Expressing key equations, like these, in manifestly covariant form means that their behavior under Lorentz transformation is trivially obvious.

$\Rightarrow$ We also see that under the transf. from $A^\alpha \xrightarrow{LT} A'^\alpha$

$\Rightarrow$ we find a mixing of $\Phi, \vec{A}$

and from $J^\alpha \to J'^\alpha$

we see a mixing of $\rho, \vec{J}$

Assertion without proof (read Sec. 11.7 on your own):

The most general proper Lorentz transformation can be written as $\Lambda = e^L$ where $L = 4\times4$ matrix. For the simplest case where the transformation is only a boost ζ along the x_1-axis,

$$\Lambda = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

while a rotation about the z-axis by ω looks like:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

But a more general boost transformation by a vector $\vec{\beta}$ looks like

$$\Lambda = e^{-\vec{\zeta}\cdot\vec{K}}$$

where $\vec{\zeta} \equiv \hat{\beta}\tanh^{-1}\beta$, and K_1, K_2, K_3 are given in 11.91, i.e.

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & & & \\ 0 & & 0 & \\ 0 & & & \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ 1 & & 0 & \\ 0 & & & \end{pmatrix}$$

and

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & & & \\ 0 & & 0 & \\ 1 & & & \end{pmatrix} = 3 \text{ generators of infinitesimal boosts}$$

whereas a pure rotation is governed by the 3 infinitesimal rotation generators,

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \; S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}; \; S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the most general proper Lorentz transf depends on 6 parameters, $\zeta_1, \zeta_2, \zeta_3, \omega_1, \omega_2, \omega_3$ and the transf. matrix is

$$\Lambda = e^{-\vec{\omega}\cdot\vec{S} - \vec{\zeta}\cdot\vec{K}}$$

see Jackson pp. 546-7

These 6 matrices obey commutation relations that define the LORENTZ GROUP, namely

$$[S_i, S_j] = \epsilon_{ijk} S_k$$
$$[K_i, K_j] = -\epsilon_{ijk} S_k$$
$$[S_i, K_j] = \epsilon_{ijk} K_k$$

Note: boosts do not in general commute

One intriguing implication of this noncommutation is the remarkable story of the

THOMAS PRECESSION EFFECT

Recall - the spin-orbit interaction in hydrogen can be viewed as the interaction energy between the e^- spin magnetic moment and the B-field created at the e^- position by the "current loop" of the "proton orbital motion" as viewed by the e^-.

This is usually derived in elementary courses by finding the $\vec{B}$-field at the e^- using the BIOT-Savart law, and then use the e^- spin magnetic moment, $\vec{\mu}_{e,spin} = -\frac{g_e e}{2 m_e c} \vec{S}$ (Gaussian units)

and then

$$H^{\text{elementary}}_{spin-orbit} = -\vec{\mu}_{e,spin} \cdot \vec{B}$$

giving

$$W^{\text{elem.}}_{spin-orbit} = \frac{g e}{2 m^2 c^2} \vec{S} \cdot \vec{L} \, \frac{1}{r} \frac{dU}{dr}$$

U = pot. energy of e-p

turns out to be approximately 2 times too large!

(e.g. in Gaussian units, for a 1-electron atom,

$$U = -\frac{Z e^2}{r}$$
)

Alternatively, this can be derived by making a Lorentz transformation of the $\vec{E}, \vec{B}$-fields from the NUCLEUS (lab) rest frame, into the e^- rest frame, using

$$\vec{B}_{e\text{-frame}} = \gamma\left(\cancel{\vec{B}} - \frac{\vec{v}}{c}\times\vec{E}\right)$$

0 here

The classical equation for spin precession is

used $\gamma \simeq 1$

$$\frac{d\vec{S}}{dt} = \vec{\mu}_s \times \vec{B}_{e\text{-frame}} = -\vec{\mu}_s \times \left(\frac{\vec{v}_e}{c}\times\vec{E}\right)$$

where $\vec{E} = -\nabla\Phi = -\frac{\vec{r}}{r}\frac{d\Phi}{dr} = +\frac{1}{e}\frac{\vec{r}}{r}\frac{dU}{dr}$

The energy associated with this precession is

$$W' = -\vec{\mu}_s \cdot \vec{B}_{e\text{-frame}} = -\vec{\mu}_s \cdot \left(-\frac{\vec{v}_e}{c}\times\vec{E}\right)$$

$$= \vec{\mu}_s \cdot \left(\frac{\vec{v}_e}{c}\times\frac{1}{e}\frac{\vec{r}}{r}\frac{dU}{dr}\right), \quad \text{but } m_e\vec{v}_e\times\vec{r} = -\vec{L}$$

so

$$W' = \frac{g_e}{2m^2c^2}\,\vec{S}\cdot\vec{L}\,\frac{1}{r}\frac{dU}{dr}$$ as before (and still wrong!)

