There is a close parallel between the E1 and M1 expressions:

	E1	M1
$\vec{H}$	$\dfrac{ck^2}{4\pi}(\hat{n}\times\vec{p})\dfrac{e^{ikr}}{r}$	$\dfrac{k^2}{4\pi}(\hat{n}\times\vec{m})\times\hat{n}\,\dfrac{e^{ikr}}{r}$
$\vec{E}$	$\dfrac{cZ_0k^2}{4\pi}(\hat{n}\times\vec{p})\times\hat{n}\,\dfrac{e^{ikr}}{r}$	$-\dfrac{Z_0k^2}{4\pi}\hat{n}\times\vec{m}\,\dfrac{e^{ikr}}{r}$
$\dfrac{dP}{d\Omega}$	$\dfrac{c^2Z_0k^4}{32\pi^2}\lvert(\hat{n}\times\vec{p})\times\hat{n}\rvert^2$	$\dfrac{Z_0k^4}{32\pi^2}\lvert(\hat{n}\times\vec{m})\times\hat{n}\rvert^2$

and there are close parallels with:

$$\vec{p}\longrightarrow\vec{m}$$
$$\vec{E}\longrightarrow\vec{H}$$
$$\vec{H}\longrightarrow\vec{E}$$

(2) Electric quadrupole radiation

$$\vec{A}_{E2}(\vec{x}) = -\frac{\mu_0}{4\pi}ck^2\frac{e^{ikr}}{r}\int\vec{x}\,'(\hat{n}\cdot\vec{x}\,')\rho(\vec{x}\,')\,d^3x' + O(r^{-2})$$

whereby $\vec{H}_{E2} = \frac{1}{\mu_0}ik\hat{n}\times\vec{A}_{E2} + O(r^{-2})$

$$= \frac{-ick^3}{8\pi}\frac{e^{ikr}}{r}\int(\hat{n}\times\vec{x}\,')(\hat{n}\cdot\vec{x}\,')\rho(\vec{x}\,')\,d^3x'$$

constants, can be factored out

Let's express this in terms of the quadrupole moment tensor (e.g., see Eq. 4.9):

$$Q_{\alpha\beta} = \int \left(3x'_\alpha x'_\beta - r^2\delta_{\alpha\beta}\right)\rho(\vec{x}')\,d^3x'$$

$$\Rightarrow \quad \int \rho\,\vec{x}'(\vec{x}'\cdot\hat{n})\,d^3x' = \frac{1}{3}\overleftrightarrow{Q}\cdot\hat{n} + \left(\frac{1}{3}\int\rho r'^2 d^3x'\right)\overleftrightarrow{I}\cdot\hat{n}$$

(where $\overleftrightarrow{I} = 3\times 3$ identity matrix
$\Rightarrow \overleftrightarrow{I}\cdot\hat{n} = \hat{n}$, since $\sum_j \delta_{ij} n_j = n_i$)

and thus

$$\hat{n}\times\int\rho\,\vec{x}'(\vec{x}'\cdot\hat{n})\,d^3x' = \frac{1}{3}\hat{n}\times(\overleftrightarrow{Q}\cdot\hat{n})$$

which allows us to write more simply,

$$\boxed{\vec{H}_{E2}(\vec{x}) = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\,\hat{n}\times\underbrace{(\overleftrightarrow{Q}\cdot\hat{n})}}$$

Jackson calls this vector $\vec{Q}(\hat{n})$

and

$$\boxed{\begin{aligned}\vec{E}_{E2}(\vec{x}) &= -Z_0\,\hat{n}\times\vec{H}_{E2}\\ &= -\frac{iZ_0 ck^3}{24\pi}\frac{e^{ikr}}{r}\left[\hat{n}\times(\overleftrightarrow{Q}\cdot\hat{n})\right]\times\hat{n}\end{aligned}}$$

and

$$\boxed{\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2}k^6\left|\left[\hat{n}\times(\overleftrightarrow{Q}\cdot\hat{n})\right]\times\hat{n}\right|^2}$$

Aside: we can simplify the E2 angular distribution of power, e.g. consider

$$\left[\hat{n}\times(\overleftrightarrow{Q}\cdot\hat{n})\right]\cdot\left[\hat{n}\times(\overleftrightarrow{Q}^{*}\cdot\hat{n})\right]$$

$$= \epsilon_{ijk}\, n_j\, Q_{kl}\, n_l\, \epsilon_{ij'k'}\, n_{j'}\, Q^{*}_{k'l'}\, n_{l'}$$

