General transformation theory

Suppose the transformation from one inertial frame to another is written as

$$x'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} x^{\beta}$$

and we can visualize $\frac{\partial x'^{\alpha}}{\partial x^{\beta}} \equiv \Lambda^{\alpha}{}_{\beta}$ as a transformation matrix

and ANY vector quantity, A^{α} whose transformation law is

$$A'^{\alpha} = \Lambda^{\alpha}{}_{\beta} A^{\beta},$$

is a contravariant vector.

Similarly, a quantity whose transformation law is $B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta}$ is a covariant vector (or tensor of rank 1)

Relativistic Doppler Shift

– a derivation by invariance arguments

Observe that in a plane wave like

$$e^{i(\vec{k}\cdot\vec{r} - \omega t)} \equiv e^{i\phi},$$

ϕ must be an INVARIANT, independent of reference frame, since it simply counts the number of wave crests

$\Rightarrow$ if we write $-\phi = \frac{\omega}{c}ct - \vec{k}\cdot\vec{x} = k^\alpha x_\alpha$

where $k^\alpha = \left(\frac{\omega}{c}, \vec{k}\right)$,

then k^α must be a 4-vector since its contraction with a known 4-vector x^α is an invariant (scalar)

Accordingly, this is all we need to know in order to write down the transformation law for k^α. First break its spatial part $\vec{k}$ into parallel + perpendicular (to $\vec{v} = c\vec{\beta}$) components, $\vec{k} = \vec{k}_\parallel + \vec{k}_\perp$

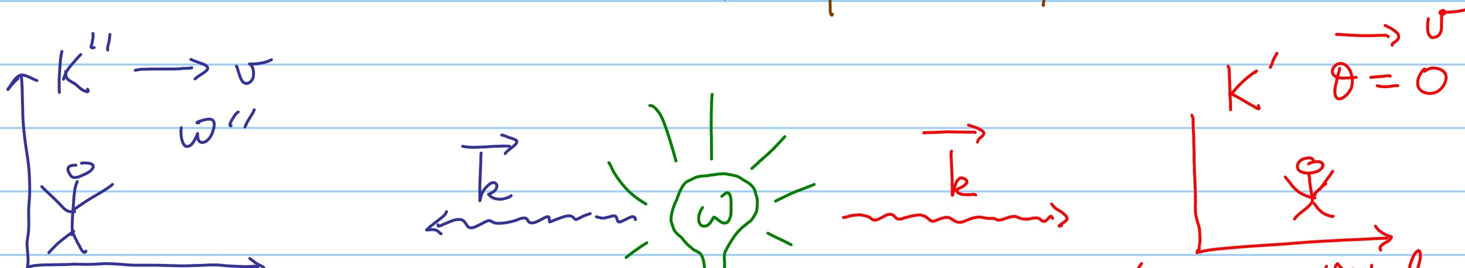
i.e. $\vec{\beta}\cdot\vec{k}_\perp = 0$, then

$$\boxed{\begin{aligned} &(1)\ \vec{k}_\perp' = \vec{k}_\perp \\ &(2)\ k_0' = \frac{\omega'}{c} = \gamma(k_0 - \beta k_\parallel) \\ &(3)\ k_\parallel' = \gamma(k_\parallel - \beta k_0) \end{aligned}}$$

Now, recall that for EM radiation in free space, $|\vec{k}| = \frac{\omega}{c} \Rightarrow v k_\parallel = v|\vec{k}|\cos\theta = \frac{v}{c}\omega\cos\theta$

where θ = angle between $\vec{v}$ and $\vec{k}$ (note: $\theta \neq \theta'$)

Thus $\omega' = \gamma\omega(1 - \beta\cos\theta)$



ω'' = blue-shifted from ω

ω' = red-shifted from ω

$\underline{\theta = 0}$ $\quad \vec{k} \parallel \vec{v} \Rightarrow \omega' = \gamma(1-\beta)\omega = \dfrac{1-\beta}{\sqrt{1-\beta^2}}\,\omega$

or $\boxed{\omega' = \left(\dfrac{1-\beta}{1+\beta}\right)^{1/2}\omega < \omega}$

