Physics 631 Electrodynamics II at Purdue

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= Course homepage

From a long view of the history of mankind—seen from, say, ten thousand years from now—there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade.

— Richard P. Feynman, In The Feynman Lectures on Physics (1964), Vol. 2, page 1-11.

Begin with the speed of light, $c = 299792458 \frac{m}{s}$ EXACTLY

= a defined constant, the same in all inertial reference frames

MAXWELL EQUATIONS

SI units		Gaussian units
$\nabla \cdot \vec{B} = 0$ $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$	Homogeneous, no sources	$\nabla \cdot \vec{B} = 0$ $\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$
$\nabla \cdot \vec{D} = \rho$ $\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$	Inhomogeneous, source eqns	$\nabla \cdot \vec{D} = 4\pi\rho$ $\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}$
$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$	in macroscopic media	$\vec{D} = \vec{E} + 4\pi\vec{P}$ $\vec{H} = \vec{B} - 4\pi\vec{M}$
$\frac{\vec{F}}{q} = \vec{E} + \vec{v} \times \vec{B}$	Lorentz force $\vec{F}$ on point charge q, $\vec{v} = d\vec{x}/dt$	$\frac{\vec{F}}{q} = \vec{E} + \frac{\vec{v}}{c} \times \vec{B}$
$\frac{1}{2m}(\vec{p} - q\vec{A})^2$	Kinetic energy of point charge q with momentum $\vec{p}$ in a vector potential $\vec{A}$	$\frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2$
$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$ $\vec{B} = \nabla \times \vec{A}$	Potentials	$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ $\vec{B} = \nabla \times \vec{A}$

To interconvert equations or specific quantities, Gaussian ↔ SI units, see Jackson Appendix, p. 775

Chapter 8 Waveguides & resonant cavities

Here we treat EM waves propagating or resonating inside metal cavities or waveguides, beginning with a discussion of the behavior of the fields at or near a boundary.

First, consider a PERFECT conductor

(On your own, read Jackson Sec 5.18, pp 218–221)

Qualitative concept: we visualize that in a "perfect conductor", charges are so free to move that they can respond instantly to any fields inside the material.

$\Rightarrow$ $\vec{E} = 0$ inside a perfect conductor, just as we discussed for electrostatics.

$\Rightarrow$ instantaneously, the surface charges Σ will move to obey $\vec{D} \cdot \hat{n} = \Sigma$

where $\vec{D}$ = electric displacement just outside the surface,

and $\hat{n}$ = surface unit normal pointing outwards from conductor

and similarly, surface currents $\vec{K}$ can respond so freely and quickly that $\hat{n} \times \vec{H} = \vec{K}$

where $\vec{H}$ = magnetic field just outside

And the fields all vanish exactly inside for a perfect conductor,

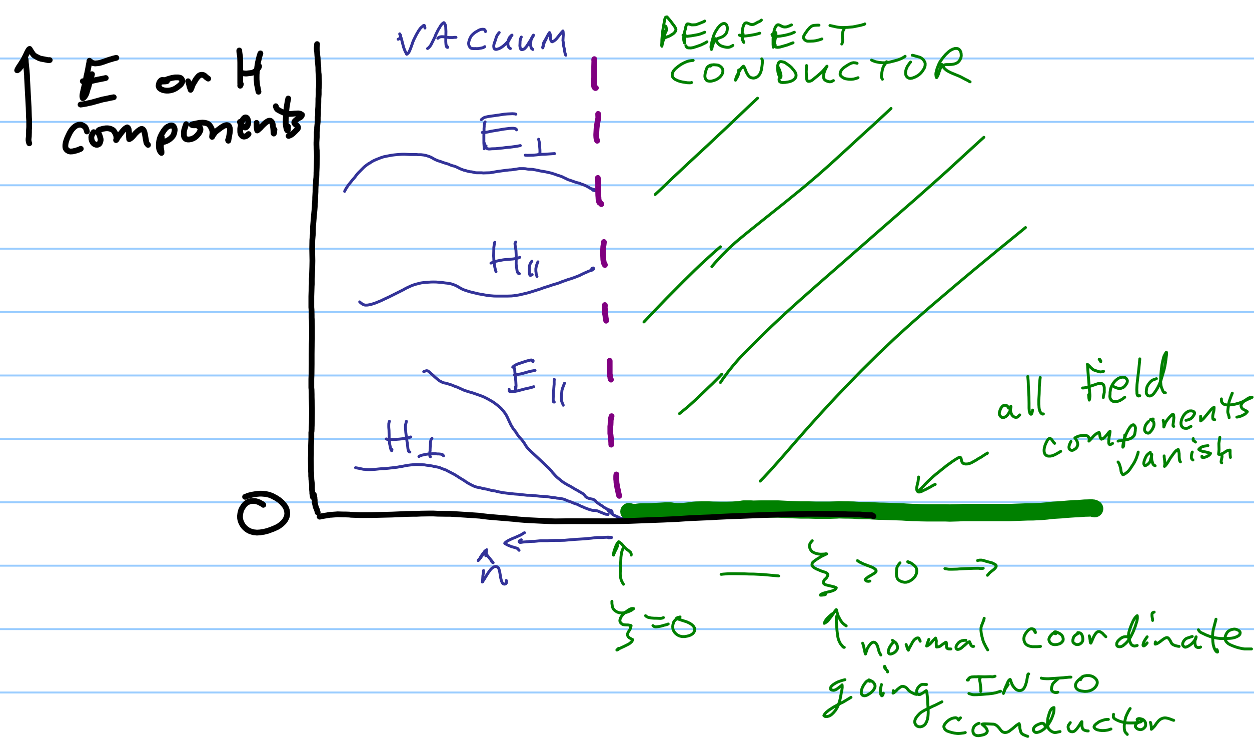
$$\Rightarrow \quad \vec{E}_c = 0 \qquad \vec{H}_c = 0$$
$$\vec{D}_c = 0 \qquad \vec{B}_c = 0$$

