Sec. 9.5 Multipole expansion for a source in a waveguide

For starters, I give an overview of results from Sec 8.12B. Specifically, we can expand $\vec{E}, \vec{H}$ inside, in terms of separately propagating waves along $\pm\hat{z}$, namely

$$\vec{E} = \vec{E}^{(+)} + \vec{E}^{(-)}$$
$$\vec{H} = \vec{H}^{(+)} + \vec{H}^{(-)}$$

where the individual pieces have a mode expansion:

$$\vec{E}^{(\pm)} = \sum_\lambda A_\lambda^{(\pm)} \vec{E}_\lambda^{(\pm)}$$

$$\vec{H}^{(\pm)} = \sum_\lambda A_\lambda^{(\pm)} H_\lambda^{(\pm)}$$

For any given problem, the expansion coefficients $A_\lambda^{(\pm)}$ can often be found using the mode orthonormality relations, e.g. 8.131 – 134. Note that the TRANSVERSE FIELDS obey an orthonormality relation. I simply state these relations here, without proof.

The full fields have the following form for each mode:

$$\vec{E}_\lambda^{(\pm)}(x,y,z) = \left[\vec{E}_\lambda(x,y) + \vec{E}_{z\lambda}(x,y)\right] e^{\pm i k_\lambda z}$$

and

$$\vec{H}_\lambda^{(\pm)}(x,y,z) = \left[\pm \vec{H}_\lambda(x,y) + \vec{H}_{z\lambda}(x,y)\right] e^{\pm i k_\lambda z}$$

and the transverse component orthonormality relations are simply, normalizing the field modes to be REAL:

$$\int \vec{E}_\lambda \cdot \vec{E}_\mu \, da = \delta_{\lambda\mu}$$

and

$$\int \vec{H}_\lambda \cdot \vec{H}_\mu \, da = \frac{1}{Z_\lambda^2} \delta_{\lambda\mu}$$

and the time-averaged power flow obeys

$$\frac{1}{2} \int (\vec{E}_\lambda \times \vec{H}_\mu) \cdot \hat{z} \, da = \frac{1}{2 Z_\lambda} \delta_{\lambda\mu}$$

The longitudinal components also obey orthonormality relations,

$$\int E_{z\lambda} E_{z\mu} \, da = -\frac{\gamma_\lambda^2}{k_\lambda^2} \delta_{\lambda\mu} \qquad \text{TM modes}$$

$$\int H_{z\lambda} H_{z\mu} \, da = -\frac{\gamma_\lambda^2}{k_\lambda^2 Z_\lambda^2} \delta_{\lambda\mu} \qquad \text{TE modes}$$

The explicit normalized modes for a rectangular wave guide are in Eqs 8.135–8.136

Another aside from Chap. 8, pp 392–4 derives two key formulas needed in Chap. 9:

(1) A localized current source $\vec{J}(\vec{x})$ in the waveguide produces fields whose expansion coefficients are

note '$\mp$' order!

$$A_\lambda^{(\pm)} = -\frac{Z_\lambda}{2}\int_V \vec{J}\cdot\vec{E}_\lambda^{(\mp)}\, d^3x \qquad 8.146$$

AND

(2) A localized aperture in a waveguide generates additional fields in addition to an unperturbed propagating field $\vec{E}$, whose expansion coefficients are:

$$A_\lambda^{(\pm)} = \frac{Z_\lambda}{2}\int_{\text{aperture}} (\vec{E}\times\vec{H}_\lambda^{(\mp)})\cdot\hat{n}\, da \qquad 8.147$$