$\underline{\theta = \pi}$ $\quad \vec{k} \parallel -\vec{v} \quad \omega'' = \gamma(1+\beta)\omega = \dfrac{\gamma(1+\beta)}{\sqrt{1-\beta^2}}\,\omega$

or $\boxed{\omega'' = \left(\dfrac{1+\beta}{1-\beta}\right)^{1/2}\omega > \omega}$

$\Rightarrow$ This is an example of the <u>longitudinal</u> Doppler shift. Note that the nonrelativistic result would be $\omega' = (1-\beta\cos\theta)\omega$, which would have $\omega' = \omega$ at $\theta = 90°$

But in relativity, even the TRANSVERSE Doppler shift is nonzero, i.e, at $\theta = 90°$, $\omega' = \gamma\omega = \dfrac{\omega}{\sqrt{1-\beta^2}}$

<u>Experiment</u> – see Saathoff et al, Phys. Rev. Lett. <u>91</u>, 190403 (2003)

– has confirmed this to a limit of $\pm 2.2 \times 10^{-7}$

Velocity Addition Theorem

Suppose a particle in frame K is observed to move from (x_1, x_2, x_3) to $(x_1+dx_1, x_2+dx_2, x_3+dx_3)$ in the time interval from t to $t+dt$.

Observer K′ sees these 2 events occurring at (x_1', x_2', x_3') and $(x_1'+dx_1', x_2'+dx_2', x_3'+dx_3')$ during the interval t' to $t'+dt'$

$\Rightarrow$ Then if frame K' moves at velocity $v\hat{x}_1$ relative to K, these observations are related by a Lorentz transformation, namely

$$dx_0 = \gamma_v \left(dx_0' + \beta \, dx_1' \right) \qquad dx_2 = dx_2'$$

$$dx_1 = \gamma_v \left(dx_1' + \beta \, dx_0' \right) \qquad dx_3 = dx_3'$$

Now, the ordinary velocity 3-vector components of the particle in each frame are:

$$u_i = c \frac{dx_i}{dx_0}, \quad u_i' = c \frac{dx_i'}{dx_0'}$$

whereby $$u_{\parallel} = c \frac{dx_1' + \beta \, dx_0'}{dx_0' + \beta \, dx_1'} = c \left(\frac{\frac{dx_1'}{dx_0'} + \beta}{1 + \beta \frac{dx_1'}{dx_0'}} \right)$$

or

$$u_{\parallel} = \frac{u_{\parallel}' + v}{1 + \frac{u_{\parallel}' v}{c^2}}$$

And $$u_{\perp} = c \frac{dx_{\perp}}{dx_0} = c \frac{dx_{\perp}'}{\gamma_v \left(dx_0' + \beta \, dx_{\parallel}' \right)}$$

or

$$u_{\perp} = \frac{u_{\perp}'}{\gamma_v \left(1 + \frac{u_{\parallel}' v}{c^2} \right)}$$

These are the Einstein velocity addition formulas.

Limiting cases

(i) $u', v \ll c \Rightarrow \vec{u} = \vec{u}' + \vec{v}$, coincides with the Galilean result

(ii) either $u' \to c$, or $v \to c$

$$\Rightarrow u \to c$$

Important: velocities do NOT transform according to a Lorentz transformation!

Terminology: SPACELIKE versus TIMELIKE separations between events

Consider 2 events, $P(t_a, \vec{x}_a)$ and $P(t_b, \vec{x}_b)$

The squared invariant interval between them is

$$s_{ab}^2 = c^2(t_a - t_b)^2 - |\vec{x}_a - \vec{x}_b|^2$$

$$\equiv (x_a - x_b)^2 = c^2 \Delta t^2 - R^2$$

Case 1 $\quad s_{ab}^2 > 0 \Rightarrow |c\Delta t| > R$

Claim: For this case we can always find an inertial frame K' for which

$$\vec{x}_a' - \vec{x}_b' = (x_0', 0),$$

in which case we say that $x_a' - x_b'$ is purely timelike.

