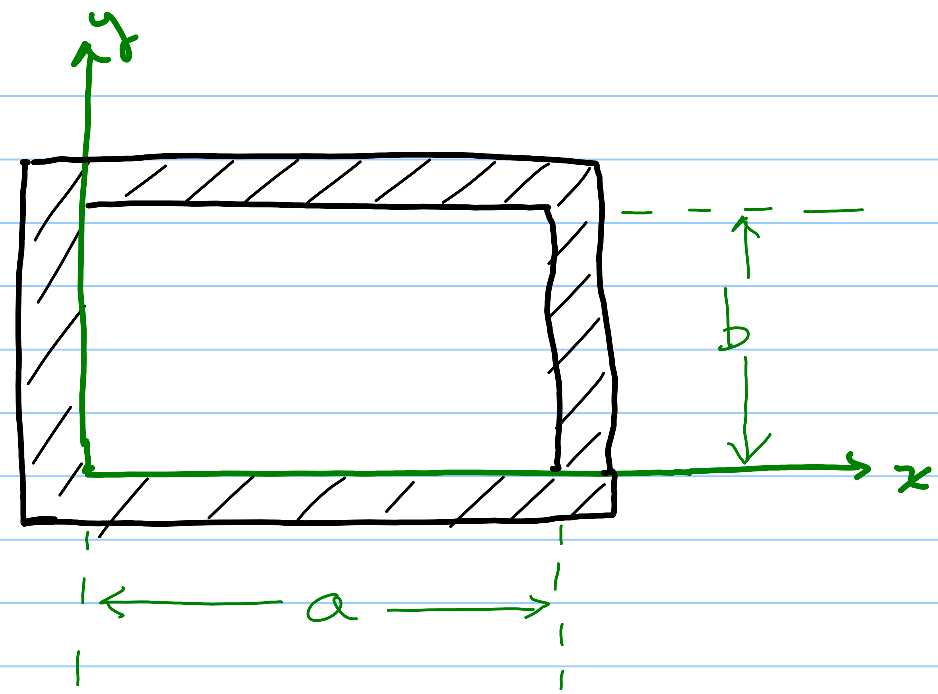
Rectangular wave guide



Solution for the TE modes

$$\psi = H_z \quad \text{obeys}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right)\psi(x,y) = 0$$

subject to the boundary conditions

$$\frac{\partial \psi}{\partial n} = 0 \quad , \text{ i.e. } \begin{cases} \dfrac{\partial \psi}{\partial x} = 0 & \text{at } x=0 \text{ and } x=a \\ \dfrac{\partial \psi}{\partial y} = 0 & \text{at } y=0 \text{ and } y=b \end{cases}$$

This is a separable PDE with separable BCs and its solutions are

$$(H_z)_{m,n} = \psi_{mn}(x,y) = H_0 \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

and $\gamma_{m,n}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$, $m,n = 0,1,2,\ldots$

The lowest frequency has $m = n = 0$

$\Rightarrow \psi_{0,0}(x,y) = \text{constant},$

which gives zero fields inside

Accordingly, this "trivial solution" is of no physical relevance, so it is omitted from the list of solutions

The cutoff frequencies are

$$\omega_{m,n} = \frac{\pi}{\sqrt{\mu\epsilon}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{1/2}$$

For definiteness, suppose $a > b$. Then the lowest mode has $m = 1$, $n = 0$, and the cutoff frequency is

$$\omega_{1,0} = \frac{\pi}{a\sqrt{\mu\epsilon}} \quad , \text{since } \gamma_{1,0} = \frac{\pi}{a}$$

and $k_{1,0} = \sqrt{\mu\epsilon} \left(\omega^2 - \omega_{1,0}^2 \right)^{1/2}$

and $\vec{H}_t = \frac{ik}{\gamma^2} \vec{\nabla}_t H_z$

$$= -\frac{i k_{m,n} H_0}{\gamma_{m,n}^2} \left\{ \hat{x} \frac{m\pi}{a} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} + \hat{y} \frac{n\pi}{b} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \right\}$$

and $\vec{H}_z \xrightarrow[n=0]{m=1} H_0 \cos\frac{\pi x}{a}\,\hat{z}$

$$\vec{H}_t^{(1,0)} \xrightarrow[n=0]{m=1} \frac{-ik_{1,0}}{\gamma_{1,0}^2} H_0 \frac{\pi}{a} \sin\frac{\pi x}{a}\,\hat{x}$$

and $\vec{E}_t^{(1,0)} = -\dfrac{\omega\mu}{k_{1,0}}\,\hat{z}\times\vec{H}_t^{(1,0)}$

$$= H_0 \frac{ik_{1,0}}{\gamma_{1,0}^2}\frac{\omega\mu}{k_{1,0}}\,\hat{y}\,\sin\frac{\pi x}{a}$$

and the final component, of course, is $E_z^{(1,0)} = 0$, since this is a TE mode.

