8.2 cylindrical waveguides + cavities

<u>Goal</u>: treat the propagation of EM waves in regions bounded by conductors

Here "cylindrical" means a constant cross section, either

1) cylinder with no end caps, which supports traveling waves

or

2) cylinder with end caps, which supports standing waves

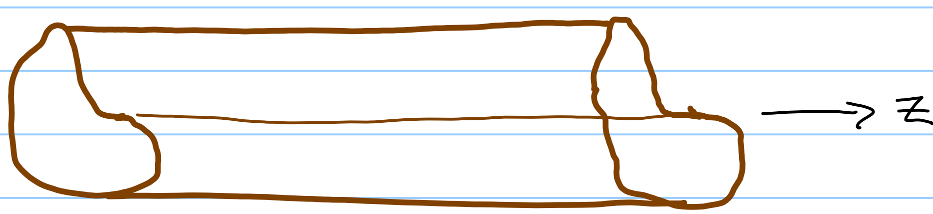
Inside, with no sources, Maxwell's equations for harmonic time dependences are:

$$\nabla \times \vec{E} = i\omega \vec{B} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = -i\mu\epsilon\omega \vec{E} \qquad \nabla \cdot \vec{E} = 0$$

So if we assume the interior is a uniform, nondissipative medium,

$$\Rightarrow \left(\nabla^2 + \mu\epsilon\omega^2\right)\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$



Look for travelling wave solutions:

i.e. try $\vec{E}(\vec{x},t) = \vec{E}(x,y)\, e^{\pm ikz - i\omega t}$

$$\vec{B}(\vec{x},t) = \vec{B}(x,y)\, e^{\pm ikz - i\omega t}$$

Plugging this in gives the following equation for the 2D amplitudes:

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right]\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$

where the transverse Laplacian is

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla^2 - \frac{\partial^2}{\partial z^2}$$

Next express the fields into a parallel component $\vec{E}_z$ and a transverse component $\vec{E}_t$

$\Rightarrow$ $\vec{E} = \vec{E}_z + \vec{E}_t$ where $\vec{E}_z = E_z \hat{z}$

and $\vec{E}_z = \hat{z}(\hat{z}\cdot\vec{E})$

$$\vec{E}_t = (\hat{z}\times\vec{E})\times\hat{z}$$

(and do this for $\vec{B}$ also)

Then in this notation Maxwell's equations are:

$$\nabla_t \cdot \vec{E}_t + \frac{\partial E_z}{\partial z} = 0$$

$$\nabla_t \cdot \vec{B}_t + \frac{\partial B_z}{\partial z} = 0$$

and

$$\left(\nabla_t + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\vec{E}_z + \vec{E}_t\right) = i\omega\left(\vec{B}_z + \vec{B}_t\right)$$

Now break this up into longitudinal ($\hat{z}$) and transverse (t) components:

LONGITUDINAL – take $\hat{z}\cdot(\nabla\times\vec{E})$

$$\Rightarrow \hat{z}\cdot\left(\nabla_t \times \vec{E}_t\right) = i\omega\, \hat{z}\cdot\vec{B}_z$$

TRANSVERSE – take $\hat{z}\times(\nabla\times\vec{E}) = i\omega\hat{z}\times\vec{B}$

$$\Rightarrow \hat{z}\times\left(\nabla_t \times \vec{E}\right) + \hat{z}\times\left(\hat{z}\frac{\partial}{\partial z}\times\vec{E}\right)$$

$$= \hat{z}\times\left(\nabla_t\times\vec{E}_z\right) + \hat{z}\times\left(\hat{z}\times\frac{\partial\vec{E}_t}{\partial z}\right)$$