Proof Align the x_1-axis along the vector

$$\vec{x}_a - \vec{x}_b \equiv \vec{x}_{ab}$$

then $\vec{x}_{ab}{}' = \gamma(\vec{x}_{ab} - \vec{v}t)$

so to find the frame where $\vec{x}_{ab}{}' = 0$, require $R = |\vec{x}_{ab}| = vt$, which is possible since $|ct| < R$,

i.e. $v < c$ which is physically acceptable

Case 2 $\quad S_{ab}^2 < 0 \Rightarrow |c\Delta t| < R$

or $|c\Delta t| < |\vec{x}_{ab}|$

Claim: we can find an inertial frame K' in which $t_a' - t_b' = 0$,

i.e. for which $x_a' - x_b' = (0, \vec{x}_{ab}{}')$ is

PURELY SPACELIKE.

Proof $\quad x_0' = \gamma(x_0 - \beta R)$

So to have $x_0' = 0$, we require $x_0 = \beta R$

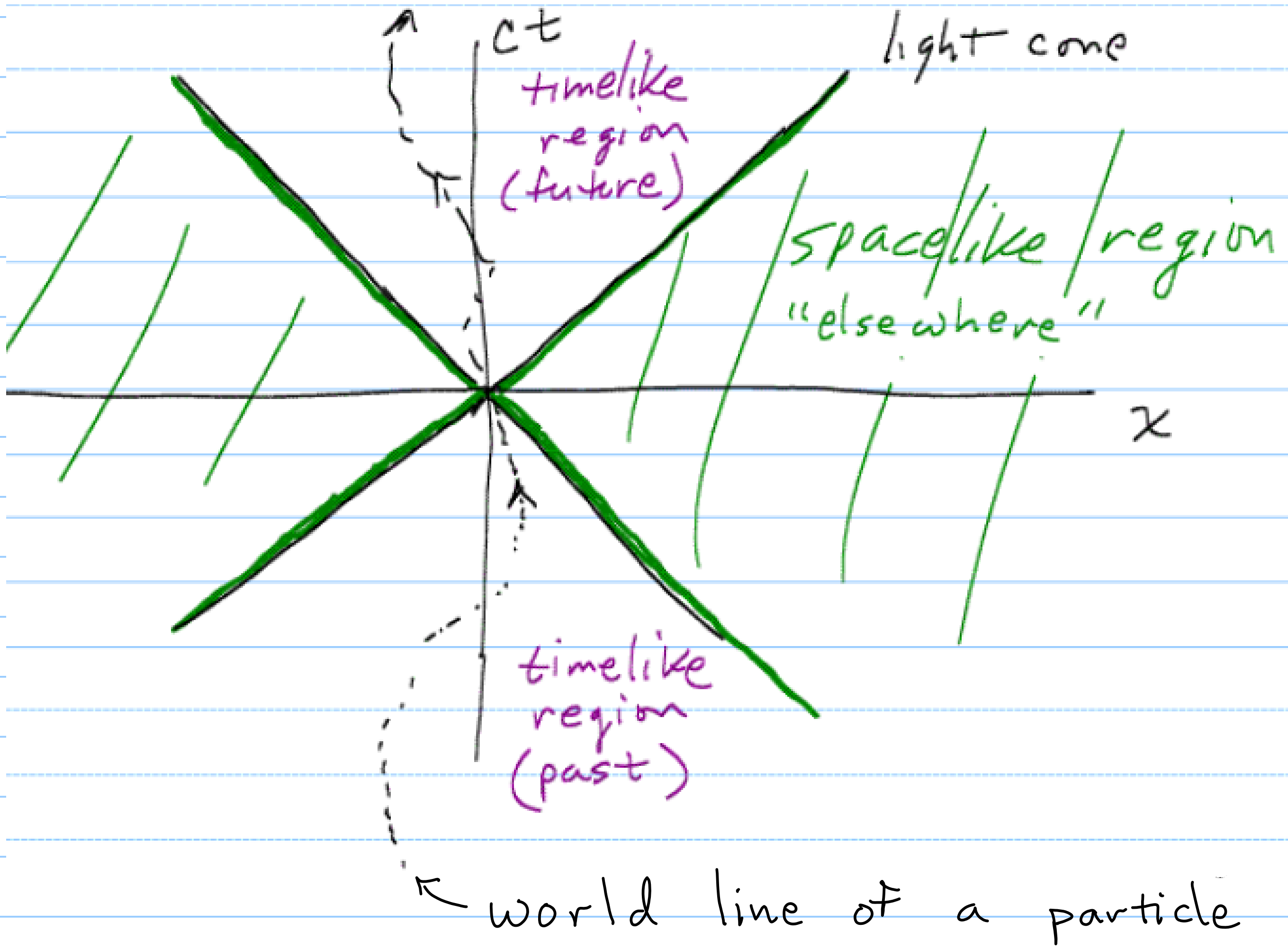
or $\quad c\Delta t = \frac{v}{c} R$, and since $c\Delta t < R$