This is a linearly-polarized E-field for the lowest TE-mode.

Observations

(1) All fields have an additional factor $e^{i(k_{1,0}z-\omega t)}$

(2) The factor of i in $\vec{H}_t$ and $\vec{E}_t$ implies a 90° phase difference between H_z and $\vec{H}_t$ (or $\vec{E}_t$)

(3) The formulas for the TM mode are obtained by setting $\Psi \rightarrow E_z$, and $\cos() \rightarrow \sin()$

and the lowest nontrivial mode has

$$m=1 \ , \ n=1$$

observe that the LOWEST TM mode has a higher cutoff frequency than does the lowest TE mode.

Energy Flow and attenuation

Next, determine the time-averaged rate of energy flow down the cavity. We are interested in $\text{Re}\left(\frac{1}{2}\vec{E}\times\vec{H}^*\right)$ for complex fields, as seen in Chap. 6.

For now, just consider $\vec{S}=\frac{1}{2}\vec{E}\times\vec{H}^*$

Then for TM,

$$\vec{S}=\frac{1}{2}\left(\vec{E}_t+\vec{E}_z\right)\times\left[\left(\frac{\epsilon\omega}{k}\right)^*\left(\hat{z}\times\vec{E}_t^*\right)\right]$$

$$=\underbrace{\frac{1}{2}\left(\frac{\epsilon\omega}{k}\right)^*\vec{E}_t\times\left(\hat{z}\times\vec{E}_t^*\right)}_{=\frac{1}{2}\left(\frac{\epsilon\omega}{k}\right)^*|\vec{E}_t|^2}+\underbrace{\frac{1}{2}\left(\frac{\epsilon\omega}{k}\right)^*\vec{E}_z\times\left(\hat{z}\times\vec{E}_t^*\right)}_{=-\frac{1}{2}\left(\frac{\epsilon\omega}{k}\right)^*E_z\vec{E}_t^*}$$

or putting it together,

$$\vec{S}=\frac{1}{2}\frac{\epsilon^*\omega^*}{k^*}\left\{\hat{z}\left|\frac{k^2}{\gamma^4}\right|\left|\nabla_t E_z\right|^2-\left(\frac{-ik^*}{\gamma^{*2}}\right)E_z\nabla_t E_z^*\right\}$$

or for ϵ,ω,k all real,

$$\boxed{\vec{S}_{TM}=\frac{1}{2}\frac{\omega k\epsilon}{\gamma^4}\left\{\hat{z}\left|\nabla_t E_z\right|^2+\frac{i\gamma^2}{k}E_z\nabla_t E_z^*\right\}}$$

Summary of the Key Chap. 8 Formulas

suggestion: print this page out for reference

Maxwell's equation solutions: $\vec{H}(x,y,z,t) = \vec{H}(x,y)e^{i(kz-\omega t)}$

$$(\nabla_t^2 + \gamma^2)\begin{pmatrix}\vec{E} \\ \vec{H}\end{pmatrix} = 0 \qquad \vec{E}(x,y,z,t) = \vec{E}(x,y)e^{i(kz-\omega t)}$$

$\nabla_t \cdot \vec{E}_t + ikE_z = 0$	$\vec{\nabla}_t E_z - ik\vec{E}_t = i\omega\mu\hat{z}\times\vec{H}_t$
$\nabla_t \cdot \vec{H}_t + ikH_z = 0$	$\vec{\nabla}_t H_z - ik\vec{H}_t = -i\epsilon\omega\hat{z}\times\vec{E}_t$

