$$\vec{E} = \frac{iZ_0}{k} \nabla \times \vec{H} \longrightarrow -Z_0 \hat{n} \times \vec{H} + O\left(\frac{1}{r^2}\right)$$

$$\Rightarrow \boxed{\vec{E} \xrightarrow[r\to\infty]{} \frac{c Z_0 k^2}{4\pi} (\hat{n} \times \vec{p}) \times \hat{n} \frac{e^{ikr}}{r}}$$

(note that $c Z_0 = \frac{1}{\epsilon_0}$)

And a particularly important quantity is the time-averaged radiated power:

$$\langle \vec{S} \rangle = \frac{1}{2} \mathrm{Re} (\vec{E} \times \vec{H}^*) \propto \frac{1}{r^2} \text{ at } r \to \infty$$

And the <u>ANGULAR DISTRIBUTION</u> of power radiated is defined as the quantity

$$\frac{dP}{d\Omega} \equiv r^2 \hat{n} \cdot \langle \vec{S} \rangle = \frac{\mathrm{Re}}{2} \frac{c^2 k^4 Z_0}{16\pi^2} \underbrace{[(\hat{n} \times \hat{p}) \times \hat{n}] \times (\hat{n} \times \vec{p}^*)}_{} \cdot \hat{n}$$

$$= -(\hat{n} \times \vec{p}^*) \times [(\hat{n} \times \vec{p}) \times \hat{n}]$$

$$= -(\hat{n} \times \vec{p}) \cancel{(\hat{n} \times \vec{p}^*) \cdot \hat{n}}_{\to 0} + \hat{n} |\hat{n} \times \vec{p}|^2 \quad \text{(BAC-CAB rule)}$$

$$= \hat{n} |\hat{n} \times (\hat{n} \times \vec{p})|^2$$

and as a special case when $\hat{n} \cdot \vec{p} = \cos\theta = \text{real}$, then $|\hat{n} \times \vec{p}|^2 = \sin^2\theta$

and finally

$$\boxed{\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2} |\vec{p}|^2 \sin^2\theta}$$

this is an important formula, but remember, it is valid ONLY if $\vec{p}$ = real electric dipole moment vector, to within an OVERALL phase factor

i.e. this is not applicable to a ROTATING dipole

Another key quantity is the <u>TOTAL</u> <u>POWER</u> <u>RADIATED</u>, namely

$$P = \int \frac{dP}{d\Omega} d\Omega = 2\pi \int_0^\pi \sin\theta \, d\theta \, \frac{dP}{d\Omega}$$

