Sections 10.3, 10.4 – scattering theory for transverse vector fields

Recall that in scalar scattering theory, e.g. for Schrödinger waves or sound waves, the following identities are useful:

$$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{l=0}^{\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y_{lm}^*(\hat{k}) Y_{lm}(\hat{x})$$

The analogous expansion for a transverse vector field is important in E+M, e.g. for an incident circularly-polarized plane wave,

$$\vec{E}^{(\pm)}(\vec{x}) = (\hat{x} \pm i\hat{y})\, e^{ikz} \qquad (a)$$

$$\hat{z}\times(\hat{x}\pm i\hat{y}) = \hat{y} \mp i\hat{x} = \mp i(\hat{x}\pm i\hat{y})$$

$$c\,\vec{B}^{(\pm)}(\vec{x}) = \hat{z}\times\vec{E}(\vec{x}) = \mp i\,\vec{E}^{(\pm)}(\hat{x}) \qquad (b)$$

then the spherical expansion of this transverse vector plane wave can generally be written as an expansion in terms of our general multipole solutions about an arbitrary origin (typically chosen at or near the scatterer "center"):

$$\vec{E}^{(\pm)}(\vec{x}) = \sum_{l,m}\left[a_{lm}^{(\pm)} j_l(kr)\vec{X}_{lm} + \frac{i}{k} b_{lm}^{(\pm)} \nabla\times j_l(kr)\vec{X}_{lm}\right] \qquad (a')$$

$$c\,\vec{B}^{(\pm)}(\vec{x}) = \sum_{l,m}\left[-\frac{i}{k} a_{lm}^{(\pm)} \nabla\times j_l(kr)\vec{X}_{lm} + b_{lm}^{(\pm)} j_l(kr)\vec{X}_{lm}\right] \qquad (b')$$

Now, equating these expansions to the plane-wave fields above, we use "Fourier's Trick" to find the coefficients $a_{\ell m}^{(\pm)}$, $b_{\ell m}^{(\pm)}$.

Three orthogonality relations are key!

(I) $\int [f_{\ell'}(r)\vec{X}_{\ell' m'}]^* \cdot [g_\ell(r)\vec{X}_{\ell m}]\, d\Omega = f_{\ell'} g_\ell\, \delta_{\ell\ell'}\delta_{mm'}$

(II) $\int [f_{\ell'}(r)\vec{X}_{\ell' m'}]^* \cdot [\nabla\times g_\ell(r)\vec{X}_{\ell m}]\, d\Omega = 0$

(III) $\frac{1}{k^2}\int [\nabla\times f_{\ell'}(r)\vec{X}_{\ell' m'}]^* \cdot [\nabla\times g_\ell(r)\vec{X}_{\ell m}]\, d\Omega$

$$= \delta_{\ell\ell'}\delta_{mm'}\left\{ f_\ell^* g_\ell + \frac{1}{k^2 r^2}\frac{\partial}{\partial r}\left[r f_\ell^* \frac{\partial}{\partial r}(r g_\ell)\right]\right\}$$

Now equate (a) to (a'), and apply $\int d\Omega\, \vec{X}_{\ell' m'}$ to both sides of the equation

$$\Rightarrow a_{\ell m}^{(\pm)} j_\ell(kr) = \int \vec{X}_{\ell m}^* \cdot \vec{E}^{(\pm)}(\vec{x})\, d\Omega$$

and doing the same to the equality between (b), (b'):

$$\Rightarrow b_{\ell m}^{(\pm)} j_\ell(kr) = c\int \vec{X}_{\ell m}^* \cdot \vec{B}^{(\pm)}(\vec{x})\, d\Omega$$

but observe that $L_\mp$

$$\vec{X}_{\ell m}^* \cdot \vec{E}^{(\pm)}(\vec{x}) = \left[\frac{(L_x \mp i L_y) Y_{\ell m}(\hat{r})}{\sqrt{\ell(\ell+1)}}\right]^* 4\pi \sum_{\ell m} i^\ell j_\ell(kr) Y_{\ell m}^*(\hat{z}) Y_{\ell m}(\hat{r})$$