Then the other boundary conditions that hold at the surface are

$$\hat{n} \cdot (\vec{B} - \vec{B}_c) = 0 \quad \Rightarrow \vec{B} \cdot \hat{n} = 0 \text{ just outside}$$

and $$\hat{n} \times (\vec{E} - \vec{E}_c) = 0 \quad \Rightarrow \vec{E} \times \hat{n} = 0 = \vec{E}_{\parallel} \quad \text{"}$$

KEY CONCLUSION: just outside a perfectly conducting surface, the $\vec{E}$-field must be NORMAL and the $\vec{H}$-field must be TANGENTIAL



Good but imperfect conductors

Inside a real conductor, the fields are attenuated over a characteristic distance called the SKIN DEPTH,

$$\boxed{\delta \equiv \left(\frac{2}{\mu_c \omega \sigma} \right)^{1/2}}$$

To understand the fields near the surface of a real conductor more realistically, begin by assuming that to a good approximation there is a normal E_c-field and a tangential H_c-field just outside + inside.

Also, just treat monochromatic fields now, assuming t-dependences are all $e^{-i\omega t}$

$\Rightarrow$ Use Maxwell's equations inside the conductor to obtain the relationships among the fields in the transition zone, $0 \lesssim \xi \lesssim \delta$

$$\Rightarrow \quad \nabla \times \vec{H}_c = \vec{J} = \sigma \vec{E}_c \quad , \text{ neglecting } \frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \quad \vec{E}_c \simeq \frac{1}{\sigma} \nabla \times \vec{H}_c$$

and

$$\vec{H}_c = -\frac{i}{\mu_c \omega} \nabla \times \vec{E}_c$$

Again, $\hat{n} \equiv$ normal unit vec pointing out of conductor

ξ = distance into conductor from surface

$\hat{\xi} = -\hat{n}$

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Then $\vec{\nabla} \simeq -\hat{n}\frac{\partial}{\partial \xi}$, neglecting derivatives in directions parallel to surface

$$\Rightarrow \vec{E}_c = \frac{-1}{\sigma}\hat{n}\times\frac{\partial \vec{H}_c}{\partial \xi} \quad (1)$$

$$\vec{H}_c = \frac{i}{\mu_c \omega}\hat{n}\times\frac{\partial \vec{E}_c}{\partial \xi} \quad (2)$$

and so (2)

$$\Rightarrow \hat{n}\times\vec{H}_c = \frac{i}{\mu_c\omega}\hat{n}\times\left(\hat{n}\times\frac{\partial \vec{E}_c}{\partial \xi}\right) = \frac{i}{\mu_c\omega}\left(\hat{n}\left(\hat{n}\cdot\frac{\partial \vec{E}_c}{\partial\xi}\right)-\frac{\partial \vec{E}_c}{\partial \xi}\right)$$

whereas (1) plugged into (2)

$$\Rightarrow \vec{H}_c = \frac{i}{\mu_c\omega}\hat{n}\times\frac{\partial}{\partial\xi}\left(-\frac{1}{\sigma}\hat{n}\times\frac{\partial \vec{H}_c}{\partial \xi}\right) = -\frac{i}{\mu_c\omega}\sigma\hat{n}\times\frac{\partial^2}{\partial\xi^2}(\hat{n}\times\vec{H}_c)$$

or $$\hat{n}\times\vec{H}_c = \frac{i}{\mu_c\omega\sigma}\frac{\partial^2}{\partial\xi^2}(\hat{n}\times\vec{H}_c), \quad \left(\text{again using } \vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})\right)$$

