Transformation of the fields

Recall that
$$\boxed{\begin{gathered}\partial^\alpha = (\partial_0, -\nabla) \\ \partial_0 = \frac{1}{c}\frac{\partial}{\partial t} \\ A^\alpha = (\Phi, \vec{A})\end{gathered}}$$

We start by writing out the fields in this notation, component-by-component:

(1) $\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \Rightarrow E^1 = \partial^1 A^0 - \partial^0 A^1$
$$= -(\partial^0 A^1 - \partial^1 A^0)$$

and likewise, $E^2 = -(\partial^0 A^2 - \partial^2 A^0)$, ...etc

(2) $\vec{B} = \nabla \times \vec{A} \Rightarrow B^1 = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$

and $B^2 = -(\partial^3 A^1 - \partial^1 A^3)$, etc.

and this suggests that a key entity in electromagnetism should be the following:

$$\boxed{F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha}$$

which we call the antisymmetric, rank-2,

FIELD-STRENGTH TENSOR

Writing out its contravariant form explicitly, this is:

$$F^{\alpha\beta} = \begin{array}{c|c} \alpha \backslash \beta & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \end{array}$$

and its covariant form is $F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta}$
which is identical, except
that everywhere $\vec{E} \to -\vec{E}$, $\vec{B} \to \vec{B}$

Another quantity of interest is the
"DUAL FIELD-STRENGTH TENSOR":

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

(simply get this by replacing $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ in $F^{\alpha\beta}$)

where $\epsilon^{\alpha\beta\gamma\delta}$ = totally antisymmetric rank-4 tensor, i.e.

$\epsilon^{0123} = 1$ and all other even permutations

$\epsilon^{1023} = -1$ " " " odd permutations

$\epsilon^{1123} = 0$ and all other elements having 2 or more equal indices

In Mathematica, e.g.:

```
epsilon = LeviCivitaTensor[4]
epsilon[[1,2,3,4]]   (returns 1)
epsilon[[2,1,3,4]]   (returns -1)
epsilon[[2,2,3,4]]   (returns 0)
```

Covariant form of Maxwell's equations (microscopic case, no media)

(a) inhomogeneous equations (remember – Gaussian units now)

1) $\nabla \cdot \vec{E} = 4\pi\rho \implies \sum_{i=1}^{3} \partial_i E^i = \frac{4\pi}{c} J^0$

and recall that $E^i = F^{i,0}$, $F^{0,0} = 0$

Hence we can write this as

$$\partial_\alpha F^{\alpha,0} = \frac{4\pi}{c} J^0$$

2) $\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$

e.g. $$\boxed{\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{\partial E_x}{\partial x^0} = \frac{4\pi}{c} J_x} \quad (*)$$

and now $B_z = F^{2,1}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial x^2} = \partial_2$

$B_y = -F^{3,1}$, $\frac{\partial}{\partial z} = \frac{\partial}{\partial x^3} = \partial_3$

$E_x = -F^{0,1}$, $\frac{\partial}{\partial x^0} = \partial_0$

Thus (*) reads:

$$\partial_0 F^{0,1} + \partial_2 F^{2,1} + \partial_3 F^{3,1} = \frac{4\pi}{c} J^1$$

for convenience, insert here $0 = \partial_1 F^{1,1}$ since $F^{1,1} = 0$

$\implies \partial_\alpha F^{\alpha,1} = \frac{4\pi}{c} J^1$, and more generally

we get the simple covariant form

of the inhomogeneous Maxwell's equations,

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$$

(b) homogeneous equations

1) $\nabla \cdot \vec{B} = 0$ 2) $-\nabla \times \vec{E} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$

and observe that these can be obtained from the INHOMOGENEOUS Maxwell equations by making these substitutions:

$$\boxed{\begin{array}{l} \vec{E} \longrightarrow \vec{B} \\ \vec{B} \longrightarrow -\vec{E} \\ \rho \longrightarrow 0 \\ \vec{J} \longrightarrow 0 \end{array}}$$

same replacement used to get $\mathcal{F}^{\alpha\beta}$ from $F^{\alpha\beta}$ above

