Sec. 11.5 Relativistic energy + momentum of a particle

Our goal: Find the relativistic generalizations of

$$\vec{p} = m\vec{u} \quad \text{and} \quad T = \tfrac{1}{2}mu^2$$

We would hope that the new generalizations will give the old (nonrelativistic) conservation laws in the limit $u \ll c$, and which hopefully identify quantities conserved even in relativity.

e.g. consider the nonrelativistic momentum conservation in a 1D collision in frame K'

$$\underset{u_1'}{①} + \underset{u_2'}{②} \longrightarrow \underset{u_3'}{③} + \underset{u_4'}{④}$$

Then the nonrelativistic equality in K' we know is

$$m_1 u_1' + m_2 u_2' = m_3 u_3' + m_4 u_4'$$

Now perform a Lorentz transf. to frame K moving at velocity $-v$ w.r.t. K':

$$\Rightarrow m_1 \frac{v + u_1'}{1 + \frac{v u_1'}{c^2}} + m_2 \frac{v + u_2'}{1 + \frac{v u_2'}{c^2}}$$

$$\stackrel{?}{=} m_3 \frac{v + u_3'}{1 + \frac{v u_3'}{c^2}} + m_4 \frac{v + u_4'}{1 + \frac{v u_4'}{c^2}}$$

It is hard to imagine that this equality could be true!

In fact it is not true!

whereas the nonrelativistic momentum conservation equation in K would read simply

$$m_1(v+u_1') + m_2(v+u_2') = m_3(v+u_3') + m_4(v+u_4')$$

which holds nonrelativistically since

$$m_1 + m_2 = m_3 + m_4,$$

and mass is conserved in Galilean kinematics

Idea of the relativistic approach:

Attempt to define the relativistic momentum in terms of the 4-velocity, i.e. try

$$\vec{P} = m\vec{U} = \gamma_u m \vec{u}\ , \text{ where } \gamma_u = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$$

This is because, if this momentum is conserved in a collision in one frame K', i.e. if the components along $\hat{x}_1$ obey: (I)

$$m_1 U_1'(1) + m_2 U_1'(2) = m_3 U_1'(3) + U_1'(4)$$

Now carry out a Lorentz transformation to K, assuming each mass m_i = Lorentz scalar

$\Rightarrow$ Let K' move at $v = \beta c$ w.r.t. K and set $\gamma_v = (1-\beta^2)^{-1/2}$, and then for each particle,

$$U_0 = \gamma_v(U_0' + \beta U_1')$$

$$U_1 = \gamma_v(U_1' + \beta U_0')$$

Now check whether momentum conservation still holds in K, see if this equality holds,

$$m_1 U_1(1) + m_2 U_1(2) \stackrel{?}{=} m_3 U_1(3) + m_4 U_1(4)$$

This holds iff

II

$$\boxed{\begin{aligned} & m_1 \gamma_v \left[U_1'(1) + \beta U_0'(1)\right] + m_2 \gamma_v \left[U_1'(2) + \beta U_0'(2)\right] \\ & \quad = m_3 \gamma_v \left[U_1'(3) + \beta U_0'(3)\right] + m_4 \gamma_v \left[U_1'(4) + \beta U_0'(4)\right] \end{aligned}}$$

and in fact this <u>does</u> hold provided I holds <u>AND</u> provided the following holds as an additional conservation law,

$$m_1 U_0'(1) + m_2 U_0'(2) = m_3 U_0'(3) + m_4 U_0'(4)$$

$\Rightarrow$ mass is clearly not conserved, but something else is. What?

Recall that $m_1 U_0'(1) = \gamma_1 m_1 c$ with $\gamma_1 = \left[1 - \frac{u_1'(1)^2}{c^2}\right]^{-\frac{1}{2}}$ and in the low velocity limit, $|u_1'(1)| \ll c$,

$$m_1 U_0'(1) \longrightarrow m_1 c$$

Thus this new conserved quantity is consistent with mass conservation in the Galilean limit

$\Rightarrow$ Look at the next order corrections:

$$m\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} m = m\left[1 + \frac{u^2}{2c^2} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\frac{u^4}{c^4} + \dots\right]$$

$$= \frac{1}{c^2}\left[mc^2 + \underbrace{\tfrac{1}{2}mu^2} + \frac{3}{8}\frac{mu^4}{c^4} + \dots\right]$$

this next correction to the mass
is the ordinary nonrelativistic kinetic energy

$$\Rightarrow \gamma mc^2 = mc^2 + T + O\left(\frac{u^4}{c^4}\right)$$

Based on this, we interpret

$mc^2 \equiv$ REST ENERGY of a particle

$\gamma mc^2 \equiv$ TOTAL ENERGY of a particle

$(\gamma - 1)mc^2 \equiv$ KINETIC ENERGY in relativity

And thus our newly-defined 3-momentum

$$\vec{p} = m\vec{U}$$

will be conserved IF our newly-defined total energy is also conserved.