Next use $\nabla\times(\vec{a}\,\psi) = \nabla\psi\times\vec{a} + \psi\nabla\times\vec{a}$

$$\Rightarrow \nabla_t\times\left(\hat{z}E_z\right) = \left(\nabla_t E_z\right)\times\hat{z}$$

and

$$\hat{z}\times\left(\hat{z}\times\frac{\partial\vec{E}_t}{\partial z}\right) = \hat{z}\,\Big(\cancelto{0}{\hat{z}\cdot\frac{\partial\vec{E}_t}{\partial z}}\Big) - \frac{\partial\vec{E}_t}{\partial z}\,\big(\cancelto{1}{\hat{z}\cdot\hat{z}}\big)$$

whereby

$$\hat{z}\times\left(\nabla\times\vec{E}\right) = -\hat{z}\times\left(\nabla_t E_z\right) - \frac{\partial\vec{E}_t}{\partial z} = i\omega\hat{z}\times\vec{B}$$

which can be further simplified to

$$\overrightarrow{\nabla_t E_z} - \frac{\partial}{\partial z}\vec{E}_t = i\omega\,\hat{z}\times\vec{B}_t \qquad (1)$$

and similarly,

$$\overrightarrow{\nabla_t B_z} - \frac{\partial}{\partial z}\vec{B}_t = -i\mu\epsilon\omega\,\hat{z}\times\vec{E}_t \qquad (2)$$

and

$$\hat{z}\cdot\left(\nabla_t\times\vec{B}_t\right) = -i\mu\epsilon\omega E_z \qquad (3)$$

If now the entire z-dependence is taken to be e^{ikz}, as expected for propagation along the cylinder, we get

$$\vec{\nabla}_t E_z - ik\vec{E}_t = i\omega\,\hat{z}\times\vec{B}_t$$

plug in here

and
$$\vec{\nabla}_t B_z - ik\vec{B}_t = -i\mu\epsilon\omega\,\hat{z}\times\vec{E}_t$$

or
$$\vec{B}_t = \frac{1}{ik}\left\{\vec{\nabla}_t B_z + i\mu\epsilon\omega\,\hat{z}\times\vec{E}_t\right\}$$

$$\Rightarrow \vec{\nabla}_t E_z - ik\vec{E}_t = i\omega\,\hat{z}\times\frac{\vec{\nabla}_t B_z}{ik} + i\omega\left(\frac{\mu\epsilon\omega}{k}\right)\hat{z}\times(\hat{z}\times\vec{E}_t)$$

$$= \frac{\omega}{k}\,\hat{z}\times\vec{\nabla}_t B_z - i\frac{\omega^2\mu\epsilon}{k}\vec{E}_t$$

and so
$$\vec{E}_t\left(-ik + \frac{i\omega^2\mu\epsilon}{k}\right) = -\vec{\nabla}_t E_z + \frac{\omega}{k}\hat{z}\times\vec{\nabla}_t B_z$$

or
$$\boxed{\vec{E}_t = \frac{ik}{k^2-\omega^2\mu\epsilon}\left\{-\vec{\nabla}_t E_z + \frac{\omega}{k}\hat{z}\times\vec{\nabla}_t B_z\right\}} \quad (4)$$

and
$$\boxed{\vec{B}_t = \frac{ik}{k^2-\omega^2\mu\epsilon}\left\{-\vec{\nabla}_t B_z - \frac{\mu\epsilon\omega}{k}\hat{z}\times\vec{\nabla}_t E_z\right\}} \quad (5)$$

The point is that knowledge of E_z, B_z will imply how to get $\vec{E}_t, \vec{B}_t$!

TEM modes Suppose we now look for transverse electromagnetic waves, for which $E_z = 0 = B_z$, i.e. $\vec{E}_{TEM} = \vec{E}_t$, $\vec{B}_{TEM} = \vec{B}_t$

$$\Rightarrow \boxed{\begin{aligned} \nabla_t\cdot\vec{E}_{TEM} &= 0 \\ \nabla_t\times\vec{E}_{TEM} &= 0 \end{aligned}}$$

← These are the same as the equations of a 2-dim^l problem in electrostatics, with $\rho = 0$!

