Chapter 9 Radiating Systems

Let's consider monochromatic sources and fields,

i.e.
$$\boxed{\begin{aligned}\rho(\vec{x},t) &= \rho(\vec{x})\, e^{-i\omega t} \\ \vec{J}(\vec{x},t) &= \vec{J}(\vec{x})\, e^{-i\omega t}\end{aligned}}$$
only real parts are relevant

Then in the Lorentz gauge, the potentials obey:

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

Then formally, these equations can be solved using a Green's function, e.g.

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \int dt'\, G(\vec{x},t;\vec{x}',t')\vec{J}(\vec{x}',t')$$

and if there are no boundary surfaces, we normally will want the RETARDED Green function,

$$G^{(+)}(\vec{x},t;\vec{x}',t') = \frac{\delta[t' - (t - \frac{R}{c})]}{R}$$

where $R = |\vec{x} - \vec{x}'|$

and t' (source) is earlier than t by $\frac{R}{c}$

$$\Rightarrow \vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi} \iint d^3x'\, dt' \; \frac{\delta[t'-(t-\frac{R}{c})]}{R} \vec{J}(\vec{x}',t')$$

or

$$\boxed{\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}}$$

In words: $\vec{A}(\vec{x},t)$ depends on $\vec{J}$ at $\vec{x}'$, but at a time earlier than t by $\frac{|\vec{x}-\vec{x}'|}{c}$, i.e. the light propagation time.

This can be evaluated further if we consider a monochromatic source,

i.e. $\vec{J}(\vec{x}',t') = \vec{J}(\vec{x}')\, e^{-i\omega t'}$

$$\Rightarrow \boxed{\vec{A}(\vec{x},t) = e^{-i\omega t} \underbrace{\left(\frac{\mu_0}{4\pi} \int d^3x' \, \vec{J}(\vec{x}') \frac{e^{\frac{i\omega}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \right)}_{\text{"} \vec{A}(\vec{x}) \text{"}}}$$

and similarly $\Phi(\vec{x},t) = \Phi(\vec{x})\, e^{-i\omega t}$ where

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{e^{ikR}}{R} \rho(\vec{x}')}$$

But in fact, in the context of radiated fields, <u>we will only need $\vec{A}$</u>, since $\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A}$

and $\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$, so in regions where $\vec{J} = 0$,

$$-\frac{i\omega}{c^2}\vec{E} = \mu_0 \nabla \times \vec{H}$$

or $$\boxed{\vec{E}(\vec{x}) = \frac{iZ_0}{k}\nabla \times \vec{H}}$$

holds for harmonic time-dependence away from sources, where $\vec{J}(\vec{x}) \to 0$

where $Z_0 \equiv \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}$ = impedance of free space.

The only Catch (22) here is that:

the integral for $\vec{A}(\vec{x})$ is difficult to evaluate

$\Rightarrow$ One systematic way to approach this is to use the partial wave expansion (9.98), namely

$$G = \frac{e^{ikR}}{4\pi R} = ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\hat{x}')Y_{lm}(\hat{x})$$

But we can begin by treating some limits by more elementary means

First note the three relevant distance scales:

$d \simeq$ physical size/extent of sources

$\lambda = \frac{2\pi c}{\omega}$ = light wavelength

r = observation distance

There are 3 different limits of interest,

Near Zone (static): $d \ll r \ll \lambda$

Intermediate Zone (induction): $d \ll r \simeq \lambda$

Far Zone (radiation): $d \ll \lambda \ll r$

Consider the NEAR ZONE: $d \ll r \ll \lambda$

$$\Rightarrow e^{ik|\vec{x}-\vec{x}'|} \simeq 1$$

$$\Rightarrow \vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

just as in magnetostatics! (Eq. 5.32)

$\Rightarrow \vec{A}(\vec{x})$ is the same as for a static, steady current source, except for an overall $e^{-i\omega t}$ time-dependence.

When helpful, we can perform a multipole expansion.