$\Rightarrow$ These combine to define the 4-momentum, or the energy-momentum 4-vector:

$$p^\alpha = mU^\alpha = (mU_0, \vec{p}) = \gamma_u m(c, \vec{u}) = \left(\frac{E}{c}, \vec{p}\right)$$

where $E = \gamma_u mc^2$ = total energy in this frame

Note also that $\vec{u} = \frac{\vec{p}\,c}{E}$

And as for any 4-vector, the "length" is INVARIANT, i.e.

$$p^\alpha p_\alpha = \frac{E^2}{c^2} - \vec{p}^{\,2} = \text{Lorentz invariant}$$

$\Rightarrow$ In practice, we can calculate it in ANY convenient reference frame, and this is a ubiquitous strategy in special relativity.

e.g. in the particle's rest frame, $\vec{p} = 0$, $\gamma = 1$, and $E = mc^2$

$$\Rightarrow p^\alpha p_\alpha = m^2c^2 = \frac{E^2}{c^2} - \vec{p}^{\,2}$$

or

$$E = \sqrt{\vec{p}^{\,2}c^2 + m^2c^4}$$

and $T = E - mc^2 = (\gamma - 1)mc^2$

Secs. 11.6, 11.9 On the idea of COVARIANCE

Recall: so far we have discussed

contravariant 4-vectors, $a^\alpha = (a^0, \vec{a})$

covariant " $a_\alpha = (a_0, -\vec{a})$ } $a^0 = a_0$

connected by the metric tensor,

$$g_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix}$$

with the properties $g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^{\ \beta}$

and $\delta_\alpha^{\ \alpha} = 4$

And the Lorentz transformation is $x'^\alpha = \Lambda^\alpha_{\ \beta} x^\beta$

where $\Lambda^\alpha_{\ \beta} = \dfrac{\partial x'^\alpha}{\partial x^\beta} = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

or $\Lambda^\alpha_{\ \beta} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for a boost along the x-axis

Other relations: $\Lambda^\alpha_{\ \beta} = g^{\alpha\eta} \Lambda_{\eta\beta} = \Lambda^{\alpha\eta} g_{\eta\beta}$

Try to stay consistent and, when possible, write these formulas with adjacent indices matching

Generalizing further, a rank 2 contravariant tensor transforms as:

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\sigma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\sigma\delta}$$

or $$F'^{\alpha\beta} = \Lambda^{\alpha}{}_{\sigma} \Lambda^{\beta}{}_{\delta} F^{\sigma\delta}$$

which can be rewritten in matrix notation as

$$F' = \Lambda F \tilde{\Lambda}$$ $\tilde{\ }$ ← means matrix transpose see Eq. 11.147.

One can similarly show that if b_β is a covariant 4-vector, then

$$Y^{\alpha} = F^{\alpha\beta} b_\beta$$ transforms as a covariant 4-vector

which is analogous to what we saw in Chap. 6, e.g. $\overleftrightarrow{T} \cdot \vec{b}$ transforms like an ordinary vector under 3D rotations

more generally transformations look like

$$F'^{\alpha'\beta'\ldots\gamma'\delta'} = \Lambda^{\alpha'}{}_{\alpha} \Lambda^{\beta'}{}_{\beta} \cdots \Lambda^{\gamma'}{}_{\gamma} \Lambda^{\delta'}{}_{\delta} F^{\alpha\beta\ldots\gamma\delta}$$

etc...

There is also a 4-vector operator that generalizes the 3D gradient operator, motivated by the 3D case. Let $\vec{a}$ = constant, 3D vector. Then we know that $\nabla(\vec{a} \cdot \vec{x}) = \vec{a}$

Similarly, consider a constant 4-vector a^α which we can use to make the invariant

$$a \cdot x = a^\alpha x_\alpha$$

$$\Rightarrow \quad \frac{\partial}{\partial x_\beta}(a \cdot x) = a^\beta$$

From this we conclude that $\frac{\partial}{\partial x_\alpha}$ transforms like a contravariant 4-vector.

$\Rightarrow$ With this motivation, let's write

$$\boxed{\frac{\partial}{\partial x_\alpha} \equiv \partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\nabla\right)}$$

and similarly $\boxed{\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla\right)}$

is a covariant 4-vector

These results further imply that the contracted operator

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x_0^2} - \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

is a Lorentz scalar / invariant

Now let's re-express some old results, to gain familiarity with this notation:

(1) The inhomogeneous wave eqns for $\Phi, \vec{A}$ in the Lorentz gauge, in Gaussian units:

Eqs (6.14 – 6.16)

$$\Box \begin{Bmatrix} \Phi \\ \vec{A} \end{Bmatrix} = -\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix} \Phi \\ \vec{A} \end{Bmatrix} = \frac{4\pi}{c}\begin{Bmatrix} c\rho \\ \vec{J} \end{Bmatrix}$$

plus the Lorentz gauge condition,

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla\cdot\vec{A} = 0$$

(2) The charge continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0$$

which suggests that we define a 4-current,

$$J^\alpha = (c\rho, \vec{J})$$

because now it is clear that the continuity equation holds in every inertial frame, i.e.

$$\partial_\alpha J^\alpha = \frac{\partial}{\partial x^0}(c\rho) + \nabla\cdot\vec{J} = 0 = \partial^\alpha J_\alpha$$

and from (1), $\Box(\Phi, \vec{A}) = \frac{4\pi}{c}(c\rho, \vec{J}) = \frac{4\pi}{c}J^\alpha$

Since $\Box$ = Lorentz scalar, this equation requires that

$(\Phi, \vec{A}) \equiv A^\alpha$ is a contravariant 4-vector,

and the Lorentz gauge condition is simply

$\partial_\alpha A^\alpha = 0$ Now the potential equations of motion and the gauge condition are expressed in covariant form.