Conclusions

(1) NO TEM modes can exist in a hollow waveguide bounded by a single conducting surface. This is because the uniqueness theorem for Laplace's equation which implies here that there is one solution having $\Phi = \text{constant}$, $\vec{E}_t = 0$, everywhere inside a conducting shell = waveguide boundary surface

(2) Referring to (1) and (2) above, and recalling that the z-dependence is e^{ikz}, we have:

$$\vec{E}_t = -\frac{\omega}{k}\hat{z}\times\vec{B}_t$$

plug in here!

$$\vec{B}_t = \underbrace{\frac{\mu\epsilon\omega}{k}\hat{z}\times\vec{E}_t}$$

So we see a consistency relation,

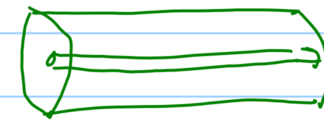
$$\vec{E}_t = -\frac{\omega}{k}\hat{z}\times(\hat{z}\times\vec{E}_t)\frac{\mu\epsilon\omega}{k} = \frac{\omega^2}{k^2}\mu\epsilon\,\vec{E}_t$$

$$\Rightarrow \boxed{\frac{\omega^2}{k^2} = \frac{1}{\mu\epsilon}}, \text{ and } \boxed{\vec{B}_t = \sqrt{\mu\epsilon}\,\hat{k}\times\vec{E}_t}$$

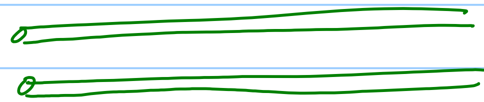
$\Rightarrow$ These coincide with the equations we found before for propagation in a homogeneous medium!

i.e. $\vec{E}_t(\vec{x},t) = \vec{E}_t^{(0)}e^{\pm ikz - i\omega t}$, $k = \frac{\omega}{c}\left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)^{1/2}$

(3) The TEM mode is the simplest mode in some waveguides with 2 or more conductors, such as a coaxial cable



or a parallel transmission line,



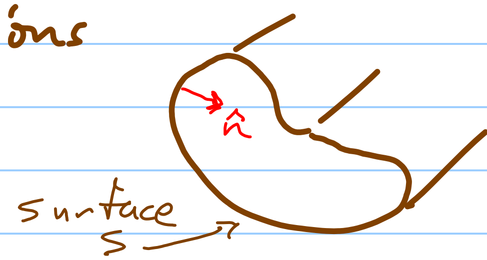
(see problems 8.2, 8.3)

Modes allowed for a perfect cylindrical conductor

recall the boundary conditions

$$\hat{n}\times\vec{E} = 0$$

$$\hat{n}\cdot\vec{B} = 0$$



These are the relevant BCs because the fields vanish in the conductor, and notice that $\hat{n}\cdot\vec{D}$ and $\hat{n}\times\vec{H}$ could be nonzero outside the conductor because there could be surface charges and/or currents.

$\Rightarrow E_z\big|_S = 0$, and another parallel component of $\vec{E}$ that must vanish is $(\hat{n}\times\hat{z})\cdot\vec{E}$
$\equiv \hat{\rho}\cdot\vec{E}$

and $\hat{n}\cdot\vec{B} = 0 \Rightarrow$ consider Eq. 8.24(b),

$$\hat{z}\cdot(\nabla_t\times\vec{B}_t) = -i\mu\epsilon\omega E_z$$, if we define $\hat{\rho} = \hat{n}\times\hat{z}$,

then $\vec{B}_t = (B_\rho, B_n, 0) \Rightarrow \nabla_t\times\vec{B}_t\big|_S = \hat{z}\left(\frac{\partial B_n}{\partial\rho} - \frac{\partial B_\rho}{\partial n}\right)\Big|_S$

More to the point, consider Eq. 8.24(a), the $\hat{n}$-component:

$$\cancel{\frac{\partial B_n}{\partial z}}^{\,0}\Big|_s - i\mu\epsilon\omega\left\{\hat{z}\times\left(E_{\ell}|_s,\ E_n|_s, 0\right)\right\}_n = \frac{\partial B_z}{\partial n}\Big|_s$$

"$E_{\ell}|_s = 0$"

$$\Rightarrow \quad \frac{\partial B_z}{\partial n}\Big|_s = 0, \quad E_z|_s = 0$$

<u>A KEY POINT</u>: The boundary condition $E_z|_s = 0$ but $E_z \neq 0$ inside, plus the BC, $B_z = 0$ everywhere, in addition to the 2D wave equation above (see notes p. 9 above), is already enough information to specify the 2D eigenvalues (k^2) for transverse magnetic TM waves.