now use

Eq. 9.104

$$L_\mp Y_{\ell m} = [(\ell \pm m)(\ell \mp m + 1)]^{1/2}\, Y_{\ell, m\mp 1}$$

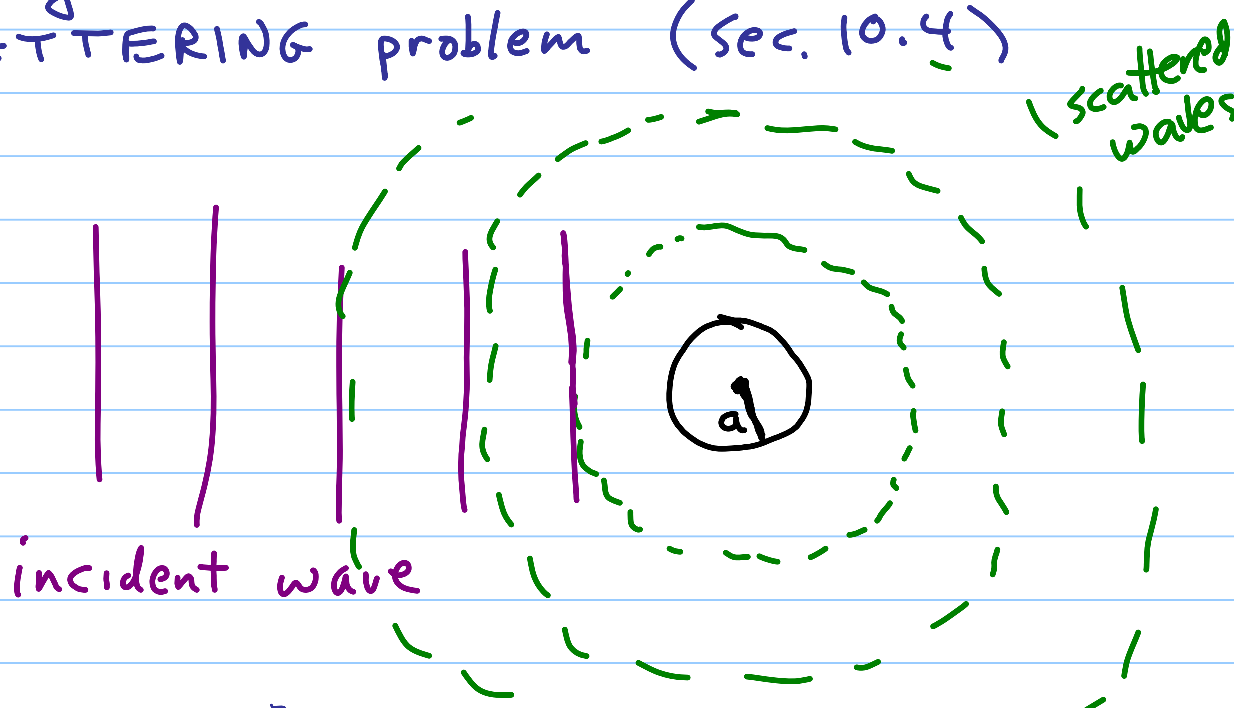
giving $\left\{ \text{From QM, recall that } J_\pm |jm\rangle = \hbar\left[j(j+1) - m(m\pm1)\right]^{1/2} |j, m\pm1\rangle \right.$

$$\boxed{\begin{aligned} a^{(\pm)}_{lm} &= i^l \left[4\pi(2l+1)\right]^{1/2} \delta_{m,\pm1} \\ b^{(\pm)}_{lm} &= \mp i\, a^{(\pm)}_{lm} \end{aligned}}$$

In conclusion, we have the spherical expansion of our vector E+M plane wave:

$$\boxed{\begin{aligned} \vec{E}^{(\pm)}(\vec{x}) &= (\hat{x} \pm i\hat{y})\, e^{ikz} = \sum_{l=1}^{\infty} i^l \left[4\pi(2l+1)\right]^{1/2} \left[j_l \vec{X}_{l,\pm1} \pm \frac{1}{k} \nabla\times j_l \vec{X}_{l,\pm1} \right] \\ c\vec{B}^{(\pm)}(\vec{x}) &= \mp i \vec{E}^{(\pm)} = \sum_{l=1}^{\infty} i^l \left[4\pi(2l+1)\right]^{1/2} \left[-\frac{i}{k} \nabla\times j_l \vec{X}_{l,\pm1} \mp i\, j_l \vec{X}_{l,\pm1} \right] \end{aligned}}$$

Now apply this to treat a SPHERICAL SCATTERING problem (Sec. 10.4)



We expect the $\vec{E}, \vec{B}$ fields should have the structure

$$\vec{E}(\vec{x}) = \vec{E}_{inc} + \vec{E}_{sc}$$

$$\vec{B}(\vec{x}) = \vec{B}_{inc} + \vec{B}_{sc}$$

And we expect that, outside the sphere, the scattered wave radial solutions must be <u>outgoing-wave</u> spherical Hankel solutions $h_\ell^{(1)}(kr)$, i.e.