$\Rightarrow$ need $\int_0^\pi \sin^3\theta \, d\theta = \frac{4}{3}$

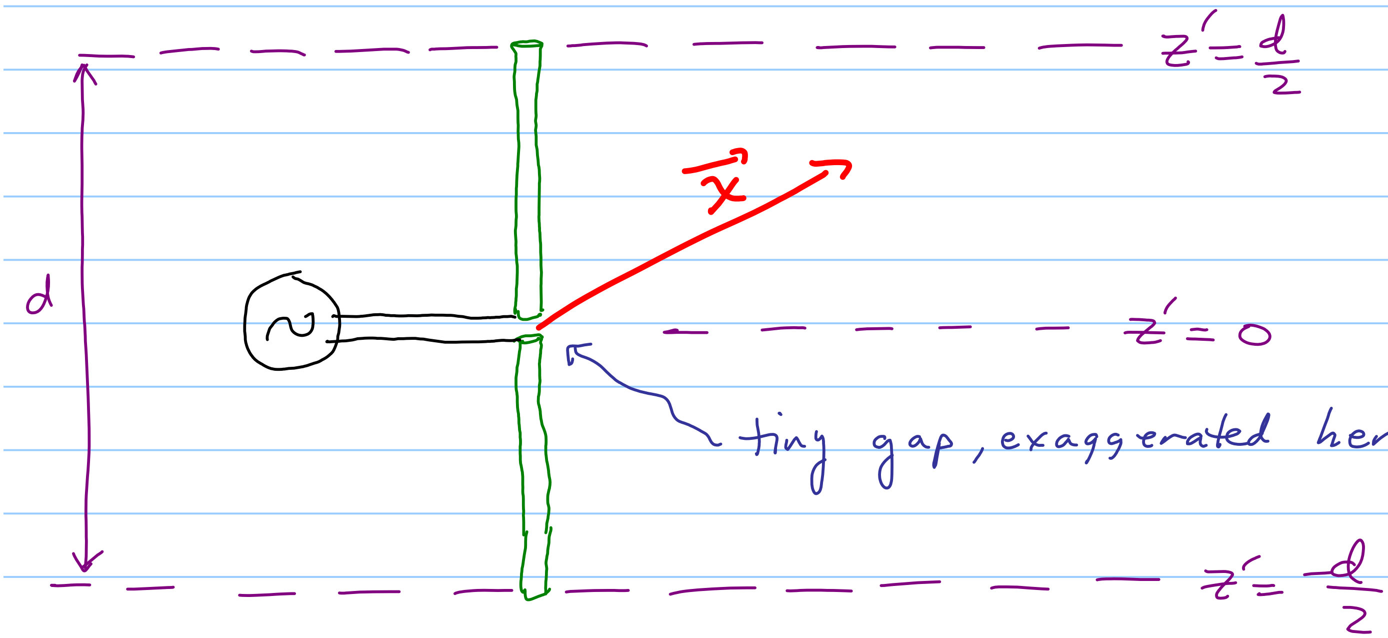
So finally

$$\boxed{P = \frac{Z_0 c^2 k^4}{12\pi} |\vec{p}|^2}$$

again for real $\vec{p}$

Example of an electric dipole radiator

– the linear, center-fed antenna with $d \ll \lambda$



Assumptions we will make:

(a) The sources and fields have a harmonic time-dependence $e^{-i\omega t}$

(b) The instantaneous current I drops linearly to zero at the ends of the antenna, and is in the same direction in each half

i.e. $$I(z)\,e^{-i\omega t} = I_0\left(1 - \frac{2|z|}{d}\right)e^{-i\omega t}$$

or the spatial current density is

$$\vec{J}(x,y,z) = \delta(x)\,\delta(y)\,I_0\left(1 - \frac{2|z|}{d}\right)\hat{z}$$
for $|z| \le d/2$

From the identity derived above (bottom of notes p. 95) we could compute the electric dipole moment using either of the following forms:

$$\vec{p} = \int \vec{x}'\rho(\vec{x}')\,d^3x' = \frac{-1}{i\omega}\int d^3x'\,\vec{J}(\vec{x}')$$

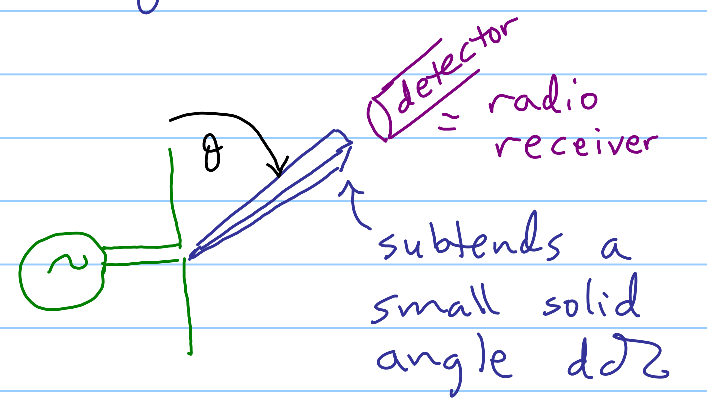
where $\rho = \frac{1}{i\omega}\nabla\cdot\vec{J} = \frac{I_0}{i\omega}\delta(x)\delta(y)\frac{\partial}{\partial z}\left(1 - \frac{2|z|}{d}\right)$

or we could write this as a charge per unit length as

$$\tilde{\rho}(z) = \int dx\,dy\,\rho = \pm\frac{2i}{\omega d}I_0 \quad \text{at} \begin{cases} 0 < z < \frac{d}{2} \\ \text{or} \\ -\frac{d}{2} < z < 0 \end{cases} \text{resp.}$$

$$\Rightarrow \vec{p} = \hat{z}\int_{-\frac{d}{2}}^{0} dz'\,z'\left(-\frac{2iI_0}{\omega d}\right) + \hat{z}\int_{0}^{\frac{d}{2}} dz'\,z'\left(\frac{2iI_0}{\omega d}\right)$$

or $\vec{p} = \hat{z}\,\frac{iI_0 d}{2\omega}$



Since the dipole moment $\vec{p}$ is real to within an overall phase factor (i), we can simply plug this in to determine the angular distribution of radiated power, namely