(using repeated index sum convention)

but

$$\epsilon_{ijk}\,\epsilon_{ij'k'} = \delta_{jj'}\,\delta_{kk'} - \delta_{jk'}\,\delta_{kj'}$$

thus [ ]$\cdot$[ ]

$$= \overbrace{(n_j n_{j'}\delta_{jj'})}^{``\hat{n}\cdot\hat{n}=1"}\, Q_{kl}\, Q^{*}_{k'l'}\, \delta_{kk'}\, n_l\, n_{l'}$$

$$- n_j\, Q^{*}_{k'l'}\, \delta_{jk'}\, n_{j'}\, Q_{kl}\, \delta_{kj'}\, n_l\, n_{l'}$$

$$= n_l\,(Q^{\dagger}Q)_{ll'}\, n_{l'} - (n_j\, Q^{*}_{jl'}\, n_{l'})(n_k\, Q_{kl}\, n_l)$$

or finally

$$\boxed{\left|\hat{n}\times(\overleftrightarrow{Q}\cdot\hat{n})\right|^2 = \hat{n}\cdot Q^{\dagger}Q\cdot\hat{n} - \left|\hat{n}\cdot\overleftrightarrow{Q}\cdot\hat{n}\right|^2}$$

We can next derive the total power radiated, as follows, using 2 identities:

$$\int d\Omega\; n_\alpha n_\beta = \frac{4\pi}{3}\,\delta_{\alpha\beta}$$

$$\int d\Omega\; n_\alpha n_\beta n_\gamma n_\delta = \frac{4\pi}{15}\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)$$

which gives finally

$$\boxed{P = \int \frac{dP}{d\Omega}\, d\Omega = \frac{c^2 Z_0 k^6}{1440\pi} \sum_{\alpha\beta} |Q_{\alpha\beta}|^2}$$

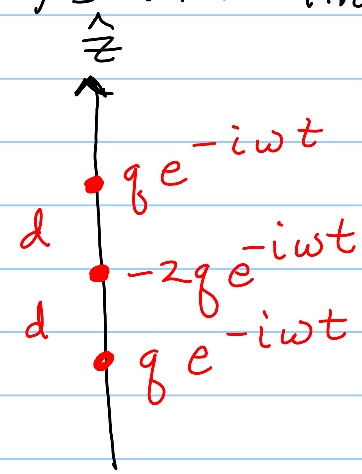
Example – electric quadrupole radiation

A simple quadrupole is 3 charges on a line:

$q_1 = q$ at $(0,0,d)$

$q_2 = -2q$ at $(0,0,0)$

$q_3 = q$ at $(0,0,-d)$



$$\Rightarrow Q_{ij} = \sum_k q_k \left(3 x_i^{(k)} x_j^{(k)} - r^{(k)2} \delta_{ij}\right)$$

$$\Rightarrow Q_{zz} = q(3d^2 - d^2)(2) = 4qd^2 \equiv Q_0$$

$$Q_{xx} = Q_{yy} = q(0 - d^2)(2) = -2qd^2 = -\tfrac{1}{2} Q_0$$

and all off-diagonal terms vanish

$$\Rightarrow \overleftrightarrow{Q} \cdot \hat{n} \equiv \vec{Q}(\hat{n}) = -\tfrac{1}{2} Q_0 (\hat{x} n_x + \hat{y} n_y) + Q_0 \hat{z} n_z$$

$$= Q_0 \left(-\tfrac{1}{2} \hat{x} \sin\theta\cos\phi - \tfrac{1}{2} \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta\right)$$

and $\hat{n} \cdot \overleftrightarrow{Q} \cdot \hat{n} = Q_0 \left(-\tfrac{1}{2}\sin^2\theta + \cos^2\theta\right) = \frac{Q_0}{4}(1 + 3\cos 2\theta)$

$$= \frac{Q_0}{2} P_2(\cos\theta)$$

and $\overleftrightarrow{Q^{+}Q}\cdot\hat{n} = |Q_0|^2\left(\frac{1}{4}\hat{x}\sin\theta\cos\phi + \frac{1}{4}\hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta\right)$