$$\vec{E}_{sc} = \frac{1}{2}\sum_{\ell=1}^{\infty} i^\ell \left[4\pi(2\ell+1)\right]^{1/2} \left\{ \alpha_\ell^{(\pm)} h_\ell^{(1)} \vec{X}_{\ell,\pm 1} \pm \frac{\beta_\ell^{(\pm)}}{k} \nabla\times h_\ell^{(1)} \vec{X}_{\ell,\pm 1} \right\}$$

and

$$c\vec{B}_{sc} = \frac{1}{2}\sum_{\ell=1}^{\infty} i^\ell \left[4\pi(2\ell+1)\right]^{1/2} \left\{ -\frac{i\alpha_\ell^{(\pm)}}{k} \nabla\times h_\ell \vec{X}_{\ell,\pm 1} \mp i\beta_\ell^{(\pm)} h_\ell^{(1)} \vec{X}_{\ell,\pm 1} \right\}$$

And the coefficients $\alpha_\ell^{(\pm)}$, $\beta_\ell^{(\pm)}$ are found by either:

(i) matching these formulas to appropriate boundary conditions at the sphere surface, e.g. if the sphere is a <u>perfect conductor</u>

or

(ii) matching these to the actual short-range solution <u>inside</u> the sphere

Our two-phase strategy:

A First solve for the complete set of TM, TE modes for the sphere by itself, imposing boundary conditions at the origin but NOT at ∞, initially.

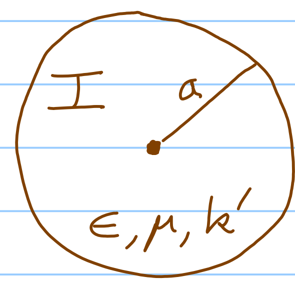
B Next, superpose those solutions to describe physical SCATTERING BCs, i.e. demanding that

$$\vec{E} \xrightarrow[r\to\infty]{} \hat{e}_0 e^{ikz} + \vec{A}_{sc}(\hat{k},\hat{e}_0)\frac{e^{ikr}}{r}$$

Application of this strategy to scattering by a sphere having ϵ, μ, radius a:

II

ϵ_0, μ_0, k



Recall

$$i\epsilon\omega\vec{E} + \nabla\times\vec{H} = 0$$
$$-i\mu\omega\vec{H} + \nabla\times\vec{E} = 0$$
$$k' = \omega\sqrt{\mu\epsilon}$$
$$k = \omega\sqrt{\mu_0\epsilon_0}$$
$$\frac{k'}{k} = \left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)^{1/2}$$

Region I, $r<a$

$$\vec{E}_{\ell m}^{(I,TE)} = j_\ell(k'r)\vec{X}_{\ell m}$$
$$\vec{H}_{\ell m}^{(I,TE)} = \frac{1}{i\mu\omega}\nabla\times j_\ell(k'r)\vec{X}_{\ell m}$$

$$\vec{H}_{\ell m}^{(I,TM)} = j_\ell(k'r)\vec{X}_{\ell m}$$
$$\vec{E}_{\ell m}^{(I,TM)} = \frac{-1}{i\epsilon\omega}\nabla\times j_\ell(k'r)\vec{X}_{\ell m}$$

Region II, $r>a$ We have the identical forms just written (I), except now $\epsilon\to\epsilon_0$, $\mu\to\mu_0$, $k'\to k$ AND we will now need to include terms with BOTH $j_\ell(kr)$ AND $n_\ell(kr)$, in order to match the BCs, which are:

$$\boxed{\begin{array}{l}(i)\ \vec{D}\cdot\hat{r} = \text{continuous}\\ (ii)\ \vec{B}\cdot\hat{r} = \text{continuous}\\ (iii)\ \hat{r}\times\vec{E} = \text{"}\\ (iv)\ \hat{r}\times\vec{H} = \text{"}\end{array}}$$

i.e.

$$\Rightarrow \vec{E}_{\ell m}^{(II, TE)} = \left(a_{\ell m}^{TE}\, j_\ell(kr) + b_{\ell m}^{TE}\, n_\ell(kr)\right)\vec{X}_{\ell m}$$

and continuity of $E_{\parallel}$ demands the B.C.