To correct the error in this derivation, recall from mechanics that the rate of change of any vector in a rotating frame relates to the rate of change in a nonrotating frame:

$$\left(\frac{d\vec{S}}{dt}\right)_{\text{nonrot}} = \left(\frac{d\vec{S}}{dt}\right)_{\text{rot}} + \vec{\omega}_T \times \vec{S}$$

where $\vec{\omega}_T$ is the angular velocity of rotation found by Thomas (1927), whereby

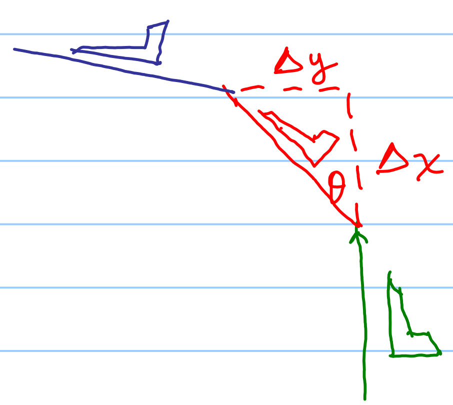
$$\left(\frac{d\vec{S}}{dt}\right)_{\text{nonrot}} = \vec{\mu}_s \times \vec{B}_{\text{e-frame}} - \vec{S} \times \vec{\omega}_T$$

and this precession rate corresponds to an interaction energy in the lab frame of

$$W = W' + \vec{S} \cdot \vec{\omega}_T$$

Next – here is a simple derivation of the "Thomas precession frequency" $\vec{\omega}_T$

$\Rightarrow$ Approximate a circular orbit of an electron (= space shuttle) by an N-sided polygon with N large. When the craft traverses ONE of the N sides, it must alter its angle of flight by $\theta = \frac{2\pi}{N}$ radians:



$$\Rightarrow \theta \simeq \frac{\Delta y}{\Delta x} \ll 1$$

$\Rightarrow$ After N segments, the shuttle is back at its original point, having rotated by 2π radians in the LAB frame.

But in the <u>shuttle</u> <u>rest</u> <u>frame</u>, the rotation angle is LARGER, namely

$$\theta' = \gamma\theta = \frac{\Delta y}{(\Delta x/\gamma)}$$

because the length Δx is Lorentz-contracted but Δy is <u>not</u>

$\Rightarrow$ After a complete revolution, the e (shuttle) frame experiences a rotation $2\pi\gamma$

$\Rightarrow$ Relativity causes an EXTRA amount of rotation in the rotating frame, equal to

$$\Delta\theta' = 2\pi(\gamma - 1),$$ or in the LAB frame,

this corresponds to a frequency ratio

$$\frac{\omega_T}{\omega} = -\frac{\Delta\theta/T}{2\pi/T} = -(\gamma - 1)$$

i.e. $$\omega_T = -(\gamma-1)\frac{v}{r} \simeq -\frac{1}{2}\frac{v^2}{c^2}\frac{v}{r}$$

and now

$$\frac{mv^2}{r} = eE = e\frac{1}{e}\frac{dU}{dr}$$

$$\Rightarrow \quad \omega_T = -\frac{1}{2}\frac{1}{mc^2}\frac{vr}{r}\frac{dU}{dr}$$

or
$$\vec{\omega}_T = -\frac{1}{2m^2c^2}\vec{L}\,\frac{1}{r}\frac{dU}{dr}$$

Hence the total energy in the lab frame is

$$W = W' + \vec{S}\cdot\vec{\omega}_T$$

or
$$W = \frac{g}{2}\frac{1}{m^2c^2}\vec{S}\cdot\vec{L}\,\frac{1}{r}\frac{dU}{dr} - \frac{1}{2}\frac{1}{m^2c^2}\vec{S}\cdot\vec{L}\,\frac{1}{r}\frac{dU}{dr}$$

so the correct spin-orbit term in the lab frame is (to order $\frac{v^2}{c^2}$):

$$\boxed{W = \frac{g-1}{2}\frac{1}{m^2c^2}\vec{S}\cdot\vec{L}\,\frac{1}{r}\frac{dU}{dr}}$$

The electron g-factor is of course

$$g \simeq 2 + \frac{\alpha}{\pi} - 0.657\left(\frac{\alpha}{\pi}\right)^2 + \ldots$$

$\Rightarrow \frac{g-1}{2} = \frac{1}{2}$ instead of the value 1, before making the Thomas precession correction!

and this modified form agrees with expt.

Note: Dirac's relativistic QM theory includes this effect "automatically", though it predicts $g=2$ exactly

$\Rightarrow$ Need QED + renormalization to get a more accurate value of g. (see QM 3)