Angular distribution of radiated power

$\Rightarrow$ go to large distances, retain the leading order terms only, use the usual short-cut that

$$\boxed{\nabla \longrightarrow ik\hat{n}}$$ in the radiation zone,

Also, for fields radiating outward from a localized source, the usual case, all radial solutions are $h_\ell^{(1)}(kr)$:

$$\Rightarrow \vec{H} = \sum_{\ell m} \left\{ a_E^{(\ell,m)} h_\ell^{(1)}(kr) \vec{X}_{\ell m} - \frac{i}{k} a_M^{(\ell,m)} \nabla \times \left[h_\ell^{(1)}(kr) \vec{X}_{\ell m} \right] \right\}$$

and

$$\vec{E} = \sum_{\ell, m} \left\{ \frac{iZ_0}{k} a_E^{(\ell,m)} \nabla \times h_\ell^{(1)}(kr) \vec{X}_{\ell m} + Z_0 h_\ell^{(1)}(kr) \vec{X}_{\ell m} a_M^{(\ell,m)} \right\}$$

and in the far zone,

$$h_\ell^{(1)}(kr) \xrightarrow[kr \gg \ell]{} (-i)^{\ell+1} \frac{e^{ikr}}{kr}$$

$$\Rightarrow \vec{H} \longrightarrow \frac{e^{ikr}}{kr} \sum_{\ell, m} (-i)^{\ell+1} \left[a_E^{(\ell,m)} \vec{X}_{\ell, m} + a_M^{(\ell m)} \hat{n} \times \vec{X}_{\ell m} \right]$$

$$\vec{E} \longrightarrow Z_0 \vec{H} \times \hat{n}$$

And the time-averaged power radiated per unit solid angle is

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{l,m} (-i)^{l+1}\left[a_E^{(lm)}\vec{X}_{lm}\times\hat{n} + a_M^{(lm)}\vec{X}_{lm}\right]\right|^2}$$

- Notice that interference between the different multipoles is generally complicated, but there is NO interference between the electric and magnetic multipoles, simply because

$$(\hat{n}\times\vec{L})\cdot\vec{L} = 0$$

- Note also that for a single multipole of a definite (l,m), the angular dependence is identical for the E, M - multipoles, because $\hat{n}\cdot\hat{\theta} = 0 = \hat{n}\cdot\hat{\phi}$

$$\Rightarrow \quad |\hat{n}\times\vec{X}_{lm}|^2 = |\vec{X}_{lm}|^2$$

$$\Rightarrow \frac{dP^{(lm)}}{d\Omega} = \frac{Z_0}{2k^2}|a^{(lm)}|^2\,|\vec{X}_{lm}|^2, \text{ or:}$$

$$\frac{dP^{(lm)}}{d\Omega} = \frac{Z_0|a^{(lm)}|^2}{2k^2 l(l+1)}\Big\{\tfrac{1}{2}(l-m)(l+m+1)|Y_{l,m+1}|^2 + m^2|Y_{lm}|^2 + \tfrac{1}{2}(l+m)(l-m+1)|Y_{l,m-1}|^2\Big\}$$

⇒ experimentally, one can in principle deduce the multipolarity of the radiation just from the angular distribution of radiated power.

- But to distinguish electric from magnetic multipoles also requires a measurement of polarization.

- We can also easily determine the TOTAL RADIATED POWER, using

$$\int \vec{X}^{*}_{l'm'} \cdot \vec{X}_{lm} = \delta_{ll'}\, \delta_{mm'},$$

i.e.

$$\boxed{P = \frac{Z_0}{2k^2} \sum_{l,m} \left(|a_E^{(l,m)}|^2 + |a_M^{(l,m)}|^2 \right)}$$

⇒ there is no interference in the total power P.

e.g. — see Table 9.1 on p. 437 for several simple cases.

