Sec 11.10 Lorentz transformation of electromagnetic fields

The fact that these fields are characterized by an antisymmetric rank 2 4-tensor,

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

tells us immediately that the Lorentz transf. from a frame K to another K' moving at velocity $\vec{v}$ w.r.t. K MUST BE:

$$F'^{\alpha\beta} = \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta F^{\gamma\delta}$$

or in matrix notation,

$$F' = \Lambda F \tilde{\Lambda}$$

where $\Lambda = \exp(-\vec{\omega}\cdot\vec{S} - \vec{\zeta}\cdot\vec{K})$

Specializing to $\vec{v} = v\hat{x}$, where

$$\Lambda^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\beta} = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $F = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$

See, e.g. the Mathematica notebook

LorentzTransformationMatrix.nb

in the lecture notes directory for the course, to do these matrix multiplications. They yield:

$$\boxed{\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - \beta B_z) & B_y' &= \gamma(B_y + \beta E_z) \\ E_z' &= \gamma(E_z + \beta B_y) & B_z' &= \gamma(B_z - \beta E_y) \end{aligned}}$$

Note also that the inverse transformation is obtained by:

(i) interchanging primed and unprimed quantities

and (ii) setting $\beta \to -\beta$

You might verify yourself that the general fields in K', moving at an arbitrary velocity $\vec{v}$ w.r.t. K are given by

$$\boxed{\begin{aligned} \vec{E}' &= \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}) \\ \vec{B}' &= \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B}) \end{aligned}}$$

Comments

- $\vec{E}$ and $\vec{B}$ get "mixed-up" when you change from one Lorentz frame to another

- What appears to be an exclusively electrical phenomenon to an observer K could be viewed as being partly or even mostly magnetic to observers in K' !

- In this sense, one might argue that $F^{\alpha\beta}$ is more "fundamental" than $\vec{E}$ and $\vec{B}$ separately

A simple but important example

- Consider a point charge q moving at constant velocity $\vec{v} = v\,\hat{x}$ relative to an observer's frame K.

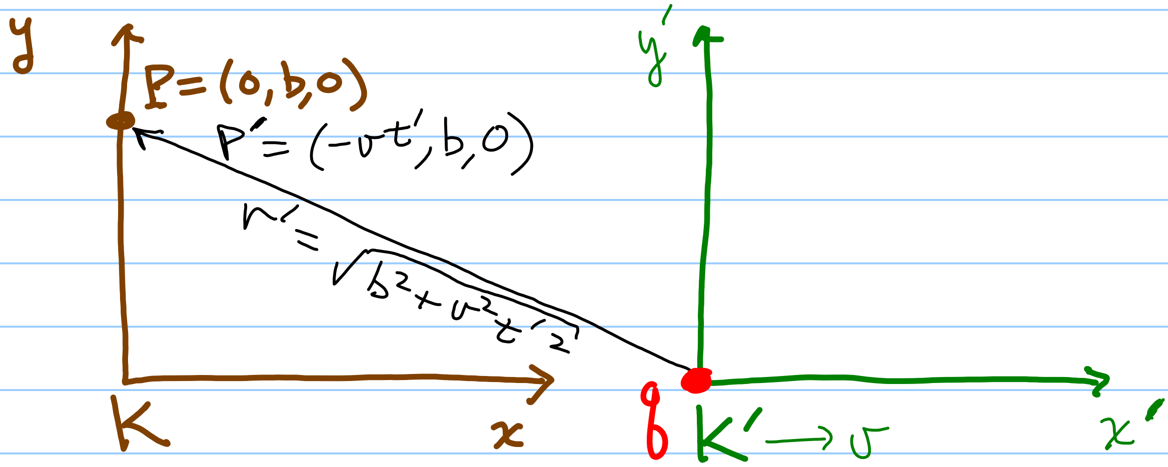
$\Rightarrow$ Find $\vec{E}$ and $\vec{B}$ seen by the observer K.

