Sec. 7.5 Frequency dispersion

A We can understand much using a simple classical model for $\epsilon(\omega)$

Assumptions

(i) material has low density, with $\mu \simeq \mu_0$

(ii) each e^- is bound harmonically, freq. ω_0, and is subjected to an E-field,

$$\vec{E}(\vec{x},t) = \vec{E}_0 e^{-i\omega t}$$

This assumes that $e^{i\vec{k}\cdot\vec{x}} \simeq$ constant over the distance scale where the e^- moves.

$$\Rightarrow \quad m\left(\ddot{\vec{x}} + \gamma \dot{\vec{x}} + \omega_0^2 \vec{x}\right) = -e\vec{E}_0 e^{-i\omega t}$$

phenomenological damping term

$\Rightarrow$ Look for a solution at the driving frequency, namely

$$\vec{x}(t) = \vec{x}_0 e^{-i\omega t}$$

plug in

$$\Rightarrow \quad m\cancel{e^{-i\omega t}}\left(-\omega^2 - i\gamma\omega + \omega_0^2\right)\vec{x}_0 = -eE_0\cancel{e^{-i\omega t}}$$

hence

$$\vec{x}_0 = -\frac{e}{m}\,\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}\,\vec{E}_0$$

and the induced electric dipole moment is

$$\vec{p} = -e\vec{x} = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \vec{E}$$

- In this model, imagine each e^- behaving as a classical damped oscillator of frequency ω_0

- Suppose there are $N \frac{\text{molecules}}{\text{unit volume}}$, each with Z electrons with different frequencies ω_j and damping constants γ_j.

- Call f_j the number of those Z electrons whose frequency is ω_j, i.e. $\boxed{\sum_j f_j = Z}$

$\Rightarrow$ The POLARIZATION is $\vec{P} = N\vec{p}$

or

$$\vec{P} = \frac{Ne^2}{m} \left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \right) \vec{E} \equiv \epsilon_0 \chi_e \vec{E}$$

and therefore

$$\boxed{\frac{\epsilon(\omega)}{\epsilon_0} + 1 = 1 + \chi_e = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}}$$

Quantum Derivation

See Fano & Rau, Atomic collisions & spectra, p. 29

Let the applied electric field be

$$\vec{E} = \hat{x} E_0 \left(e^{-i\omega t} + e^{i\omega t}\right)$$

$\Rightarrow$ the potential energy is $V = -exE_0\left(e^{-i\omega t} + e^{i\omega t}\right)$

1st-order time-dependent perturbation theory states that

energy of j^{th} atomic level

$$\Psi = \Psi^{ground} + \sum_j \Psi_j \, e^{-i\epsilon_j t/\hbar} \, c_j(t)$$

where $c_j(t) = \frac{eE_0}{\hbar} \langle \Psi_j | x | \Psi^{ground} \rangle$

$$\times \left[\frac{e^{i(\omega + \omega_j - \frac{1}{2} i\gamma_j)t}}{\omega_j + \omega - \frac{1}{2} i\gamma_j} + \frac{e^{-i(\omega - \omega_j + \frac{i\gamma_j}{2})t}}{\omega_j - \omega - \frac{1}{2} i\gamma_j} \right]$$

where γ_j = decay rate of excited state Ψ_j

and $\frac{\epsilon_j - \epsilon_{ground}}{\hbar} = \omega_j - \frac{1}{2} i\gamma_j$

simplified notation: call $\Psi^{ground} \longrightarrow |0\rangle$

$\Psi_j \longrightarrow |j\rangle$

Then the mean dipole moment is:

$$\langle p \rangle = e\langle x \rangle = e\sum_j \left\{ \langle 0|x|j\rangle c_j(t)\, e^{-i(\omega_j - \frac{i\gamma_j}{2})t} + c_j^*(t)\, e^{i(\omega_j + \frac{i\gamma_j}{2})t} \langle j|x|0\rangle \right\}$$

$$\Rightarrow \langle p \rangle = \frac{e^2 E_0}{\hbar} \sum_j \langle 0|x|j\rangle \left(\frac{e^{i\omega t}}{\omega_j^2 - \omega^2 + i\omega\gamma_j} + \frac{e^{-i\omega t}}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right)$$

$$\equiv E_0\, \alpha(\omega) e^{-i\omega t} + E_0\, \alpha^*(\omega) e^{i\omega t}$$

and the polarizability is

$$\boxed{\alpha(\omega) = \frac{e^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}}$$

where the quantal oscillator strength is

$$\boxed{f_j = \frac{2m\omega_j}{\hbar} |\langle j|x|0\rangle|^2}$$

which is dimensionless

Aside: can prove in nonrelativistic QM that

$$\sum_j f_j = Z,$$

which is the Thomas – Reiche – Kuhn sum rule.