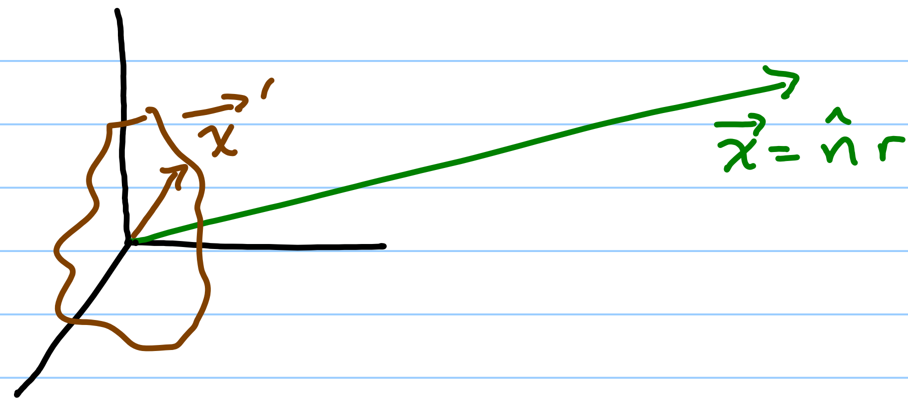
Next, the FAR ZONE, $d \ll \lambda \ll r$

Observe that in all 3 zones, $r \gg d$,

$$\Rightarrow |\vec{x}-\vec{x}'| = |\vec{x}| \left(1 + \frac{x'^2}{x^2} - 2\frac{\vec{x}\cdot\vec{x}'}{x^2}\right)^{1/2}$$ 047

or $\quad |\vec{x} - \vec{x}'| = r\left(1 - \frac{\vec{x}}{r}\cdot\frac{\vec{x}'}{r} + \ldots\right.$

or $\quad |\vec{x} - \vec{x}'| \simeq r - \hat{n}\cdot\vec{x}'$



Then far from the source,

$$\vec{A}(\vec{x}) \xrightarrow[r \gg d]{} \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')\, e^{-ik\hat{n}\cdot\vec{x}'}\, d^3x'$$

and we see that $\vec{A}(\vec{x})$ far from the source is an <u>OUTGOING</u> spherical wave, decaying like $\frac{1}{r}$.

The angle dependence comes from $\hat{n} = (\theta, \phi)$, and we can write asymptotically,

$$\vec{A}(\vec{x}) = \vec{F}^{(k)}(\theta,\phi)\,\frac{e^{ikr}}{r}$$

From this, we obtain the fields:

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A} \xrightarrow[kr \gg 1]{} \hat{\theta} \frac{e^{ikr}}{r} (-ik) F_\phi^{(k)}(\theta, \phi)$$

$$+ \hat{\phi} \frac{e^{ikr}}{r} (ik) F_\theta^{(k)}(\theta, \phi) + O(r^{-2})$$

and

$$\vec{E} = \frac{iZ_0}{k} \nabla \times \vec{H} \xrightarrow[kr \gg 1]{} \frac{iZ_0 k}{\mu_0} \frac{e^{ikr}}{r} \left(\hat{\theta} F_\theta^{(k)}(\theta, \phi) + \hat{\phi} F_\phi^{(k)}(\theta, \phi) \right) + O(r^{-2})$$

<u>NOTES</u> These fields obey:

$$\vec{H} \cdot \hat{n} = 0$$

$$\vec{E} \cdot \hat{n} = 0$$

$$\vec{H} \cdot \vec{E} = 0$$

$\Rightarrow$ The electric and magnetic fields are <u>transverse</u> to the propagation axis, and to each other, as expected.

Moreover, they fall off like $\frac{1}{r}$.

If $kx' \simeq kd \ll 1 \Rightarrow e^X = 1 + X + \frac{X^2}{2!} + \dots$

and

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{l=0}^{\infty} \frac{1}{l!} \int d^3x' \vec{J}(\vec{x}')(-ik\hat{n}\cdot\vec{x}')^l$$

successive terms are of order $(kd)^l$.