$\gamma^2 = \mu\epsilon\omega^2 - k^2$	$k^2 = \mu\epsilon\omega^2 - \gamma^2$
$\gamma_\lambda^2 = \mu\epsilon\omega_\lambda^2$	$k_\lambda^2 = \mu\epsilon(\omega^2 - \omega_\lambda^2)$

$$\vec{E}_t = \frac{i}{\gamma^2}\left(k\vec{\nabla}_t E_z - \mu\omega\hat{z}\times\vec{\nabla}_t H_z\right)$$

$$\vec{H}_t = \frac{i}{\gamma^2}\left(k\vec{\nabla}_t H_z + \epsilon\omega\hat{z}\times\vec{\nabla}_t E_z\right)$$

$$\vec{H}_t = \frac{\epsilon\omega}{k}\hat{z}\times\vec{E}_t = \frac{\hat{z}}{Z}\times\vec{E}_t \quad (TM);\ Z = \frac{k}{\epsilon\omega}$$

$$\vec{H}_t = \frac{k}{\mu\omega}\hat{z}\times\vec{E}_t = \frac{\hat{z}}{Z}\times\vec{E}_t \quad (TE);\ Z = \frac{\mu\omega}{k}$$

$$\vec{E}_t = \frac{ik}{\gamma^2}\vec{\nabla}_t E_z = \frac{ik}{\gamma^2}\vec{\nabla}_t\psi \quad (TM)$$

$$\vec{H}_t = \frac{ik}{\gamma^2}\vec{\nabla}_t H_z = \frac{ik}{\gamma^2}\vec{\nabla}_t\psi \quad (TE)$$

$v_p v_g = \frac{1}{\mu\epsilon}$	$v_g = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}$

Next derive $\vec{S}_{TE}$:

$\boxed{TE}$

$$\vec{S} = \frac{1}{2}\vec{E}_t \times (\vec{H}_t^* + \vec{H}_z^*)$$

$$= \frac{1}{2}\vec{E}_t \times (\hat{z}\times\vec{E}_t^*)\left(\frac{k}{\mu\omega}\right)^*$$

which we write in terms of $H_z = \psi$,

by using $\vec{H}_t = \frac{ik}{\gamma^2}\nabla_t\psi$, and 8.31,

$$\vec{H}_t = \frac{k}{\mu\omega}\hat{z}\times\vec{E}_t$$

Then taking $\hat{z}\times(8.31)$,

$$\Rightarrow \hat{z}\times\vec{H}_t = \frac{k}{\mu\omega}\hat{z}\times(\hat{z}\times\vec{E}_t) = -\frac{k}{\mu\omega}\vec{E}_t$$

or $\boxed{\vec{E}_t = -\frac{\mu\omega}{k}\hat{z}\times\vec{H}_t}$

whereby $\vec{S} = \frac{1}{2}\left(-\frac{\mu\omega}{k}\right)(\hat{z}\times\vec{H}_t)\times(\vec{H}_t^* + \vec{H}_z^*)$

$$= \frac{\mu\omega}{2k}\left\{\hat{z}\,|\vec{H}_t|^2 - H_z^*\vec{H}_t\right\}$$

$$= \frac{\mu\omega}{2k}\left\{\hat{z}\left|\frac{k^2}{\gamma^4}\right|\,|\vec{\nabla}_t H_z|^2 - \frac{ik}{\gamma^2}H_z^*\vec{\nabla}_t H_z\right\}$$

and if k, γ, μ are all real,

$$\Rightarrow \boxed{\vec{S}^{(TE)} = \frac{\omega k}{2\gamma^4}\left\{\hat{z}\mu\,|\vec{\nabla}_t H_z|^2 - \frac{i\gamma^2\mu}{k}H_z^*\nabla_t H_z\right\}}$$

Observe, moreover, that since $\psi = \begin{cases} E_z, & \text{TM modes} \\ H_z, & \text{TE modes,} \end{cases}$

obeys a <u>REAL</u> PDE with <u>REAL</u> BCs,

we may take ψ to be <u>REAL</u> without any loss of generality.

Then the terms $i E_z \nabla_t E_z^*$ and $i H_z^* \nabla_t H_z$ <u>do not contribute</u> to the energy flow along the guide, <u>after we take</u> $\mathrm{Re}\,\vec{S}$.

And the total time-averaged power flow is

$$P = \int_A \vec{S} \cdot \hat{z}\, da = \frac{\omega k}{2\gamma^4} \begin{Bmatrix} \epsilon \\ \text{or} \\ \mu \end{Bmatrix} \int_A |\nabla_t \psi|^2 da$$

for $\begin{Bmatrix} \text{TM} \\ \text{or} \\ \text{TE} \end{Bmatrix}$ modes

Next, application of Green's 1st identity in 2D gives:

$$P = \frac{\omega k}{2\gamma^4} \begin{Bmatrix} \epsilon \\ \text{or} \\ \mu \end{Bmatrix} \left(\cancel{\oint_C \psi^* \frac{\partial \psi}{\partial n} dl}^{\,0 \text{ for TM and TE modes}} - \int_A \psi^* \underbrace{\nabla_t^2 \psi}_{= -\gamma^2 \psi} da \right)$$

or $P = \dfrac{\omega k}{2\gamma^2} \begin{Bmatrix} \epsilon \\ \text{or} \\ \mu \end{Bmatrix} \displaystyle\int_A |\psi|^2 da$, and using