$\Rightarrow \beta = \frac{v}{c} < 1$, this is physically acceptable

Case 3 $\quad$ If $S_{ab}^2 = 0 \Rightarrow$ A LIGHTLIKE separation

$\Rightarrow$ These 2 events can be connected only by light pulses.

CAUSALITY   2 events with a spacelike separation cannot be causally related, because they would require faster-than-c communication, which is not possible.



Causality – consider a particle at the origin

$\Rightarrow$ 1. It can only be affected then by events in the timelike past.

2. It can only affect events in the future timelike region.

Proper time and 4-velocity

The Einstein velocity addition formula differs from the Lorentz transformation because:

velocity = ratio between the spatial components of a 4-vector x and the timelike component of a 4-vector

Observe, however, that the ratio of a 4-vector to ANY invariant (scalar) would transform like a 4-vector.

In particular, an appropriate invariant to consider is the PROPER TIME.

e.g. recall that $ds^2 = dx^\alpha dx_\alpha$ = invariant

or $ds^2 = c^2 dt^2 - d\vec{x}^{\,2}$, which equals, in the particle's instantaneous rest frame, $ds^2 \rightarrow c^2 dt'^2$

But this dt' is precisely the proper time between the "two events", when the particle is at $\vec{x}', t'$ and when it is at $\vec{x}' + d\vec{x}'$, at time $t' + dt'$.

Hence we can write this as $ds^2 = c^2 d\tau^2$

Now, to express $d\tau$ in terms of dt in any other frame, we apply our time dilation result, i.e.

$$dt = \gamma\, d\tau = \text{time interval in } K$$

Alternative argument: we can instead consider ds^2, which must be the same in the two frames, i.e.

$$ds^2 = \underbrace{c^2 d\tau^2}_{\text{in particle's rest frame}} = \underbrace{c^2 dt^2 - \left(\frac{d\vec{x}}{dt}\right)^2 dt^2}_{\text{in reference frame } K} = c^2(1-\beta^2)\,dt^2 = \frac{c^2 dt^2}{\gamma^2}$$

$$\Rightarrow \boxed{dt = \gamma\, d\tau} \text{ by this argument too}$$

Thus we define the 4-velocity as $\boxed{U^\alpha = \frac{dx^\alpha}{d\tau}}$ which now transforms as a 4-vector!

Interpretation

Since $x^\alpha = (ct, \vec{x})$, the space part of U^α is

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \gamma_u \frac{d\vec{x}}{dt} \Rightarrow \boxed{\vec{U} = \gamma_u \vec{u}}$$

while the time part is

$$U^0 = \frac{dx^0}{d\tau} = c\,\gamma_u \frac{dt}{dt} = c\,\gamma_u$$

so in other words, $U^\alpha = \gamma_u(c, \vec{u})$

with $\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

Notation: capital U^α is the 4-velocity, l.c. $\vec{u}$ = ordinary velocity