– see Fano + Cooper, Rev. Mod. Phys. 40, 441 (1968).

(B) Anomalous Dispersion

Recall, from Chap. 4, $\chi_e = \dfrac{\alpha N}{1 - \frac{1}{3}\alpha N} \simeq \alpha N$

And typically $\gamma_j \ll \omega_j$, so damping is not very important except when ω is near resonance, $\omega_j^2 \simeq \omega^2$

First look at two limits:

Low ω

$$\frac{\epsilon(\omega)}{\epsilon_0} \xrightarrow[\omega \to 0]{} 1 + \frac{e^2 N}{\epsilon_0 m} \sum_{j>0} \frac{f_j}{\omega_j^2} + \underbrace{\frac{i e^2 N}{\epsilon_0 m} \frac{f_0}{\omega(\gamma_0 - i\omega)}}$$

this term is present if an $\omega_0 = 0$ resonant frequency exists

High ω

$$\frac{\epsilon(\omega)}{\epsilon_0} \xrightarrow[\substack{\omega \to \infty \\ \omega \gg \text{all } \omega_j}]{} 1 - \frac{N e^2 Z}{m \epsilon_0 \omega^2} \equiv 1 - \frac{\omega_p^2}{\omega^2}$$

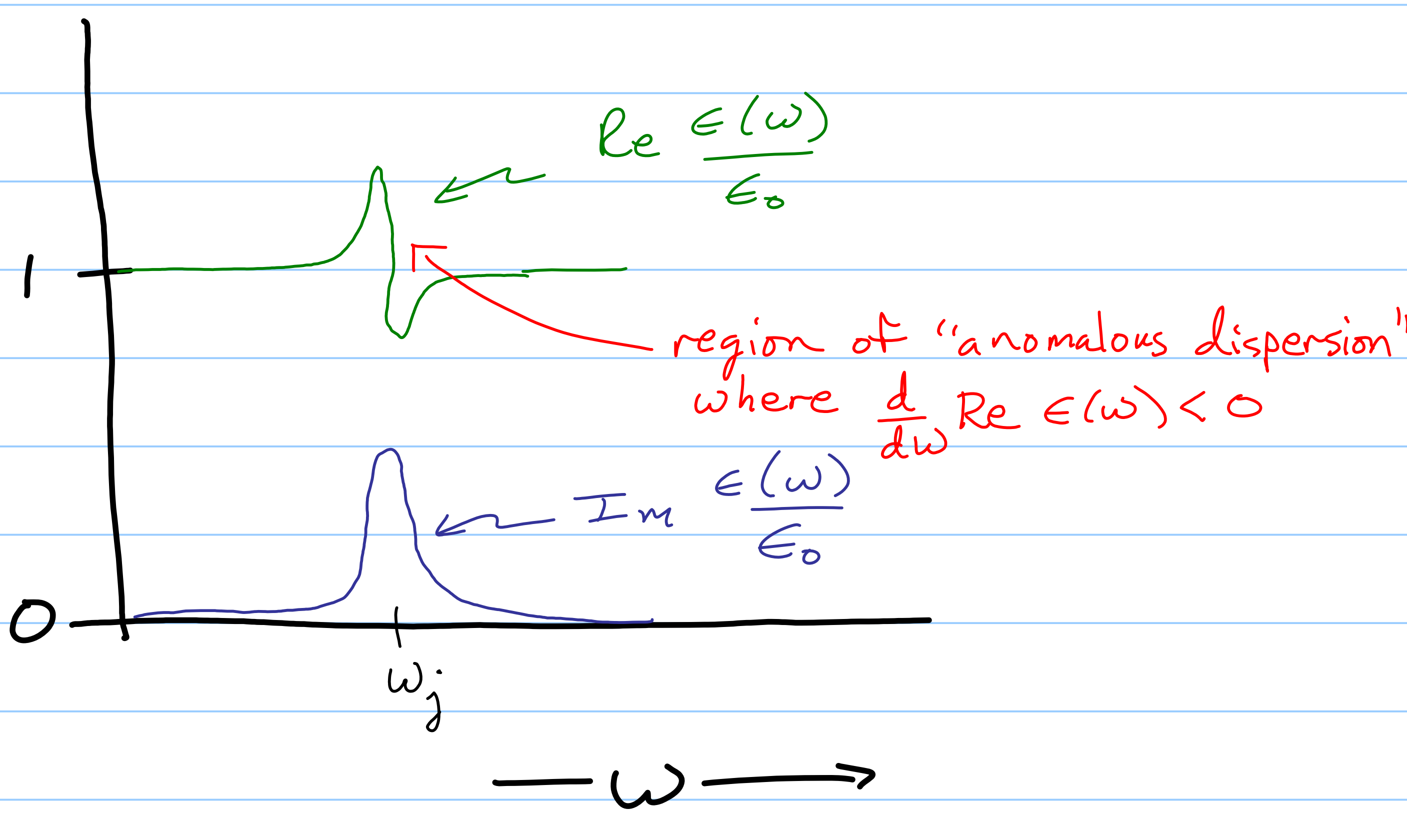
where $\omega_p^2 \equiv \dfrac{N Z e^2}{\epsilon_0 m}$ is called the PLASMA FREQUENCY (squared)