For many physical systems, such as those in thermodynamic equilibrium, with no direction in space singled out, the source radiates an INCOHERENT sum of all m-components of the l-th multipole

We express this mathematically as

$$\frac{dP^{(\ell)}}{d\Omega} \longrightarrow \frac{Z_0}{2k^2}\sum_m \left|\vec{X}_{\ell,m}\right|^2 \left|a^{(\ell)}\right|^2$$

where $a^{(\ell m)} \longrightarrow a^{(\ell)}$, independent of m

but since $\sum_m \left|\vec{X}_{\ell m}\right|^2 = \frac{2\ell+1}{4\pi}$ = independent of θ, ϕ

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \text{\underline{isotropic} in this case}}$$

Sec. 9.10 Sources of multipole radiation

Let's continue to assume purely harmonic $e^{-i\omega t}$ time-dependence for all sources and fields, and write down Maxwell's equations for $\vec{E}$ and $\vec{H}' \equiv \frac{\vec{B}}{\mu_0}$ including the sources:

$$\nabla \cdot \vec{H}' = 0 \qquad \nabla \times \vec{E} - ikZ_0 \vec{H}' = 0$$

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{H}' + \frac{ik\vec{E}}{Z_0} = \vec{J} + \nabla \times \vec{M}$$

where $\vec{M}$ = magnetization

and we assume that there are specified sources, namely $\rho(\vec{x})e^{-i\omega t}$, $\vec{J}(\vec{x})e^{-i\omega t}$, $\vec{M}(\vec{x})e^{-i\omega t}$

but omitting an explicit POLARIZATION source $\cancel{\vec{P}(\vec{x})e^{-i\omega t}}$

As always, we have the continuity equation too,
$$i\omega\rho = \nabla\cdot\vec{J}$$

Jackson's trick: Set $\boxed{\vec{E}' \equiv \vec{E} + \frac{i}{\omega\epsilon_0}\vec{J}}$,

which allows us to recast these equations for divergenceless "fields", and of course, outside the range of the sources,
$$\vec{E}' \to \vec{E} \quad \text{and} \quad \vec{H}' \to \vec{H}$$

whereas inside the source,
$$\vec{H}' = \frac{\vec{B}}{\mu_0} = \frac{\mu_0(\vec{H}+\vec{M})}{\mu_0} = \boxed{\vec{H}+\vec{M} = \vec{H}'}$$

This gives new primed equations:

$$\boxed{\begin{aligned}\nabla\cdot\vec{H}' &= 0\\ \nabla\cdot\vec{E}' &= 0\end{aligned}} \qquad \boxed{\begin{aligned}\nabla\times\vec{E}' - ikZ_0\vec{H}' &= \frac{i}{\omega\epsilon_0}\nabla\times\vec{J}\\ \nabla\times\vec{H}' + \frac{ik}{Z_0}\vec{E}' &= \nabla\times\vec{M}\end{aligned}}$$

Taking the curl of these equations gives:

$$\boxed{\begin{aligned}(\nabla^2+k^2)\vec{H}' &= -\nabla\times(\vec{J}+\nabla\times\vec{M})\\ (\nabla^2+k^2)\vec{E}' &= -iZ_0k\,\nabla\times\left(\vec{M}+\frac{1}{k^2}\nabla\times\vec{J}\right)\end{aligned}}$$

= inhomogeneous versions of Eqs. 9.108, 9.109, relevant when sources are present

Now, we saw from Eqs. 9.122, 9.123 that it is enough to know $\vec{r}\cdot\vec{E}'$ and $\vec{r}\cdot\vec{H}'$ to determine the multipole expansion coefficients.