$$j_\ell(k'a) = a_{\ell m}^{TE}\, j_\ell(ka) + b_{\ell m}^{TE}\, n_\ell(ka)$$

and since the continuity of $D_\perp$ is irrelevant for any TE mode, consider $H_{\parallel}$, (iv):

$$\vec{H}_{\ell m}^{(II, TE)} = \frac{1}{i\mu_0\omega}\nabla\times\left(a_{\ell m}^{TE}\, j_\ell(kr) + b_{\ell m}^{TE}\, n_\ell(kr)\right)\vec{X}_{\ell m}$$

and note the identity:

$$\hat{n}\times\left(\nabla\times Z_\ell\vec{X}_{\ell m}\right) = \frac{k}{2\ell+1}\left(\ell Z_{\ell+1} - (\ell+1)Z_{\ell-1}\right)\vec{X}_{\ell m}$$

provided $Z_\ell(kr)$ = <u>any</u> spherical Bessel function of order ℓ

and to simplify notation, introduce

$$u_\ell(x) \equiv (\ell+1)\, j_{\ell-1}(x) - \ell\, j_{\ell+1}(x)$$

$$v_\ell(x) \equiv (\ell+1)\, n_{\ell-1}(x) - \ell\, n_{\ell+1}(x)$$

So the $H_{\parallel}$ continuity B.C. reads:

$$\frac{k'}{i\mu\omega}u_\ell(k'a) = \frac{k}{i\mu_0\omega}\left(a_{\ell m}^{TE}\, u_\ell(ka) + b_{\ell m}^{TE}\, v_\ell(ka)\right)$$

For notational brevity, define also $\boxed{\gamma \equiv \frac{k' \mu_0}{k \mu}}$

and our 2 equations, 2 unknowns can be solved:

$$\boxed{\begin{aligned} a_\ell^{TE} &= \frac{(ka)^2}{2\ell+1}\left[j_\ell(k'a)\, v_\ell(ka) - \gamma\, u_\ell(k'a)\, n_\ell(ka) \right] \\ b_\ell^{TE} &= \frac{(ka)^2}{2\ell+1}\left[-j_\ell(k'a)\, u_\ell(ka) + \gamma\, u_\ell(k'a)\, j_\ell(ka) \right] \end{aligned}}$$

note identity: $n_\ell(z)\, u_\ell(z) - j_\ell(z)\, v_\ell(z) = -(2\ell+1)/z^2$

Phaseshifts: It is convenient & physically useful to replace a_ℓ^{TE} and b_ℓ^{TE} by an overall amplitude A_ℓ^{TE} and a phaseshift δ_ℓ^{TE}, i.e. set

$$\boxed{\begin{aligned} a_\ell^{TE} &= A_\ell^{TE} \cos \delta_\ell^{TE} \\ b_\ell^{TE} &= -A_\ell^{TE} \sin \delta_\ell^{TE} \end{aligned}} \quad \text{or} \quad \boxed{\tan \delta_\ell^{TE} = \frac{-b_\ell^{TE}}{a_\ell^{TE}}}$$

We do this because it simplifies the asymptotic form of these waves, namely:

$$a_\ell^{TE} j_\ell(kr) + b_\ell^{TE} n_\ell(kr) \xrightarrow[r\to\infty]{} \frac{A_\ell^{TE}}{kr} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell^{TE}\right)$$

where we have used
$$j_\ell \longrightarrow \frac{\sin(x - \frac{\ell\pi}{2})}{x}$$
$$n_\ell \longrightarrow \frac{-\cos(x - \frac{\ell\pi}{2})}{x}$$

and observe that

δ_ℓ^{TE} = TE-mode scattering phaseshift for partial wave ℓ is IMPORTANT!

whereas the amplitude $\cancel{A_\ell^{TE}}$! is largely unimportant, and can be divided out, or set to unity

Of course one can similarly introduce phaseshifts δ_ℓ^{TM} for the TM-modes, whereby the most general fields outside the sphere have the form

$$\vec{E} = \sum_{\ell m} \left(d_{\ell m}^{TE} \vec{E}_{\ell m}^{TE} + d_{\ell m}^{TM} \vec{E}_{\ell m}^{TM} \right)$$

$$\vec{H} = \sum_{\ell m} \left(d_{\ell m}^{TE} \vec{H}_{\ell m}^{TE} + d_{\ell m}^{TM} \vec{H}_{\ell m}^{TM} \right)$$

which have the following asymptotic forms at $r \to \infty$:

$$\vec{E} \xrightarrow[r\to\infty]{} \sum_{\ell m} d_{\ell m}^{TE} \frac{\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell^{TE}\right)}{kr} \vec{X}_{\ell m}$$

$$+ \sum_{\ell m} \left(-\frac{d_{\ell m}^{TM}}{i\epsilon_0 \omega} \right) \nabla \times \frac{\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell^{TM}\right)}{kr} \vec{X}_{\ell m}$$