Typically, the dominant radiated term is the lowest-ℓ term that is nonzero

Explicit treatment of the lowest few multipoles:

ELECTRIC MONOPOLE

Consider $\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}', t-\frac{R}{c})}{R}$

$$\xrightarrow[R\to r\to\infty]{} \frac{1}{4\pi\epsilon_0 r}\int d^3x' \rho(\vec{x}', t-\tfrac{R}{c})$$

$$= \frac{Q(t-\frac{R}{c})}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r},$$

since the total charge is t-independent

$\Rightarrow$ We conclude that:

- a charge monopole produces static fields only
- Fields with an $e^{-i\omega t}$ time-dependence NEVER have a monopole term
- the $\frac{1}{r}$ potential generates a $\frac{1}{r^2}$ E-field, not a $\frac{1}{r}$ field expected for radiation.

ELECTRIC DIPOLE RADIATION

The leading term in our expansion of the vector potential is the $l=0$ term:

$$\vec{A}(\vec{x}) \xrightarrow[l=0]{} \vec{A}_{dipole}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \, \vec{J}(\vec{x}')$$

At this point we derive a useful identity:

Consider
$$\int d^3x' \, J_i(\vec{x}') = \int d^3x' \, (\nabla' x_i') \cdot \vec{J}(\vec{x}')$$

and using $\nabla' \cdot (\psi \vec{a}) = \nabla' \psi \cdot \vec{a} + \psi \nabla' \cdot \vec{a}$

with $\psi = x_i'$, $\vec{a} = \vec{J}(\vec{x}')$

$$\Rightarrow \int d^3x' \, J_i(\vec{x}') = \int d^3x' \left[\nabla' \cdot (x_i' \vec{J}(\vec{x}')) - x_i' \nabla' \cdot \vec{J}(\vec{x}') \right]$$

use divergence theorem

$= \oint da' \, x_i' \, \hat{n}' \cdot \vec{J}(\vec{x}')$

$= 0$ normally, though not necessarily, e.g. for an infinite waveguide

$\nabla' \cdot \vec{J}(\vec{x}') = -\frac{\partial \rho}{\partial t} = +i\omega\rho$ by the charge continuity equation

So we have the following identity for a localized, monochromatic source,

$$\boxed{\int d^3x' \, \vec{J}(\vec{x}') = -i\omega \int \vec{x}' \rho(\vec{x}') \, d^3x'}$$

Hence we can rewrite the vector potential for electric dipole radiation as

$$\vec{A}_{dipole}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\frac{e^{ikr}}{r}\underbrace{\int d^3x'\,\vec{x}'\rho(\vec{x}')}$$

" $\vec{p}$, the electric dipole moment!

and we have the key result:

$$\boxed{\vec{A}_{dipole}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\vec{p}\,\frac{e^{ikr}}{r}}$$

Now we calculate the fields, using

$$\nabla\times(\psi\vec{p}) = \nabla\psi\times\vec{p} + \psi\,\cancel{\nabla\times\vec{p}} \rightarrow 0$$

since $\vec{p}$ = constant

$$\Rightarrow \vec{H} = \frac{1}{\mu_0}\nabla\times\vec{A} = -\frac{ikc}{4\pi}\left(\nabla\frac{e^{ikr}}{r}\right)\times\vec{p}$$

where $\nabla\frac{e^{ikr}}{r} = \hat{n}\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}$, $\hat{n} = \hat{r}$

$$\Rightarrow \boxed{\vec{H} \xrightarrow[r\to\infty]{} \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}} + O\left(\frac{1}{r^2}\right)$$

Notice that the dominant term at $r \to \infty$ came here from $\nabla e^{ikr} \rightarrow ik\hat{r}e^{ikr}$ and we will use to simplify subsequent derivations

e.g. the E-field in the far zone is