$\Rightarrow$ Let's utilize this idea, through identities, namely

$$\nabla^2(\vec{r}\cdot\vec{A}) = \vec{r}\cdot\nabla^2\vec{A} + \cancel{2\nabla\cdot\vec{A}}$$

0 if $\vec{A}$ = divergenceless

and

$$\vec{r}\cdot(\nabla\times\vec{A}) = (\vec{r}\times\nabla)\cdot\vec{A} = i\vec{L}\cdot\vec{A} \quad \left(\text{for any vector field } \vec{A}\right)$$

whereby we have

$$(\nabla^2 + k^2)(\vec{r}\cdot\vec{H}') = -i\vec{L}\cdot(\vec{J} + \nabla\times\vec{M})$$

and the solution having outgoing waves at $r\to\infty$ is

$$\boxed{\vec{r}\cdot\vec{H}' = \frac{i}{4\pi}\int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\vec{L}'\cdot\left[\vec{J}(\vec{x}') + \nabla'\times\vec{M}(\vec{x}')\right]d^3x'}$$

and likewise

$$(\nabla^2 + k^2)(\vec{r}\cdot\vec{E}') = Z_0 k\vec{L}\cdot\left(\vec{M} + \frac{1}{k^2}\nabla\times\vec{J}\right)$$

whose relevant solution is:

$$\boxed{\vec{r}\cdot\vec{E}' = -\frac{Z_0 k}{4\pi}\int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\vec{L}'\cdot\left[\vec{M}(\vec{x}') + \frac{1}{k^2}\nabla'\times\vec{J}(\vec{x}')\right]d^3x'}$$

Now, plugging in the spherical expansion of the Green's function, and setting $r_> = r$, $r_< = r'$,

i.e. $$\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = ik\sum_{lm} h_l^{(1)}(kr)Y_{lm}(\theta,\phi)j_l(kr')Y^*_{lm}(\theta',\phi')$$

and invoking 9.123 again, gives

the multipole coefficients,

$$a_M^{(lm)} = \frac{k}{\sqrt{l(l+1)}} \int Y_{lm}^* \vec{r}\cdot\vec{H}\, d\Omega \Big/ h_l^{(1)}(kr)$$

$$a_E^{(lm)} = \frac{k}{Z_0\sqrt{l(l+1)}} \int Y_{lm}^* \vec{r}\cdot\vec{E}\, d\Omega \Big/ h_l^{(1)}(kr)$$

and since

$$\int d\Omega\, Y_{lm}^* \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = 4\pi i k\, h_l^{(1)}(kr)\, j_l(kr')\, Y_{lm}(\theta',\phi'),$$

these multipole coefficients reduce to:

$$\boxed{\begin{aligned} a_E^{(lm)} &= \frac{ik^3}{\sqrt{l(l+1)}} \int j_l(kr)\, Y_{lm}^*\, \vec{L}\cdot\left(\vec{M} + \frac{1}{k^2}\nabla\times\vec{J}\right) d^3x \\ &\text{and} \\ a_M^{(lm)} &= \frac{-k^2}{\sqrt{l(l+1)}} \int j_l(kr)\, Y_{lm}^*\, \vec{L}\cdot\left(\vec{J} + \nabla\times\vec{M}\right) d^3x \end{aligned}}$$

Which Jackson further simplifies via identities to

$$\boxed{a_E^{(lm)} = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^* \left\{ c\rho \frac{\partial}{\partial r}\left(r j_l(kr)\right) + ik(\vec{r}\cdot\vec{J})\, j_l(kr) - ik\, \nabla\cdot(\vec{r}\times\vec{M})\, j_l(kr) \right\} d^3x}$$

shows how "electric multipoles" are mostly determined by ρ, and "magnetic multipoles" determined mainly by $\vec{J}, \vec{M}$

$$\boxed{a_M^{(lm)} = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^* \left\{ \nabla\cdot(\vec{r}\times\vec{J})\, j_l(kr) + \nabla\cdot\vec{M} \frac{\partial}{\partial r}\left(r j_l(kr)\right) - k^2(\vec{r}\cdot\vec{M})\, j_l(kr) \right\} d^3x}$$