And similarly, the BC. $\frac{\partial B_z}{\partial n}\Big|_s = 0$ and $E_z = 0$ everywhere, is sufficient to specify the eigenvalues allowed for k^2 in the case of transverse electric (TE) waves. Since the BCs are different, the eigenvalues will in general be different k^2 for TE versus TM waves.

These are the appropriate BCs to be imposed on solutions of the 2D Helmholtz equations:

$$(\nabla_t^2 + \gamma^2)E_z(x,y) = 0$$

$$(\nabla_t^2 + \gamma^2)B_z(x,y) = 0,$$

subject to two different sets of BCs,

TRANSVERSE MAGNETIC WAVES (TM)

$B_z = 0$ everywhere inside

$E_z|_s = 0$ but $E_z \neq 0$ inside

OR

TRANSVERSE ELECTRIC WAVES (TE)

$E_z = 0$ everywhere inside

$\left.\frac{\partial B_z}{\partial n}\right|_s = 0$ but $B_z \neq 0$ inside

And one can show that the full set of all TM and TE modes (plus the TEM mode if it exists) are COMPLETE and can be superposed to describe any EM phenomena inside the waveguide.

A unified notation for both the TM and TE modes: Call the desired z component to be calculated Ψ, i.e.

For TM WAVES where $B_z = 0$, recall 8.26a

$$\Rightarrow \vec{E}_t = \frac{ik}{\gamma^2} \vec{\nabla}_t E_z \text{ , so define } \boxed{E_z = \Psi} \text{ for TM waves}$$

where $\gamma^2 = \mu\epsilon\omega^2 - k^2$

and then $\vec{B}_t = \frac{\mu\epsilon\omega}{k} \hat{z} \times \vec{E}_t = \mu \vec{H}_t$

or $\vec{H}_t = \frac{\epsilon\omega}{k} \hat{z} \times \vec{E}_t \equiv \frac{\hat{z}}{Z_e} \times \vec{E}_t$

while for $\underline{\text{TE WAVES}}$ where $E_z = 0$, (8.26b)

$$\Rightarrow \vec{B}_t = \frac{ik}{\gamma^2} \vec{\nabla}_t B_z \text{, so define } \boxed{\Psi = H_z \text{ for TE waves}}$$

$$\text{and (8.23a)} \Rightarrow \vec{E}_t = -\frac{\omega}{k} \hat{z} \times \vec{B}_t$$

$$\Rightarrow \hat{z} \times \vec{E}_t = -\frac{\omega}{k} \hat{z} \times (\hat{z} \times \vec{B}_t) = \frac{\omega}{k} \vec{B}_t = \frac{\mu\omega}{k} \vec{H}_t$$

$$\text{whereby } \vec{H}_t = \frac{k}{\mu\omega} \hat{z} \times \vec{E}_t \equiv \frac{\hat{z}}{Z} \times \vec{E}_t$$

$$\text{where } Z \equiv \begin{cases} \dfrac{k}{\epsilon\omega}, & \text{TM waves} \\ \dfrac{\mu\omega}{k}, & \text{TE waves} \end{cases} = \text{"wave impedance"}$$

And in this notation, both TM and TE waves obey $(\nabla^2 + \gamma^2)\Psi = 0$, $\gamma^2 = \mu\epsilon\omega^2 - k^2$

subject to either BC:

$$\Psi|_S = 0 \ (\text{TM}) \quad \underline{\text{OR}} \quad \frac{\partial \Psi}{\partial n}\bigg|_S = 0 \ (\text{TE})$$

From previous experience, we expect that Ψ must be oscillatory to satisfy BCs on opposite sides of a hollow cylinder

$$\Rightarrow \gamma^2 \geq 0$$

$\Rightarrow$ This will define a differential eigenvalue problem, with eigenvalues γ_λ^2 and corresponding eigenfunctions Ψ_λ,

$$\lambda = 1, 2, 3, \ldots$$

These eigensolutions are called the MODES of the waveguide

And recall that, for any frequency ω, the z-propagation wavenumber in mode λ is:

$$k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2$$

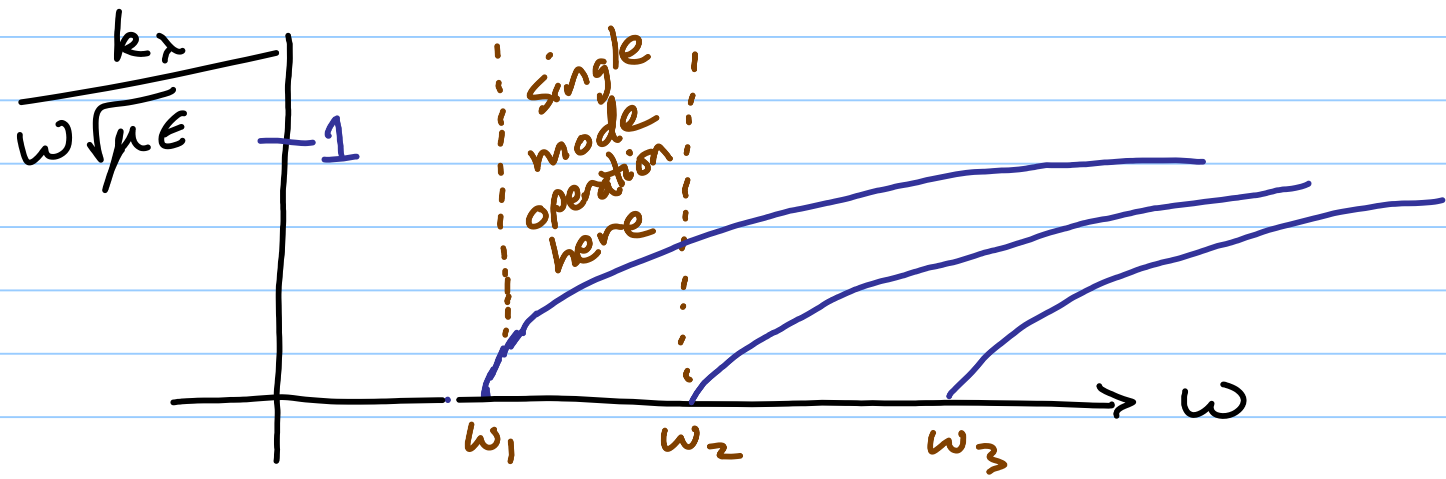
and k_λ^2 should be nonnegative to allow propagation of mode λ.

$\Rightarrow$ Thus there is a CUTOFF FREQUENCY, (for mode λ)

$$\omega_\lambda \equiv \frac{\gamma_\lambda}{\sqrt{\mu\epsilon}}$$

and we can write $k_\lambda = (\mu\epsilon)^{1/2}(\omega^2 - \omega_\lambda^2)^{1/2}$

Then for $\omega > \omega_\lambda$, k_λ = real and waves in mode λ can propagate along z, while for $\omega < \omega_\lambda$, k_λ = imaginary, and mode λ supports only evanescent modes (or cutoff modes) that decay exponentially in z.



We can find the phase velocity along the waveguide, v_p, which is mode-dependent,

$$e^{ik_\lambda x - i\omega t} = e^{ik_\lambda (x - v_p t)}$$

where $$v_p = \frac{\omega}{k_\lambda} = \frac{1}{\sqrt{\mu\epsilon}} \frac{1}{\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}} > \frac{1}{\sqrt{\mu\epsilon}}$$

Notice that the phase velocity can exceed the speed of light, c, since

$$v_p \xrightarrow[\omega \to \omega_\lambda]{} \infty$$

and $$\omega(k_\lambda) = (k_\lambda^2 + \gamma_\lambda^2)^{1/2} \frac{1}{\sqrt{\mu\epsilon}}$$

so the group velocity of a wavepacket is

$$v_g = \frac{d\omega}{dk_\lambda} = \frac{d}{dk_\lambda}\left[\frac{1}{\sqrt{\mu\epsilon}}(k_\lambda^2 + \gamma_\lambda^2)^{1/2}\right]$$

$$\Rightarrow \quad v_g = \frac{1}{\sqrt{\mu\epsilon}} \frac{k_\lambda}{\sqrt{k_\lambda^2 + \gamma_\lambda^2}} = \frac{c}{n} \frac{k_\lambda}{(k_\lambda^2 + \gamma_\lambda^2)^{1/2}}$$

Notice that in the usual case where $n > 1$, this translates into $v_g < c$ for all real γ_λ, k_λ.