Intermediate ω near resonant ω_j:

$\Rightarrow$ for $\omega \simeq \omega_j$,

$$\alpha(\omega) \simeq \frac{e^2}{m} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}$$

$$\boxed{\begin{aligned} \Rightarrow \mathrm{Re}\,\alpha(\omega) &\simeq \frac{e^2}{m} f_j \frac{\omega_j^2 - \omega^2}{(\omega_j^2-\omega^2)^2 + \gamma_j^2\omega^2} \\ \mathrm{Im}\,\alpha(\omega) &= \frac{e^2}{m} f_j \frac{\gamma_j \omega}{(\omega_j^2-\omega^2)^2 + \gamma_j^2\omega^2} \end{aligned}}$$



On your own, read Jackson, Sec. 7.5(E) pp 314-316, on the importance of the WATER WINDOW, where water has virtually NO absorption between $\nu \simeq 4\times10^{14}$ Hz $- 8\times10^{14}$ Hz in the visible region!

Sec. 7.5 Wavepackets, dispersion, group velocity

Recall that the monochromatic plane waves discussed up to now are a mathematical abstraction, a δ-function in $\vec{k}$-space and in ω $\Rightarrow$ they extend over all space and exist for all time

Of course, any EM waves occurring in nature will have finite extent & duration
$\Rightarrow$ these require a superposition of frequencies

Moreover, in general $\epsilon(\omega)$ is complex, which has the following implications:

(1) The phase velocity depends on ω
$\Longrightarrow$ different frequency components in a wavepacket disperse, i.e. drift out of phase

(2) Energy flows at the group velocity, rather than the phase velocity c/n.

(3) For $\text{Im}\,\epsilon(\omega) \neq 0$ there is also dissipation

Sec 7.8

For starters, consider the 1D wave eqn,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0$$

This equation has monochromatic plane wave solutions, $u(x,t) = u_0 e^{i(kx-\omega t)}$,

where $\omega = kv$, or in general $\dfrac{\omega(k)}{k} = v(k)$
= phase velocity

Assume for now that μ, ϵ are real

$\Rightarrow$ k, $\omega(k)$ are real

$\Rightarrow$ DISSIPATION is ignored for now

Next, Form a general wavepacket solution by superposing monochromatic plane waves,

i.e. $$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx - i\omega(k)t}\, dk$$

In general, to determine $A(k)$, we must specify BOTH $u(x,0)$ AND $\left.\dfrac{\partial u}{\partial t}(x,t)\right|_{t=0} \equiv u_t(x,0)$

because this wave equation is 2^{nd}-order in t;

(this is in contrast to the Schrödinger equation, first-order in t, which only requires knowledge of $\psi(x,0)$ to determine $\psi(x,t)$ at all later t.)

Then for any real wavepacket $u(x,t)$ we must have:

$$\Rightarrow \quad u(x,0) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk + c.c.$$

and
$$u_t(x,0) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(-i\omega(k)\right) A(k)\, e^{ikx}\, dk + c.c.$$

Next apply "Fourier's Trick" to find $A(k)$,

$$\Rightarrow \text{set} \quad C_{k'} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-ik'x}\, dx = \frac{A(k') + A^*(-k')}{2}$$

and $$D_{k'} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,0)\, e^{-ik'x}\, dx$$