These results are essentially EXACT! Read Jackson pp 441-2

Sec. 9.11 Multipole radiation in atoms and nuclei

Next explore radiative lifetimes of atoms and nuclei, using quantum ideas in the semiclassical approximation. For instance, we can estimate the lifetime τ through the formula

$$P\tau = \hbar\omega = \text{energy radiated}$$

or setting

$$\Gamma = \frac{1}{\tau} = \frac{\text{transition probability}}{\text{unit time}}$$

$$\Rightarrow \quad \Gamma = \frac{P}{\hbar\omega}$$

Now, combining Jackson's Eq. 9.155, 9.169–9.172 we have for an electric multipole:

$$\Gamma_E^{(lm)} = \frac{\omega Z_0 k^{2l}}{2\hbar\left[(2l+1)!!\right]^2}\frac{l+1}{l}\left|Q_{lm} + Q'_{lm}\right|^2$$

and the same expression holds for a magnetic multipole decay rate $\Gamma_M^{(lm)}$ except for the replacement of

$$Q_{lm} + Q'_{lm} \longrightarrow \frac{1}{c}\left(M_{lm} + M'_{lm}\right)$$

order-of-magnitude estimates:

$$Q_{\ell m} \equiv \int \rho\, r^{\ell} Y_{\ell m} d^3x \simeq eR^{\ell}$$

where $R \simeq$ size of the system, whereas

$$Q'_{\ell m} \equiv \frac{-ik}{(\ell+1)c} \int r^{\ell} Y^{*}_{\ell m} \nabla\cdot(\vec{r}\times\vec{M})\, d^3x$$

so assuming that the relevant particles have mass $m \Rightarrow$ these magnetic moments are of order

$$\frac{e\hbar}{m} \Rightarrow |\vec{M}| \simeq \frac{e\hbar}{mR^3}$$

$$\Rightarrow Q'_{\ell m} \simeq \frac{\hbar\omega}{mc^2} eR^{\ell}$$

And since for both atoms & nuclei, $\hbar\omega \ll mc^2$,

$$\Rightarrow \quad Q_{\ell m} \gg Q'_{\ell m}$$

A similar estimate for magnetic multipoles gives $\frac{1}{c}|M_{\ell m} + M'_{\ell m}| \simeq \frac{e\hbar}{mc} R^{\ell-1}$

$$\Rightarrow \frac{\Gamma_M^{(\ell m)}}{\Gamma_E^{(\ell m)}} \simeq \frac{\left(\frac{e\hbar}{mc} R^{\ell-1}\right)^2}{e^2 R^{2\ell}} \simeq \frac{\hbar^2}{m^2c^2} \frac{1}{R^2}$$

For an atomic electron in a shell that sees an "average effective charge" Z_{eff}, the radius is about $R \simeq a_{Bohr}/Z_{eff}$

$$\Rightarrow \frac{\Gamma_M^{(\ell)}}{\Gamma_E^{(\ell)}} \simeq \left(\frac{Z_{eff}}{137}\right)^2$$, since $c \simeq 137$ atomic units

e.g. $Z_{eff} \simeq 1$ for valence shell transitions

$Z_{eff} \simeq Z$ for deep inner-shell X-ray transitions

$\Rightarrow$ For valence shell radiative transitions, electric multipoles dominate over magnetic multipoles of the same order, but for deep inner shell x-ray transitions, they can become comparable

Also note that lower-ℓ multipoles usually dominate, since

$$\frac{\Gamma_E^{(\ell+1)}}{\Gamma_E^{(\ell)}} \simeq k^2 R^2 \simeq \frac{\omega^2}{c^2} R^2 \sim (Z_{eff}\,\alpha)^2 = \left(\frac{Z_{eff}}{137}\right)^2$$

$\Rightarrow$ for atoms, $\Gamma_{M1} \simeq \Gamma_{E2}$, etc.

For NUCLEI, on the other hand, these estimates are rather inaccurate for $l=1$.

In particular, <u>magnetic</u> <u>dipole</u> transitions are far more common than for atoms, and usually just as intense as electric dipole transitions

see Jackson pp 443–444