Now multiply by $-i\mu_c\omega\sigma$

$$\Rightarrow \frac{\partial^2}{\partial\xi^2}(\hat{n}\times\vec{H}_c) + \underbrace{i\mu_c\omega\sigma}_{\text{this is } 2i/\delta^2}(\hat{n}\times\vec{H}_c) = 0$$

$$\left(\text{and } \hat{n}\cdot\vec{H}_c = 0\right)$$

Then this simple differential equation can be solved, giving

$$\vec{H}_c = \vec{H}_{\parallel}\, e^{i(i\mu_c\omega\sigma)^{1/2}\xi}$$

and recalling that $i^{1/2} = (e^{i\pi/2})^{1/2} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$,

we have $$\vec{H}_c = \vec{H}_{\parallel}\, e^{i\frac{\xi}{\delta}}\, e^{-\xi/\delta}$$

where $\vec{H}_{\parallel}$ = parallel $\vec{H}$-field in vacuum at surface

and $\delta = \left(\frac{2}{\mu_c\omega\sigma}\right)^{1/2} \Rightarrow \mu_c\omega = \frac{2}{\delta^2\sigma}$

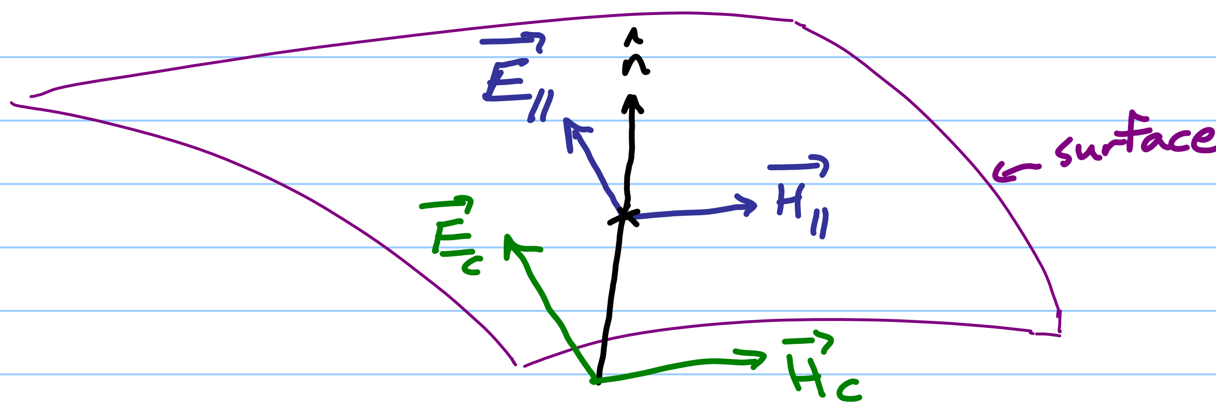
Then $\vec{E}_c = -\frac{1}{\sigma}\hat{n}\times\frac{\partial \vec{H}_c}{\partial \xi} = -\frac{1}{\sigma}\,\frac{i-1}{\delta}\,\hat{n}\times\vec{H}_{\parallel}\,e^{i\xi/\delta}e^{-\xi/\delta}$

which can be rewritten as:

$$\vec{E}_c = \left(\frac{\mu_c\omega}{2\sigma}\right)^{1/2}(1-i)\,\hat{n}\times\vec{H}_{\parallel}\,e^{i\xi/\delta}\,e^{-\xi/\delta}$$

= also tangential to surface
(orthogonal to both $\hat{n}$ and $\vec{H}_{\parallel}$)

Aside: there is also a small component of $\vec{E}_c$ normal to the surface, $\vec{E}_c\cdot\hat{n} \simeq \frac{i\omega\epsilon}{\sigma}E_{\perp}$, but this is one order smaller than $\vec{E}_{c,\parallel}$. See footnote, p. 355. Neglect $\vec{E}_c\cdot\hat{n}$ here.



So just outside the surface, (rather than 0 as for a perfect conductor)

$$\vec{E}_{\parallel} = \left(\frac{\mu_c\omega}{2\sigma}\right)^{1/2}(1-i)\,\hat{n}\times\vec{H}_{\parallel}$$