so

$$\hat{n}\cdot\overleftrightarrow{Q^{+}Q}\cdot\hat{n} = Q_0^2\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right) = \frac{Q_0^2}{8}\left(5 + 3\cos 2\theta\right)$$

giving

$$\hat{n}\cdot\overleftrightarrow{Q^{+}Q}\cdot\hat{n} - |\hat{n}\cdot\overleftrightarrow{Q}\cdot\hat{n}|^2 = \frac{9}{4}Q_0^2\sin^2\theta\cos^2\theta = \frac{9}{16}Q_0^2\sin^2 2\theta$$

and putting this together gives

$$\boxed{\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{1152\pi^2}\,\frac{9}{16}\,Q_0^2\sin^2 2\theta}$$

interestingly, this vanishes at $\theta = 0,\ \frac{\pi}{2},$ and π

```
In[2]:= ParametricPlot3D[9/16 Qo^2 Sin[2 t]^2 {Sin[t] Cos[p], Sin[t] Sin[p], Cos[t]} /. Qo → 1,
  {t, 0, Pi}, {p, 0, 2 Pi}, PlotPoints → {40, 40}, PlotRange → All,
  ViewPoint → {10, 10, 10}]
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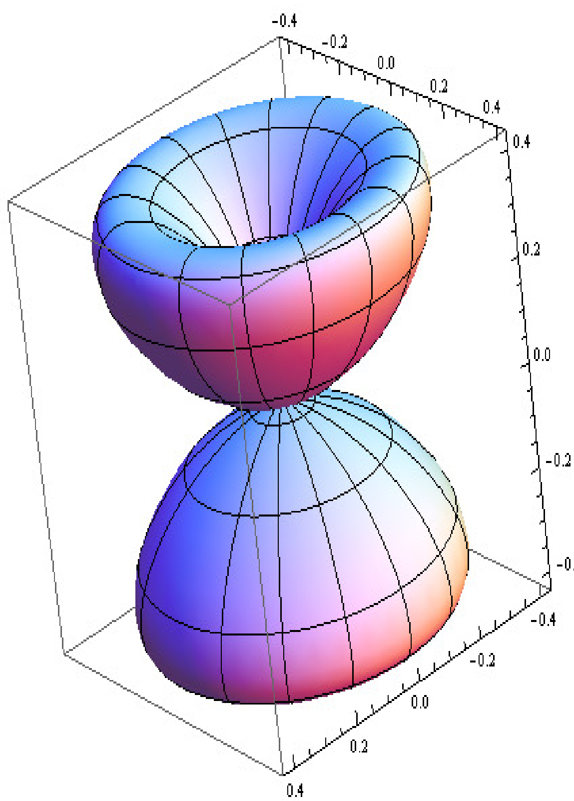
Out[2]=



angular distribution of radiated power from a linear quadrupole aligned with the z-axis.

Note that this is a spherical polar plot, which means that for each θ, ϕ, what is plotted is the value of $\frac{dP}{d\Omega}$ as the "distance" from $(0,0,0)$

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In the long-wavelength limit, the multipole expansion converges rapidly.

i.e. consider $\left|\dfrac{A_{E2}}{A_{E1}}\right| \simeq \dfrac{kQ}{P} \simeq \dfrac{kqd^2}{qd} \simeq kd \ll 1$

and similarly

$\frac{1}{2}\int d^3x'\, \vec{x}' \times \vec{J} \xrightarrow[\vec{J}\to\rho\vec{v}]{} \simeq qvd$

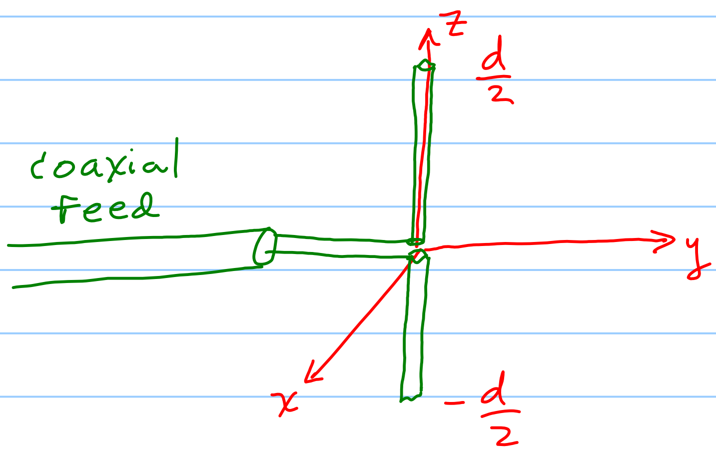
$$\left|\frac{A_{M1}}{A_{E1}}\right| \simeq \frac{m}{pc} \simeq \frac{qvd}{qdc} \simeq \frac{v}{c} \simeq \frac{\omega d}{c}$$

$$= kd \ll 1 \quad \text{also.}$$

Sec. 9.4 Explicit treatment of the center-fed antenna, without a multipole expansion

For this case, the full integral for $\vec{A}(\vec{x})$ can be evaluated in the radiation zone

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, e^{ik\hat{n}\cdot\vec{x}'} \vec{J}(\vec{x}')$$



We will assume that the current is given by

$$\vec{I}(z') = \hat{z} I_0 \sin\left(\frac{kd}{2} - k|z|\right) \times \theta\left(\frac{d}{2} - |z|\right)$$

for an infinitely-thin antenna

As in Chap. 5, we can make the usual replacement, $\int \vec{J} d^3x' (\cdots) \rightarrow \int \vec{I} d\vec{l}' (\cdots)$

