Generalization – walls of finite but high conductivity

We will see that for σ large but $< \infty$, now

$$k_\lambda = \underbrace{k_\lambda^{(0)}}_{\text{wavenumber for perfectly conducting mode}} + \underbrace{\alpha_\lambda}_{\text{small shift in real part, mainly important when } k_\lambda^{(0)} \to 0} + i\underbrace{\beta_\lambda}_{\text{acquires a small imaginary part } i\beta_\lambda}$$

Energy conservation argument

Expectation: the power flow down the waveguide should be attenuated exponentially,

$$\Rightarrow \quad P(z) = P_0 e^{-2\beta_\lambda z}$$

i.e.

$$\beta_\lambda = \frac{-\dfrac{dP(z)}{dz}}{2P(z)}$$

but in Sec. 8.1 we derived the rate of energy loss into a wall of finite conductivity as

$$\frac{-dP}{dz} = \frac{1}{2\sigma\delta}\oint_C |\hat{n}\times\vec{H}|^2 dl = -\oint_C \frac{dP}{da}\, dl$$

Let's work this out explicitly for a TM mode:
$\hookrightarrow H_z = 0$

$$|\hat{n}\times\vec{H}|^2 = |\vec{H}_t|^2 = |\hat{n}\times\vec{H}_t|^2$$

$$= |\hat{n}\times(\hat{z}\times\nabla_t\psi)|^2 \frac{k^2}{\gamma^4}\frac{\epsilon^2\omega^2}{k^2}$$

$$= |\hat{z}(\hat{n}\cdot\nabla_t\psi)|^2 \frac{\epsilon^2\omega^2}{\gamma^4} = \left|\frac{\partial\psi}{\partial n}\right|^2 \frac{\omega^2}{\mu^2\omega_\lambda^4}$$

and so

$$\boxed{-\frac{dP^{TM}}{dz} = \frac{1}{2\sigma\delta}\frac{\omega^2}{\mu^2\omega_\lambda^4}\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl \quad \text{for TM}}$$

and similarly

$$\boxed{-\frac{dP^{TE}}{dz} = \frac{1}{2\sigma\delta}\frac{\omega^2}{\omega_\lambda^2}\oint_C\left\{\frac{1}{\mu\epsilon\omega_\lambda^2}\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)|\hat{n}\times\nabla_t\psi|^2 + \frac{\omega_\lambda^2}{\omega^2}|\psi|^2\right\}dl}$$

Order-of-magnitude estimates

Idea: estimate the approximate magnitude of the transverse derivatives, very roughly, using:

$$(\nabla_t^2 + \mu\epsilon\omega_\lambda^2)\psi = 0$$

$$\Rightarrow \text{take } \left\langle\left|\frac{\partial\psi}{\partial n}\right|^2\right\rangle \simeq \mu\epsilon\omega_\lambda^2\langle|\psi|^2\rangle$$

$$= \langle|\hat{n}\times\nabla_t\psi|^2\rangle$$

whereby

$$\oint_C \frac{1}{\omega_\lambda^2}\left|\frac{\partial\psi}{\partial n}\right|^2 dl \simeq \mu\epsilon\, C\,\langle|\psi|^2\rangle$$

C: circumference of the cross-section area

$$\simeq \frac{\mu\epsilon C}{A}\int_A |\psi|^2 da$$

Or to be more precise, we can introduce a dimensionless constant ξ_λ of order unity, defined by:

$$\xi_\lambda \equiv \frac{\oint_C \frac{1}{\omega_\lambda^2}\left|\frac{\partial \psi}{\partial n}\right|^2 dl}{\frac{\mu \epsilon C}{A}\int_A |\psi|^2 da}$$

Next recall the following relations for TM modes, from above:

$$\boxed{-\frac{dP^{TM}}{dz} = \frac{1}{2\sigma\delta}\frac{\omega^2}{\mu^2\omega_\lambda^4}\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl \quad \text{for TM}}$$

and

$$\boxed{P = \frac{\omega}{2\sqrt{\mu\epsilon}\,\omega_\lambda^2}(\omega^2-\omega_\lambda^2)^{1/2}\left\{{\epsilon \atop \text{or } \mu}\right\}\int_A |\psi|^2 da}$$

which yields:

$$\boxed{\beta_\lambda^{TM} = -\frac{1}{2P}\frac{dP}{dz} = \left(\frac{\epsilon}{\mu}\right)^{1/2}\frac{1}{\sigma\delta_\lambda}\frac{C}{2A}\frac{(\omega/\omega_\lambda)^{1/2}}{\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}}\xi_\lambda}$$

where δ_λ = skin depth at the cutoff frequency ω_λ

If this derivation is repeated for TE modes, a similar expression results, except that

$$\xi_\lambda \longrightarrow \xi_\lambda + \frac{\omega_\lambda^2}{\omega^2}\eta_\lambda$$

where η_λ is also dimensionless and of order unity

Example take the microwave oven frequency,

$$\omega = (2\pi)\,2.5\ \text{GHz}$$

and conductivity $\sigma \simeq 6\times 10^{7}\ \Omega^{-1} m^{-1}$

$$\mu_c \simeq \mu_0 = 4\pi \times 10^{-7}\ \frac{N}{A^2}$$

Then the skin depth is $\delta = \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2} \simeq 1.3\,\mu m$