$k \equiv \sqrt{\mu\epsilon}\,(\omega^2 - \omega_\lambda^2)^{1/2}$, $\gamma^2 \equiv \mu\epsilon\,\omega_\lambda^2$,

$$\Rightarrow \boxed{P = \frac{\omega}{2\sqrt{\mu\epsilon}\,\omega_\lambda^2} (\omega^2 - \omega_\lambda^2)^{1/2} \begin{Bmatrix} \epsilon \\ \text{or} \\ \mu \end{Bmatrix} \int_A |\psi|^2 da}$$

Similarly, the time-averaged electric and magnetic energies per unit volume stored inside is (6.133):

$$W_e = \frac{1}{4}\vec{E}\cdot\vec{D}^* \quad , \quad W_m = \frac{1}{4}\vec{B}\cdot\vec{H}^*$$

and thus the energy stored per unit length inside the guide is, for real ϵ, μ:

$$\frac{dU}{dl} = \int da \left(\frac{\epsilon}{4}|\vec{E}|^2 + \frac{\mu}{4}|\vec{H}|^2 \right)$$

and we can recast this in terms of ψ:

TM case

$$\vec{H}_t = \frac{\epsilon\omega}{k}\hat{z}\times\vec{E}_t \quad \text{and} \quad \omega_\lambda^2 = \frac{\gamma_\lambda^2}{\mu\epsilon}$$

and $k_\lambda^2 = \mu\epsilon(\omega^2 - \omega_\lambda^2) = \mu\epsilon\omega^2 - \gamma_\lambda^2$

$$\Rightarrow \frac{dU}{dl} = \int da \left\{ \frac{\epsilon}{4}|\vec{E}_t|^2 + \frac{\epsilon}{4}|E_z|^2 + \frac{\mu}{4}|\vec{H}_t|^2 \right\}$$

where $\vec{E}_t = \frac{ik}{\gamma^2}\nabla_t\psi$, $\vec{H}_t = i\frac{\epsilon\omega}{\gamma^2}\hat{z}\times\nabla_t\psi$

and $E_z = \psi$

$$\Rightarrow \frac{dU}{dl} = \int da \left\{ \frac{\epsilon}{4}|\psi|^2 + \frac{\epsilon}{4\gamma^4}|\nabla_t\psi|^2(2k^2+\gamma^2) \right\}$$

and after again employing Green's identity,

$$\Rightarrow \int da |\nabla_t\psi|^2 = \xcancel{\oint dl\, \psi^* \frac{\partial\psi}{\partial n}}^{\,\to 0} - \int da\, \psi^* \nabla_t^2\psi$$

$$= \gamma^2 \int |\psi|^2 da$$

So $$\frac{dU}{dl} = \int da \frac{\epsilon}{4} |\psi|^2 \left(1 + \frac{2k^2\gamma^2 + \gamma^4}{\gamma^4}\right)$$

$$(\ ) = \frac{2\gamma^2 + 2k^2}{\gamma^2} = \frac{2\mu\epsilon\omega^2}{\gamma^2} = 2\frac{\omega^2}{\omega_\lambda^2}$$

$$\Rightarrow \boxed{\frac{dU}{dl} = \int da \frac{\epsilon}{2} \frac{\omega^2}{\omega_\lambda^2} |\psi|^2 \text{ for TM modes}}$$

And a similar derivation for TE modes gives:

$$\boxed{\frac{dU}{dl} = \frac{\mu}{2} \frac{\omega^2}{\omega_\lambda^2} \int da |\psi|^2 \text{ for TE modes}}$$

The ratio between power flow rate and energy density is interpreted as the velocity at which energy flows down the cavity, i.e.

$$\frac{P}{\frac{dU}{dl}} = \frac{\omega(\omega^2 - \omega_\lambda^2)^{1/2}}{\omega^2\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}}\left(1 - \frac{\omega^2}{\omega_\lambda^2}\right)^{1/2} = v_g$$

$\Rightarrow$ as expected, $P = \left(\frac{dU}{dl}\right) v_g$, confirming that energy flows at the group velocity, and $v_g \le c$