Solution call K' the rest frame of q, and let the origins of K, K' coincide at $t = t' = 0$

Let's further specify that the observer in K who measures the fields will be located at point $P = (0, b, 0)$ in K, i.e. on the y-axis a distance b from the origin.

Then in K' the observer's position at time t' is

$$(-vt', b, 0)$$



and $t' = \gamma\left(t - \frac{vx}{c^2}\right) \to \gamma t$ here, since $x=0$ at $\underline{P}$

Of course, in K' there is only an electric field at P', namely

$$E'_x = \frac{q}{r'^2}\left(\frac{-vt'}{r'}\right) = -\frac{qvt'}{r'^3} = \frac{-qv\gamma t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

and

$$E'_y = \frac{qb}{r'^3} = \frac{qb}{(b^2+\gamma^2 v^2 t^2)^{3/2}}, \quad E'_z = 0$$

$$(\text{and } \vec{B}' = 0)$$

To transform these into frame K, we need the <u>inverse</u> of the above transformation, namely

$$\boxed{\begin{array}{ll} E_x = E'_x & B_x = B'_x \\ E_y = \gamma(E'_y + \beta B'_z) & B_y = \gamma(B'_y - \beta E'_z) \\ E_z = \gamma(E'_z - \beta B'_y) & B_z = \gamma(B'_z + \beta E'_y) \end{array}}$$

Applying these formulas to the present example:

$$E_x = E'_x = \frac{-q\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$E_y = \gamma E'_y = \frac{\gamma q b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$B_z = \gamma \beta E'_y = \frac{\gamma \beta q b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$E_z = 0$

$B_x = 0$

$B_y = 0$

Observations

(1) To observer K at P, $\dfrac{E_x}{E_y} = -\dfrac{vt}{b}$

meaning that $\vec{E}$ points radially AWAY from the "present" position of the particle; i.e. NOT from its position at the retarded time.

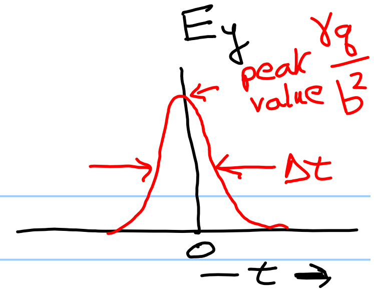
(2) $B_z \neq 0$, as expected since this moving charge represents a CURRENT element to K.

i.e. recall the Biot-Savart law (Eq. 5.5 converted to Gaussian units) which says $\vec{B} = \frac{q}{c}\frac{\vec{v}\times\vec{r}}{r^3}$, and for $\vec{v} = v\hat{x}$ here,

and $\vec{r} = -vt\,\hat{x} + b\hat{y} \Rightarrow \vec{B}(\text{Biot-Savart}) = \frac{\beta q b}{r^3}\hat{z}$

Equivalent to the above result at low velocities, when $\beta \ll 1$, $\gamma \simeq 1$

(3) At $t=0$, $E_y^{max} = \frac{\gamma q}{b^2} \xrightarrow[\beta\to 1]{} \infty$



$\Rightarrow$ Observer K sees a pulse of E_y $\Rightarrow$ <u>and</u> a pulse of $B_z = \beta E_y$