and take the cross-section radius to be

$$R = 10^{-2}\ m$$

$$\Rightarrow \quad \omega \simeq \omega_0 \sqrt{3}$$

This leads to $\beta \simeq \left(\frac{\epsilon}{\mu}\right)^{1/2} \frac{1}{\sigma\delta}\,\frac{2\pi R}{2(\pi R^2)}\,\frac{3^{1/4}}{(2/3)^{1/2}} \simeq 0.005\ m^{-1}$

and consequently $P(z) = P(0)e^{-2\beta z}$ decays by a factor $\frac{1}{e}$ over a distance of

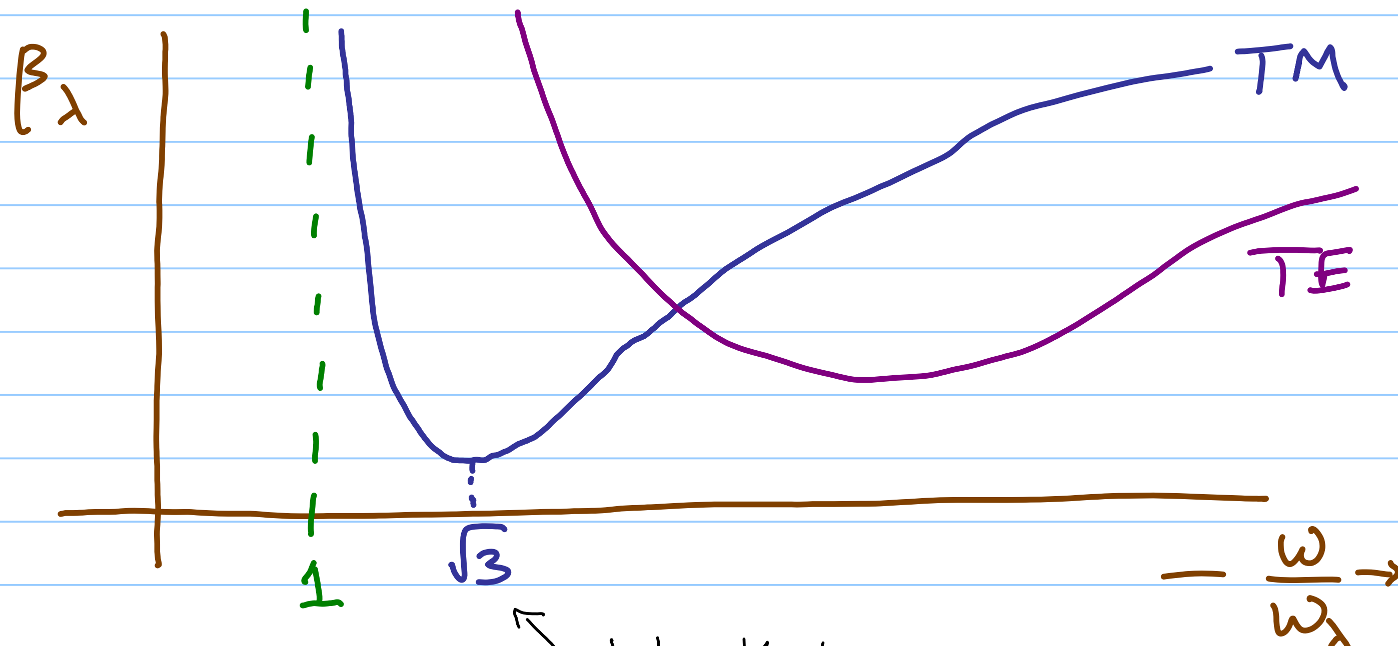
$$\Delta z \simeq 100\ m$$

Jackson says that 200–400 m is typical

Limiting Behavior

For high frequency,
$\beta \propto \omega^{1/2}$

Also, for ω near the cutoff frequency ω_λ, β_λ gets large and our approximate treatment fails



Note that
$\omega_{min} = \sqrt{3}\, \omega_\lambda$ always for TM, but for TE modes there is no simple, general formula for ω_{min}.

On your own, read Sec. 8.6 to learn about a more general and more accurate treatment.

<u>**Resonant Cavity**</u> = a waveguide with conducting endcaps.

$\Rightarrow$ Where we had travelling waves for a waveguide, now we will obtain STANDING WAVES.

i.e. $\psi(x,y)e^{\pm ikz} \longrightarrow \psi(x,y)\begin{cases}\cos kz\\ \sin kz\end{cases}$

A resonant cavity could have any shape, but we will only treat cylinders of constant cross-sectional area

<u>e.g.</u> Consider the TM mode and take the ends at $z=0$, $z=d$.