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4\sin^2\theta\,|\vec{p}|^2 = \frac{\cancel{k^2c^2}\,I_0^2 Z_0\,(kd)^2}{(32\pi^2)(4\cancel{\omega^2})}\sin^2\theta$$

or $$\boxed{\frac{dP}{d\Omega} = \frac{I_0^2 Z_0}{128\pi^2}(kd)^2 \sin^2\theta}$$

and the total average power radiated is

$$P = \frac{Z_0 c^2 k^4}{12\pi} |\vec{p}|^2 \quad \text{or} \quad \boxed{P = \frac{Z_0 I_0^2 (kd)^2}{48\pi}}$$

note: $P \propto \omega^2$ or $P \propto \left(\frac{d}{\lambda}\right)^2$ for $\lambda >> d$

If we view the antenna as an AC-circuit component with ohmic losses, i.e. with an effective resistance, we can equate the radiated power P to

$$P = \frac{1}{2} |I_0|^2 R_{rad}$$

where we denote R_{rad} = radiation resistance

Then for this example,

$$R_{rad} = \frac{Z_0 (kd)^2}{24\pi} \simeq 5(kd)^2 \text{ ohms}$$

(assuming the antenna is a perfect conductor)

Magnetic Dipole and Electric Quadrupole Radiation

Both contributions are included in the $l=1$ term of Eq. 9.11:

$$\vec{A}(\vec{x}) = \mu_0 ik \sum_{l,m} h_l^{(1)}(kr)\, Y_{lm}(\hat{x}) \int \vec{J}(\vec{x}')\, j_l(kr')\, Y^*_{lm}(\hat{x}')\, d^3x'$$

keeping $l=1$ only

$$\mu_0 ik\, h_1^{(1)}(kr) \sum_{m=-1}^{1} Y_{1m}(\hat{n}) \int \vec{J}(\vec{x}')\, j_1(kr')\, Y^*_{1m}(\hat{x}')\, d^3x'$$

Any standard reference can be consulted to see that:

$$h_1^{(1)}(kr) = -\frac{e^{ikr}}{kr}\left(1 + \frac{i}{kr}\right)$$

and

$$j_1(kr) \xrightarrow[kr \ll 1]{} \frac{kr}{3} + O(kr)^3$$

and the spherical harmonic addition implies that

$$\sum_m Y_{lm}(\hat{n})\, Y^*_{lm}(\hat{x}') = \frac{2l+1}{4\pi} P_l(\hat{n}\cdot\hat{x}') \xrightarrow[l\to 1]{} \frac{3}{4\pi}\, \hat{n}\cdot\hat{x}'$$

$$\Rightarrow \vec{A}(\vec{x}) = -\frac{3\mu_0}{4\pi} ik \frac{e^{ikr}}{kr}\left(1 + \frac{i}{kr}\right) \int d^3x'\, \vec{J}(\vec{x}')\, \frac{kr'}{3}\, \hat{n}\cdot\hat{x}'$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right) \int d^3x'\, \vec{J}(\vec{x}')\, \hat{n}\cdot\vec{x}'$$

We will see that this expression includes both the magnetic dipole AND electric quadrupole bits.

To separate them requires a bit of study. The CONCEPT of this derivation is the same as before, namely to use $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$ to replace $\vec{J}$ by ρ.

However, recall that any vector field such as $\vec{J}$ can be written as the SUM of a longitudinal ($\vec{J}_\ell$) and a transverse ($\vec{J}_t$) part, such that $\nabla \times \vec{J}_\ell = 0$ and $\nabla \cdot \vec{J}_t = 0$ (see pp. 241-2)

Our previous logic led to terms like $\nabla \cdot \vec{J} = i\omega\rho$ thus works ONLY for the longitudinal $\vec{J}_\ell$ and NOT for $\vec{J}_t$.

$\Rightarrow$ We must treat $\vec{J}_t$ explicitly (this will turn out to be the magnetic dipole part)

WARNING This development may seem messy and ad hoc. It is more elegant + systematic to do this using vector spherical harmonics, Sec. 9.7.