Therefore we have at once that the covariant form of Maxwell's homogeneous equations are

$$\boxed{\partial_\alpha \mathcal{F}^{\alpha\beta} = 0}$$

Aside Jackson asserts that this can also be recast as Eq. 11.143, $\boxed{\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0}$

with $\alpha, \beta, \gamma =$ any 3 nonequal indices of $(0,1,2,3)$

e.g. $\alpha, \beta, \gamma = 0, 1, 2$ would give

$$\partial_0 F_{1,2} + \partial_1 F_{2,0} + \partial_2 F_{0,1} \stackrel{?}{=} 0$$

i.e. $\frac{\partial}{\partial x^0} F_{12} + \frac{\partial}{\partial x^1} F_{20} + \frac{\partial}{\partial x^2} F_{01} \stackrel{?}{=} 0$

and $F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$ (11.138)

$$= \frac{1}{c}\frac{\partial}{\partial t}(-B_z) + \frac{\partial}{\partial x}(-E_y) + \frac{\partial}{\partial y} E_x$$

$$= -\frac{1}{c}\frac{\partial B_z}{\partial t} - (\nabla \times \vec{E})_z \quad \text{, which checks!}$$

Summary Covariant electromagnetism can be expressed EITHER using potentials:

$$\boxed{\begin{aligned} \Box A^\alpha &= \frac{4\pi}{c} J^\alpha \\ \partial_\alpha J^\alpha &= 0 \\ \partial_\alpha A^\alpha &= 0 \end{aligned}}$$

OR using fields, in the form

$$\boxed{\begin{aligned} \partial_\alpha F^{\alpha\beta} &= \frac{4\pi}{c} J^\beta \\ \partial_\alpha \mathcal{F}^{\alpha\beta} &= 0 \end{aligned}}$$

Relativistic dynamics

Next let's put the equations of motion for a charged particle into covariant form. In any chosen inertial reference frame the appropriate form of Newton's 2nd law is

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}\right)$$

note: the Lorentz force has the same form in the relativistic limit!

and recall that the relativistic momentum 4-vector is

$$p^\alpha = (p_0, \vec{p}) = m(U^0, \vec{U})$$

with $p^0 = E/c$, $U^\alpha = \gamma(c, \vec{u})$

To get a version of this $\frac{d\vec{p}}{dt} = \vec{F}$ equation having simpler transformations, it would be better to work instead with $\frac{d\vec{p}}{d\tau}$.

$\Rightarrow$ Consider $\gamma \frac{d\vec{p}}{dt} = q\left(\gamma\vec{E} + \frac{\vec{U}}{c} \times \vec{B}\right)$

and, using $d\tau = \frac{dt}{\gamma}$, we obtain

$$\frac{d\vec{p}}{d\tau} = \frac{q}{c}\left(\gamma c\vec{E} + \vec{U} \times \vec{B}\right)$$

or $$\frac{d\vec{p}}{d\tau} = \frac{q}{c}\left(U_0\vec{E} + \vec{U}\times\vec{B}\right)$$

where, for instance, the x-component looks like

$$\frac{dp^1}{d\tau} = \frac{q}{c}\left(F^{1,0}U_0 - B_y\overbrace{(\vec{U})_z}^{''\,-U_3} + B_z\overbrace{(\vec{U})_y}^{''\,-U_2}\right)$$

now recall that

$$F^{\alpha\beta} = \begin{array}{c} \alpha\backslash\beta \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & \;\;1 & \;\;\;2 & \;\;\;3 \end{array} \\ \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \end{array}$$

or with these identifications, we see that

$$\frac{dp^1}{d\tau} = \frac{q}{c}\left(F^{10}U_0 + F^{11}U_1 + F^{12}U_2 + F^{13}U_3\right)$$

or $\dfrac{dp^1}{d\tau} = \dfrac{q}{c}F^{1\beta}U_\beta$, which clearly generalizes for 2, 3 spatial components too

What about the TIME component (0) of this equation?

Eq. 6.110 implies that for a single charge q moving at velocity $\vec{u}$, and if $\mathcal{E}$ = mechanical energy, then

$$\frac{d\mathcal{E}}{dt} = q\,\vec{E}\cdot\vec{u}$$

$\leftarrow$ describes the rate of change of mechanical energy and multiplying through by γ_u/c gives

$$\frac{dp^0}{d\tau} = \frac{q}{c}\vec{E}\cdot\vec{U} = \frac{q}{c}\sum_{i=1}^{3} F^{0i}U_i$$

← since $F^{0i} = (-\vec{E})_i$ and $U_i = (-\vec{u})_i$

$$= \frac{q}{c}F^{0\alpha}U_\alpha$$

← since $F^{00} = 0$

And we see that the equations of motion are the following, now in manifestly covariant form, for any charged particle in an electromagnetic field:

$$\boxed{\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta}$$

plus, of course →

$$\boxed{U^\alpha = \frac{d}{d\tau}x^\alpha}$$

⇒ These are coupled differential equations whose solution gives the position 4-vector $x^\alpha(\tau)$ as a function of proper time, assuming that the fields are known and specified versus position & time. We also would have in classical Newtonian mechanics that $F^{\alpha\beta} = F^{\alpha\beta}(x(\tau))$ should already be known, to find the trajectory of charge q.