where $\vec{E}$ = exact tangential $\vec{E}$-field in the aperture,

and

$\hat{n}$ = OUTWARDLY-directed normal unit vector

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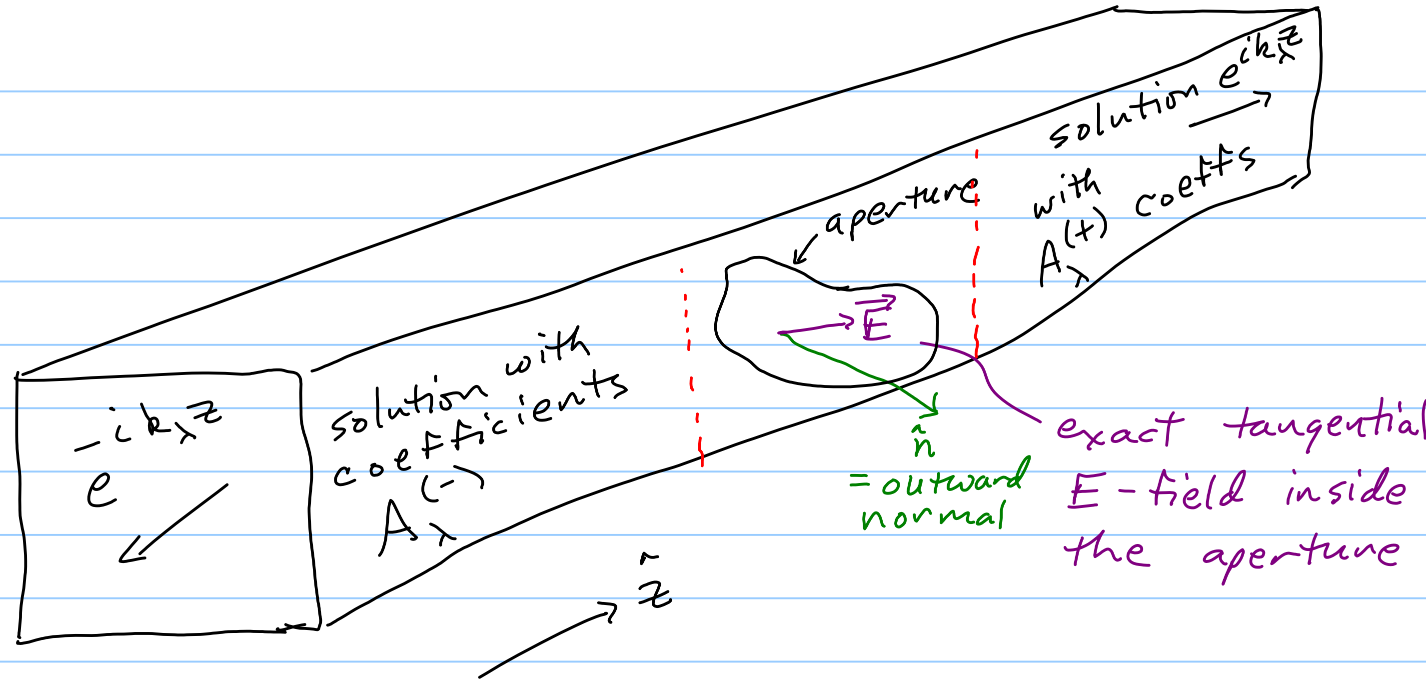

Now return to Sec 9.5, the multipole exp. for a small source inside a waveguide

↖ means small compared to the distance over which the fields vary

This "smallness" suggests that we can expand the $\vec{E}_\lambda^{(\pm)}$ into a Taylor series.

- use repeated index summation convention
- omit the labels λ and $(\pm)$ for simplicity:

$$\Rightarrow \int \vec{J}\cdot\vec{E}\, d^3x = \int \underbrace{J_\alpha(\vec{x})\Big[E_\alpha(0)}_{\text{term 1}} + \overbrace{(\vec{x}\cdot\nabla)E_\alpha(0)}^{\text{term 2}} + \ldots\Big]\, d^3x$$

term $1 = \vec{E}(0)\cdot\int \vec{J}\, d^3x = -i\omega\,\vec{p}\cdot\vec{E}(0)$ where

$\vec{p} = \frac{i}{\omega}\int \vec{J}(\vec{x})\, d^3x$ = source electric dipole moment Eq. 9.14

$$\text{term } 2 = \int d^3x \left\{ J_\alpha x_\beta \frac{\partial E_\alpha(0)}{\partial x_\beta} \right\}$$ ⟸ now analyze this as we did Eq. 9.30

$$= \int d^3x \left\{ \frac{1}{4}\left(J_\alpha x_\beta - J_\beta x_\alpha\right)\left(\frac{\partial E_\alpha(0)}{\partial x_\beta} - \frac{\partial E_\beta(0)}{\partial x_\alpha}\right) + \frac{1}{2}\left(J_\alpha x_\beta + J_\beta x_\alpha\right)\frac{\partial E_\alpha(0)}{\partial x_\beta} \right\}$$

gives a magnetic dipole term

gives an electric quadrupole contribution

or
$$\text{term } 2 = i\omega\, \vec{m}\cdot\vec{B}(0) - \frac{i\omega}{6}\left[\nabla\cdot\left(\overleftrightarrow{Q}\cdot\vec{E}\right)\right]_{\vec{x}\to 0}$$

So finally, this gives

$$A_\lambda^{(\pm)} = \frac{i\omega Z_\lambda}{2}\left\{\vec{p}\cdot\vec{E}_\lambda^{(\mp)}(0) - \vec{m}\cdot\vec{B}_\lambda^{(\mp)}(0) + \frac{1}{6}\left[\nabla\cdot\left(\overleftrightarrow{Q}\cdot\vec{E}_\lambda^{(\mp)}\right)\right]_{\vec{x}\to 0} + \ldots\right\} \quad (9.69)$$

Similarly, if there is a localized <u>aperture</u> in a waveguide carrying EM-fields, we can expand in terms of the <u>effective</u> dipole moments associated with the aperture, (9.71)

$$A_\lambda^{(\pm)} = \frac{i\omega Z_\lambda}{4}\left[\vec{P}_{eff}\cdot\vec{E}_\lambda^{(\mp)}(0) - \vec{m}_{eff}\cdot\vec{B}_\lambda^{(\mp)}(0) + \ldots\right]$$

where the "effective dipole moments" are

$$\vec{P}_{eff} = \epsilon \hat{n} \int \vec{x} \cdot \vec{E}_{tangential} \, da$$

and

$$\vec{m}_{eff} = \frac{2}{i\mu\omega} \int \hat{n} \times \vec{E}_{tangential} \, da$$

and $\vec{E}_{tangential}$ = exact tangential $\vec{E}$-field in the aperture