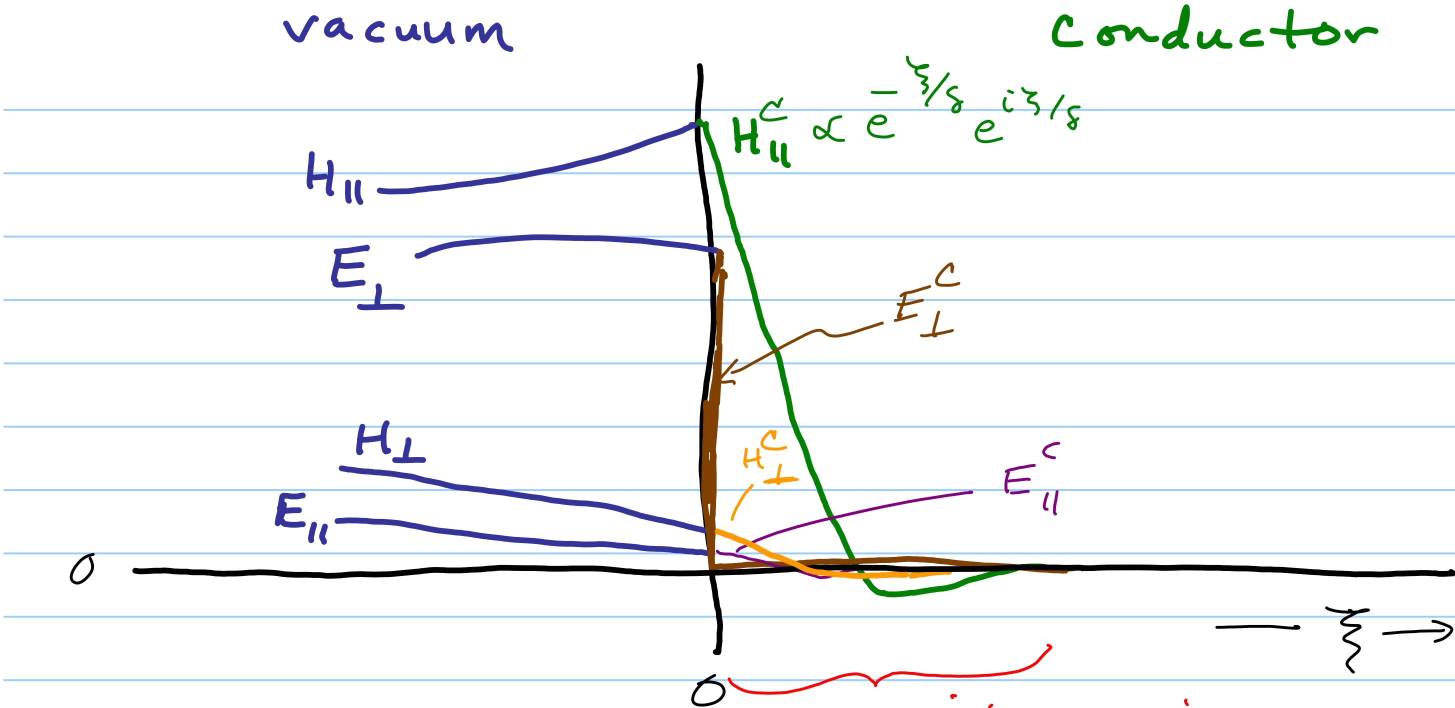
And so we have a time-averaged energy flow INTO the conductor, which per unit (area·time) equals

$$\frac{dP_{loss}}{da} = -\frac{1}{2}\mathrm{Re}\left(\hat{n}\cdot(\vec{E}_{\parallel}\times\vec{H}_{\parallel}^{*})\right) = \frac{1}{2}\mathrm{Re}\left(\frac{\mu_c\omega}{2\sigma}\right)^{1/2}(1-i)\,|\vec{H}_{\parallel}|^2$$

$$= \frac{1}{2}\,\frac{1}{\delta\sigma}\,|\vec{H}_{\parallel}|^2 = \frac{1}{2}\,\frac{\mu_c\omega\delta^2}{2\delta}\,|\vec{H}_{\parallel}|^2$$

giving $\boxed{\frac{dP_{loss}}{da} = \frac{\mu_c\omega\delta}{4}\,|\vec{H}_{\parallel}|^2}$

"Real" Conductor, Finite σ



transition region near conductor surface, over which fields decrease to 0, whose depth is of order

$$\Delta \xi \sim \delta$$

Ohm's Law implies that a current exists near the surface of the conductor, namely

$$\vec{J} = \sigma \vec{E}_c = \frac{1-i}{\delta} \hat{n} \times \vec{H}_{\parallel} e^{i\xi/\delta} e^{-\xi/\delta}$$

and if you visualize this as a true surface current, it can be expressed in terms of $\vec{H}_{\parallel}$ outside:

$$\vec{K}_{\text{effective}} = \int_0^\infty \vec{J} \, d\xi = \hat{n} \times \vec{H}_{\parallel}$$

Then it is convenient to express the power loss per unit area in terms of only this $\vec{K}_{\text{eff}}$:

$$\frac{dP_{\text{loss}}}{da} = \frac{1}{2\sigma\delta} |\vec{K}_{\text{eff}}|^2$$