and

$$\vec{H} \xrightarrow[r\to\infty]{} \sum_{\ell m} d_{\ell m}^{TM} \frac{\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell^{TM}\right)}{kr} \vec{X}_{\ell m}$$

$$+ \sum_{\ell m} \frac{d_{\ell m}^{TE}}{i\mu_0 \omega} \nabla \times \frac{\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell^{TE}\right)}{kr} \vec{X}_{\ell m}$$

and if we now demand that, for $\vec{E}_{inc} = \hat{e}_{\pm} e^{ikz}$,

$$\vec{E} - \vec{E}_{inc} \xrightarrow[r\to\infty]{} \vec{A}_{sc}(\hat{k}, \hat{e}_0) \frac{e^{ikr}}{r},$$ i.e. with no $\underline{\text{incoming waves}}$ $\propto e^{-ikr}/r$

and then plug in the spherical expansion for $\vec{E}_{inc}$, we can immediately apply "Fourier's Trick" to determine the $d_{\ell m}$ above, and find the scattering amplitude

i.e. $$\vec{E}^{(\pm)}_{inc} \xrightarrow[r\to\infty]{} \sum_\ell i^\ell [4\pi(2\ell+1)]^{1/2} \left\{ \frac{\sin(kr - \frac{\ell\pi}{2})}{kr} \vec{X}_{\ell,\pm 1} \pm \frac{1}{k} \nabla \times \frac{\sin(kr - \frac{\ell\pi}{2})}{kr} \vec{X}_{\ell,\pm 1} \right\}$$

So clearly, the application of Fourier's trick tells us that only $m = \pm 1$ contribute, giving:

$$\vec{E} - \vec{E}^{(\pm)}_{inc} \xrightarrow[r\to\infty]{} \sum_\ell \left\{ d^{TE}_{\ell,\pm 1} \frac{\sin(kr - \frac{\ell\pi}{2} + \delta_\ell^{TE})}{kr} - \frac{i^\ell [4\pi(2\ell+1)]^{1/2}}{kr} \sin(kr - \frac{\ell\pi}{2}) \right\} \vec{X}_{\ell,\pm 1}$$
$$+ \sum_\ell \left\{ d^{TM}_{\ell,\pm 1} \left(\text{---- TM terms in } \vec{E} - \vec{E}_{inc} \right) \right.$$

So after writing out as exponentials, using
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
one readily sees that, to kill the incoming TE waves proportional to e^{-ikr}/r, we must have
$$d^{TE}_{\ell,\pm 1} = i^\ell [4\pi(2\ell+1)]^{1/2} e^{i\delta_\ell^{TE}}$$

and plugging this in & simplifying gives

$$\vec{E} - \vec{E}_{inc}^{(\pm)} \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{r} \left(-\frac{i}{k}\right) \sum_{\ell} \left[\pi(2\ell+1)\right]^{1/2} \left(e^{2i\delta_\ell^{TE}} - 1\right)$$

$$+ \left(\text{TM terms in } \vec{E} - \vec{E}_{inc}\right)$$

and similarly

to kill the incoming-waves in the TM terms of $\vec{H}$, one must have $d_{\ell,\pm 1}^{TM} = \pm\left[4\pi(2\ell+1)\right]^{1/2} i^{\ell} e^{i\delta_\ell^{TM}}$,

giving after some algebra:

$$\vec{H} - \vec{H}_{inc} \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{r} \left\{ \mp \sum_{\ell} \left(\frac{\epsilon_0}{\mu_0}\right)^{1/2} \left[\pi(2\ell+1)\right]^{1/2} \frac{e^{2i\delta_\ell^{TM}} - 1}{k} \vec{X}_{\ell,\pm 1} \right\}$$

$$+ \left(\text{TE terms in } \vec{H} - \vec{H}_{inc}\right)$$

Then the analysis in Jackson, p. 474 gives the scattering amplitude, and the integrated scattering cross section, e.g. Eq. 10.61:

$$\sigma_{sc} = \frac{\pi}{2k^2} \sum_{\ell=1}^{\infty} (2\ell+1) \left\{ \left| e^{2i\delta_\ell^{TE}} - 1 \right|^2 + \left| e^{2i\delta_\ell^{TM}} - 1 \right|^2 \right\}$$

so if the δ_ℓ are all real (nondissipative media) we get the simple result

$$\boxed{\sigma_{sc} = \frac{2\pi}{k^2} \sum_{\ell=1}^{\infty} (2\ell+1) \left(\sin^2\delta_\ell^{TE} + \sin^2\delta_\ell^{TM} \right)}$$