Note 1 This procedure is only approximately valid, to pre-specify the currents and then find the radiation fields produced. In reality the fields can act back and influence the currents, but this effect is usually small and we neglect it here

Note 2 This is basically the same problem we treated before, except there we solved it in the limit $\lambda >> d$

Here we write the exact solution in the far zone as

$$\vec{A}(\vec{x}) = \hat{z}\,\frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\,I_0 \int_{-\frac{d}{2}}^{\frac{d}{2}} dz'\, \sin\left(\frac{kd}{2} - k|z'|\right) e^{-ikz'\cos\theta}$$

(using $k\hat{n}\cdot\vec{x}' = kz'\cos\theta$)

or

$$\vec{A}(\vec{x}) = \frac{\hat{z}\mu_0 I_0}{4\pi}\,\frac{e^{ikr}}{r}\,\frac{1}{2i}\left\{\int_0^{d/2} e^{ik\frac{d}{2} - ikz'(1+\cos\theta)}\,dz' - \int_0^{\frac{d}{2}} e^{-ikd/2 + ikz'(1-\cos\theta)}\,dz' + \int_{-\frac{d}{2}}^{0} e^{ik\frac{d}{2} + ikz'(1-\cos\theta)}\,dz' - \int_{-\frac{d}{2}}^{0} e^{-ik\frac{d}{2} - ikz'(1+\cos\theta)}\,dz'\right\}$$

These integrals are all simple, giving:

$$\vec{A}(\vec{x}) = \hat{z}\,\frac{\mu_0}{4\pi}\,I_0\,\frac{e^{ikr}}{r}\,\frac{2}{k}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin^2\theta}\right]$$

and the fields derived from:

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A} \xrightarrow[\text{rad. zone}]{} \frac{1}{\mu_0} ik\hat{n} \times \vec{A}$$

$$\vec{E} = Z_0 \vec{H} \times \hat{n}$$

and the angular distribution of radiated power is

$$\frac{dP}{d\Omega} = \lim_{r\to\infty} \frac{1}{2} \mathrm{Re}\left[r^2 \hat{n}\cdot \vec{E} \times \vec{H}^*\right]$$

The vector structure is:

$$\hat{n}\cdot\left[-\hat{n} \times (\hat{n}\times\hat{z})\right] \times (\hat{n}\times\hat{z})^* = |\hat{n}\times(\hat{n}\times\hat{z})|^2$$

$$= \sin^2\theta$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{Z_0 I_0^2}{8\pi^2}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta}\right]^2}$$

This contains ALL multipole contributions

$\Rightarrow$ An important special case is the long-wavelength limit $kd = \frac{2\pi d}{\lambda} \ll 1$

$$\Rightarrow \frac{dP}{d\Omega} \xrightarrow[kd\ll 1]{} \frac{Z_0 I_0^2}{572\pi^2}(kd)^4 \sin^2\theta,$$ which agrees with 9.28,

although we must observe that comparing 9.25 and 9.183 requires the currents to be related by

$$I_0^{\text{linear model}} = \frac{kd}{2} I_0(\text{sinusoidal model})$$

Note also that the ang. distrib. pattern from a half-wave antenna is <u>more</u> <u>directional</u> than the dipole pattern.