But we will tackle it here using identities, e.g.:

$$\hat{n} \times (\vec{x}' \times \vec{J}) = \vec{x}'(\hat{n} \cdot \vec{J}) - \vec{J}(\hat{n} \cdot \vec{x}')$$

or
$$(\hat{n} \cdot \vec{x}')\vec{J} = \vec{x}'(\hat{n} \cdot \vec{J}) - \hat{n} \times (\vec{x}' \times \vec{J})$$

add $(\hat{n} \cdot \vec{x}')\vec{J}$ to both sides

$$\Rightarrow 2(\hat{n} \cdot \vec{x}')\vec{J} = \vec{x}'(\hat{n} \cdot \vec{J}) + (\hat{n} \cdot \vec{x}')\vec{J} - \hat{n} \times (\vec{x}' \times \vec{J})$$

$$\Rightarrow (\hat{n}\cdot\vec{x}\,')\vec{J} = \frac{1}{2}\left[(\hat{n}\cdot\vec{x}\,')\vec{J} + \vec{x}\,'(\hat{n}\cdot\vec{J})\right]$$
$$- \frac{1}{2}\hat{n}\times(\vec{x}\,'\times\vec{J})$$

observe: this term is symmetric under interchange of $\vec{x}\,'$ and $\vec{J}$

this term is antisymmetric under $\vec{x}\,' \longleftrightarrow \vec{J}$

Initially, just consider these 1^{st} two symmetric terms

$\Rightarrow$ ELECTRIC QUADRUPOLE

Recall the identity we proved,

$$\int d^3x'\, J_i(\vec{x}\,') = -\int d^3x'\, x_i'\, \nabla'\cdot\vec{J}(\vec{x}\,')$$

but replace $J_i \longrightarrow (\hat{n}\cdot\vec{x}\,')J_i$

$$\Rightarrow \int d^3x'\,(\hat{n}\cdot\vec{x}\,')J_i(\vec{x}\,') = -\int d^3x'\, x_i'\, \nabla'\cdot\left[(\hat{n}\cdot\vec{x}\,')\vec{J}\right]$$
$$= -\int d^3x'\, x_i'\left\{\nabla'(\hat{n}\cdot\vec{x}\,')\cdot\vec{J} + (\hat{n}\cdot\vec{x}\,')\nabla'\cdot\vec{J}\right\}$$

and

$$\underbrace{\left[\nabla'(\hat{n}\cdot\vec{x}\,')\right]}\cdot\vec{J} = \hat{n}\cdot\vec{J}$$

$$= (\hat{n}\cdot\nabla')\vec{x}\,' = \hat{x}'(\hat{n}\cdot\hat{x}') + \hat{y}'(\hat{n}\cdot\hat{y}') + \hat{z}'(\hat{n}\cdot\hat{z}')$$
$$= \hat{n}$$

$$\Rightarrow \int d^3x'\,(\hat{n}\cdot\vec{x}\,')\vec{J}(\vec{x}\,')$$
$$= -\int d^3x'\,\vec{x}\,'\left[\hat{n}\cdot\vec{J} + (\hat{n}\cdot\vec{x}\,')(\nabla'\cdot\vec{J})\right]$$

iwp

or finally we obtain

$$\boxed{\int d^3x' \left[(\hat{n}\cdot\vec{x}')\vec{J} + \vec{x}'(\hat{n}\cdot\vec{J}) \right] = -i\omega \int d^3x' \, \vec{x}'(\hat{n}\cdot\vec{x}')\rho(\vec{x}')}$$

Now, plugging these identities into our $l=1$ expression for $\vec{A}$ gives

$$\vec{A}_{l=1}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\left(1+\frac{i}{kr}\right)$$

$$\times \left\{ -\frac{i\omega}{2}\int d^3x' \, \vec{x}'(\hat{n}\cdot\vec{x}')\rho(\vec{x}') - \hat{n}\times\int \frac{\vec{x}'\times\vec{J}}{2} d^3x' \right\}$$

components of electric quadrupole

magnetic dipole

Now consider these two contributions separately:

(1) MAGNETIC DIPOLE

$\vec{m} = \frac{1}{2}\int \vec{x}'\times\vec{J} \, d^3x'$ as in Eq. 5.54

$$\vec{A}_{M1} = \frac{\mu_0}{4\pi} ik \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\hat{n}\times\vec{m}$$

So in the radiation zone, $\left|\frac{i}{kr}\right| \ll 1$, we obtain

$$\boxed{\begin{aligned} \vec{H}_{M1} &= \frac{1}{\mu_0}\nabla\times\vec{A}_{M1} = \frac{k^2}{4\pi}(\hat{n}\times\vec{m})\times\hat{n}\,\frac{e^{ikr}}{r} \\ &\text{and} \\ \vec{E}_{M1} &= Z_0\vec{H}_{M1}\times\hat{n} = -\frac{Z_0 k^2}{4\pi}\hat{n}\times\vec{m}\,\frac{e^{ikr}}{r} \end{aligned}}$$