$

Thus the natural generalization to finite velocity is

$$\gamma V_{int} = \frac{q}{c} U^\alpha A_\alpha$$

$$\Rightarrow \quad \gamma L \longrightarrow \gamma L_{free} - \frac{q}{c} U^\alpha A_\alpha$$

or finally

$$\boxed{L = -mc^2\sqrt{1 - \frac{u^2}{c^2}} - \frac{q}{c}\left(c\Phi - \vec{u}\cdot\vec{A}\right)}$$

This is completely correct, and γL is a Lorentz invariant, even though it is not manifestly covariant

And the Euler-Lagrange equations do give the correct equations of motion, as is readily verified: $\frac{d}{dt}\frac{\partial L}{\partial u_i} = \frac{\partial L}{\partial x_i}$

$$\Rightarrow \frac{d}{dt}\left(\gamma m u_i + \frac{q}{c}A_i\right) = -q\frac{\partial \Phi}{\partial x_i} + \frac{q}{c}\sum_j u_j \frac{\partial A_j}{\partial x_i}$$

Now recall that $\vec{A} = \vec{A}(\vec{x}(t), t)$

$$\Rightarrow \quad \frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + (\vec{u}\cdot\nabla)A_i$$

$$\Rightarrow \frac{d}{dt}(\gamma m u_i) = q\left[(-\nabla\Phi)_i - \frac{1}{c}\frac{\partial A_i}{\partial t}\right]$$
$$+ \frac{q}{c}\sum_j\left(u_j\frac{\partial A_j}{\partial x_i} - u_j\frac{\partial A_i}{\partial x_j}\right)$$

and $\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

Moreover $\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = \sum_k \epsilon_{ijk}B_k$

whereby $\sum_j u_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) = \sum_{jk}\epsilon_{ijk}u_j B_k = (\vec{u}\times\vec{B})_i$

and this gives finally the expected result,

$$\boxed{\frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{u}) = q\left(\vec{E} + \frac{\vec{u}}{c}\times\vec{B}\right)}$$

Next, with an eye toward quantum physics, the <u>Hamiltonian</u> is constructed

First of all, the momentum p_i conjugate to any generalized coordinate q_i is defined to be

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial u_i} = \gamma m u_i + \frac{q}{c} A_i$$

Then

$$H = \sum_i p_i \dot{q}_i - L = \vec{p} \cdot \vec{u} - L$$

and to use this H in either classical or QM, we should express H only in terms of p_i, q_i and eliminate $u_i = \dot{q}_i$, i.e. using $\boxed{\vec{p} = \gamma m \vec{u} + \frac{q}{c} \vec{A}}$

e.g. $\vec{p}^{\,2} = \gamma^2 u^2 m^2 + \frac{q^2}{c^2} A^2 + 2\frac{q}{c} \gamma \vec{u} \cdot \vec{A}$

or $\gamma^2 u^2 m^2 = \left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 = \dfrac{m^2 u^2}{1 - \frac{u^2}{c^2}}$

$\Rightarrow \vec{u} = \dfrac{c\left(\vec{p} - \frac{q}{c}\vec{A}\right)}{\left[\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + m^2 c^2\right]^{1/2}}$, so plugging this in and simplifying gives

$\Rightarrow H = \vec{p} \cdot \vec{u} - L$ equals

$$\boxed{H = \sqrt{\left(c\vec{p} - q\vec{A}\right)^2 + m^2 c^4} + q\Phi(\vec{x})}$$

Aside: The so-called "first quantization" involves the replacement $\vec{p} \longrightarrow -i\hbar\nabla$, in QM