$\Rightarrow$ $\psi = E_z$, and the BCs are:

(i) $E_z = 0$ on the sides, just as for the waveguides

and (ii) $\vec{E}_t = 0$ on the ends, so that $\vec{E}_{\parallel} = 0$

We already implemented BC (i) in our solution above for the TM modes. To implement BC(ii), consider a solution of the form

$$E_z = \psi(x,y)\cos kz\, e^{-i\omega t}$$
$$= \psi(x,y)\,\frac{e^{ikz}+e^{-ikz}}{2}\,e^{-i\omega t}$$

To see why the relevant solution is $\cos kz$, recall Eq.(8.23 a), $\dfrac{\partial \vec{E}_t}{\partial z} + i\mu\omega\,\hat{z}\times\vec{H}_t = \vec{\nabla}_t E_z$

and 8.26 b says that

$$\vec{H}_t = \frac{i}{\gamma^2}\left(\xcancel{\pm k\nabla_t H_z}^{\,0 \text{ for TM}} + \epsilon\omega\,\hat{z}\times\nabla_t E_z\right)$$

So, if $E_z = \psi(x,y)\cos kz$, then

$$\frac{\partial \vec{E}_t}{\partial z} - i\mu\omega\left(\frac{i\epsilon\omega}{\gamma^2}\right)\nabla_t E_z = \nabla_t E_z$$

or
$$\frac{\partial \vec{E}_t}{\partial z} = \frac{\gamma^2-\mu\epsilon\omega^2}{\gamma^2}\nabla_t E_z = \frac{-k^2}{\gamma^2}\nabla_t E_z$$

and integrating this over z gives

$$\vec{E}_t = -\frac{k^2}{\gamma^2}\nabla_t\psi(x,y)\,\frac{\sin kz}{k}$$

So we see that the endcap BCs will be satisfied if $k = p\frac{\pi}{d}$, $p = 0, 1, 2, \ldots$ and then

$$\boxed{\begin{aligned} \vec{E}_t &= \frac{-p\pi}{d\gamma^2}\sin\left(\frac{p\pi z}{d}\right)\nabla_t \psi \\ \vec{H}_t &= \frac{i\epsilon\omega}{\gamma^2}\cos\left(\frac{p\pi z}{d}\right)\hat{z}\times\nabla_t\psi \end{aligned}}$$

for TM modes

$p = 0, 1, 2 \ldots$

And by a similar argument (derive on your own)

$$\boxed{\begin{aligned} \vec{E}_t &= -\frac{i\omega\mu}{\gamma^2}\sin\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi \\ \vec{H}_t &= \frac{p\pi}{d\gamma^2}\cos\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi \end{aligned}}$$

for TE modes

$p = 1, 2, \ldots$

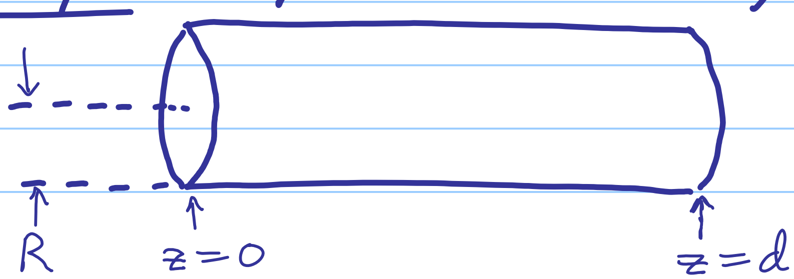
($p = 0$ would give the trivial 0 solution only for TE)

and $\quad \gamma^2 = \mu\epsilon\omega^2 - \left(\frac{p\pi}{d}\right)^2$

so the allowed eigenfrequencies of the perfectly conducting cavity are ONLY the following discrete values:

$$\omega_{\lambda p}^2 = \frac{1}{\mu\epsilon}\left(\gamma_\lambda^2 + \left(\frac{p\pi}{d}\right)^2\right)$$

Example right-circular cylinder cavity



Ψ must obey:

$$\left(\nabla_t^{\,2} + \gamma^2\right)\Psi(\rho,\phi) = 0$$

$$\Rightarrow \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \gamma^2\right)\Psi(\rho,\phi) = 0$$

$-m^2$, $m = 0, 1, 2, \ldots$

and this is solved by separable solutions,

$$\Psi(\rho,\phi) = R(\rho)\begin{Bmatrix}\cos m\phi \\ \text{or} \\ \sin m\phi\end{Bmatrix}$$

and the ρ solutions are Bessel functions, namely

$$R(\rho) = J_m(\gamma\rho)$$

must be J_m to be regular at 0.

The appropriate BC for a TM mode is

$$\Psi\big|_s = 0 \Rightarrow J_m(\gamma R) = 0$$

$$\Rightarrow \gamma = \frac{x_{mn}}{R} \quad \text{where } x_{mn} = n^{th} \text{ root of } J_m(x_{mn}) = 0$$

and then

$$\Psi_{m,n}^{\binom{s}{c}}(\rho,\phi) = J_m\left(x_{mn}\frac{\rho}{R}\right)\begin{Bmatrix}\sin m\phi \\ \text{or} \\ \cos m\phi\end{Bmatrix}$$

and $$\gamma_{mn}^2 = \frac{x_{mn}^2}{R^2} = \mu\epsilon\omega^2 - \left(\frac{p\pi}{d}\right)^2$$