or $B_z \simeq E_y$ for $\beta \simeq 1$

The duration of this pulse is of order

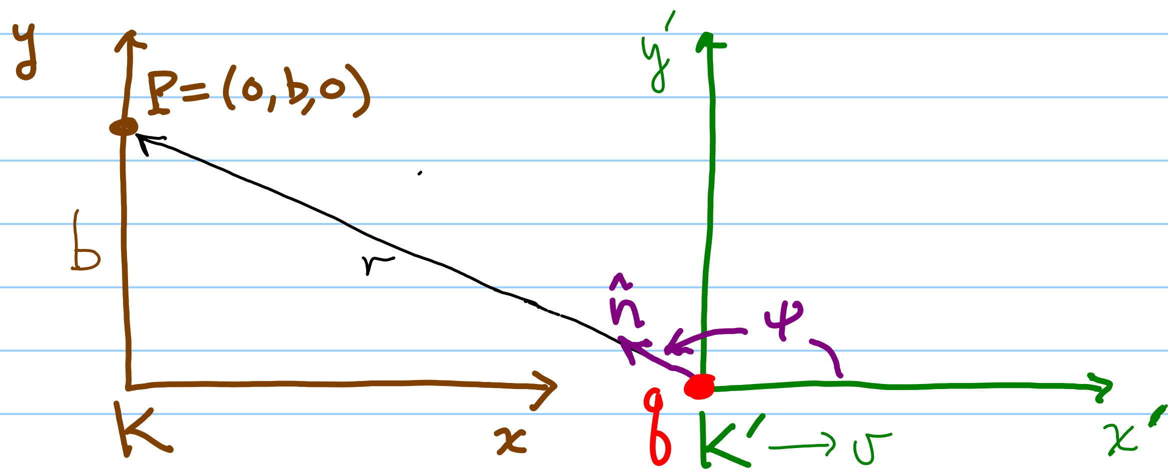
$$\Delta t \simeq \frac{b}{\gamma v} \xrightarrow[\beta\to 1]{} 0$$

and the integrated pulse strength is

$$\underset{q\text{'s path}}{\int} E_y\, dx = \int E_y\, v\, dt = \int \frac{\gamma q v b\, dt}{(b^2+\gamma^2 v^2 t^2)^{3/2}} = \frac{2q}{b}$$

and this, interestingly, is <u>independent</u> of v!

(4) Consider Fig 11.8 of Jackson



and observe that $b = r\sin(\pi - \psi) = r\sin\psi$

$$vt = r\cos(\pi-\psi) = -r\cos\psi$$

and thus we can rewrite the $\vec{E}$-Field as

$$E_x = \frac{q r \cos\psi}{\gamma^2 r^3 (1-\beta^2 \sin^2\psi)^{3/2}}$$

$$E_y = \frac{q r \sin\psi}{\gamma^2 r^3 (1-\beta^2 \sin^2\psi)^{3/2}}$$

Why? $b^2 + \gamma^2 v^2 t^2 = r^2 [\sin^2\psi + \gamma^2(1-\sin^2\psi)]$

$$= \gamma^2 r^2 \{1 + \sin^2\psi \underbrace{(\gamma^{-2} - 1)}_{= -\beta^2}\}$$

$$= \gamma^2 r^2 (1 - \beta^2 \sin^2\psi)$$

So we deduce a simple, compact expression for the full $\vec{E}$-vector in K (lab frame):

$$\boxed{\vec{E} = \frac{q \vec{r}}{\gamma^2 r^3 (1-\beta^2 \sin^2\psi)^{3/2}}}$$

Thus: (a) $\vec{E} \xrightarrow[\beta \to 0]{} \frac{q \hat{r}}{r^2}$ as expected

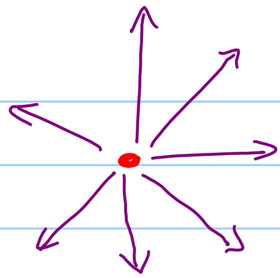
(b) $\vec{E} \propto \hat{r}$, i.e. radially outward from the particle's present position, as noted above.

(c) $\vec{E}$ is not isotropic in frame K

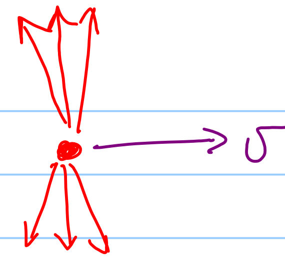
$\Rightarrow$ at any fixed r-value, $|\vec{E}|$ is maximum at $\sin^2\psi = 1 \Rightarrow \psi = \pm\frac{\pi}{2}$

while $|\vec{E}|$ is minimum at $\sin^2\psi = 0 \Rightarrow \psi = 0, \pi$

$\beta << 1$, isotropic



$\beta \simeq 1$, anisotropic



$\Rightarrow$ This compression of field lines in the direction of motion is somewhat analogous to a "length contraction"