$$= -\frac{i}{2}\left(\omega(k') A(k') - \omega^*(-k') A^*(-k') \right)$$

$$= \frac{1}{2} i\omega(k') \left(-A(k') + A^*(-k') \right),$$

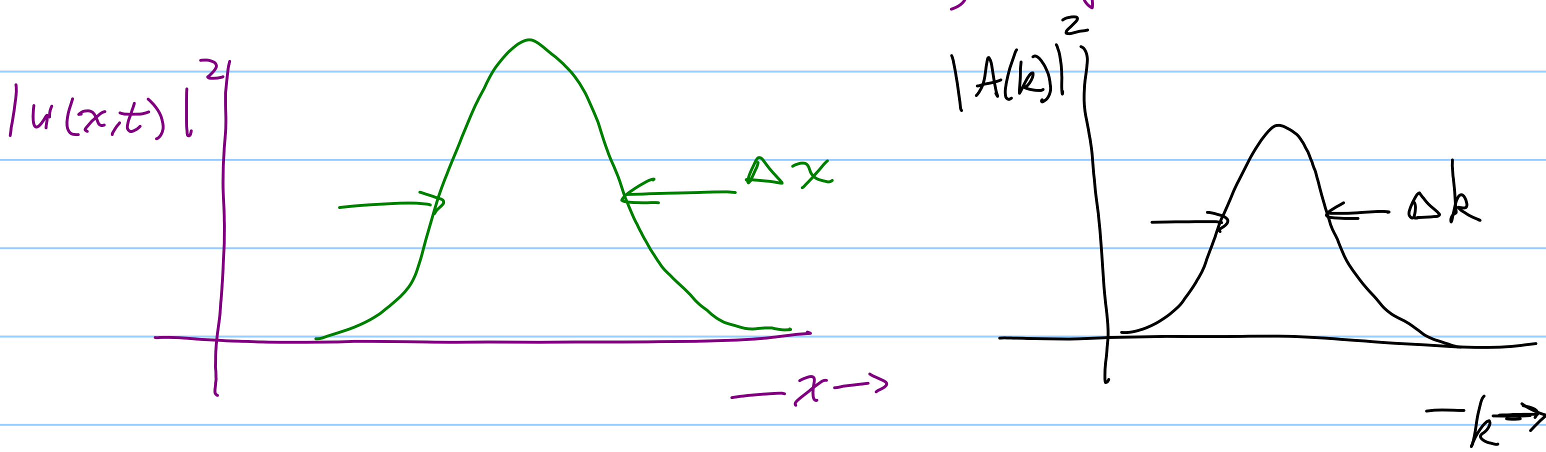
(since frequencies here are real and direction-independent)

These 2 equations can be solved, giving

$$C_k + \frac{D_k}{i\omega(k)} = A^*(-k) \qquad C_k - \frac{D_k}{i\omega(k)} = A(k)$$

$$\Rightarrow \boxed{A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left[u(x,0) + \frac{i}{\omega(k)} u_t(x,0) \right] dx}$$

These waves obey an uncertainty relation, both in time and in space, e.g.



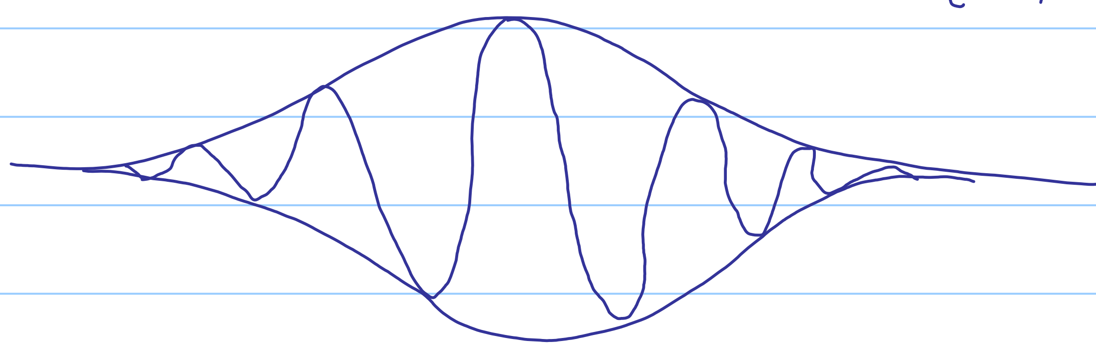
Then from Fourier analysis, $\Delta k \Delta x \geq \frac{1}{2}$ with equality only for a Gaussian pulse

To proceed farther, consider an $A(k)$ that is peaked around some value, say, k_0, and Taylor expand $\omega(k)$ about k_0:

$$\omega(k) \simeq \omega_0 + \left.\frac{d\omega}{dk}\right|_{k_0}(k-k_0) + \frac{1}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k_0}(k-k_0)^2 + \dots$$

Then the following model can be solved exactly: suppose that the $t=0$ solution is

$$u(x,0) = e^{-x^2/\gamma^2}\cos(k_0 x),$$ and neglect $u_t(x,0) \simeq 0$



(This actually corresponds to 2 different pulses moving in the $\pm x$-directions)

Then $$A(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} e^{-x^2/\gamma^2} \cos(k_0 x)$$

$$= \frac{1}{2\sqrt{2\pi}} \int dx \left\{ e^{-i(k-k_0)x - x^2/\gamma^2} + e^{-i(k+k_0)x - \frac{x^2}{\gamma^2}} \right\}$$

To do this integral, complete the square, i.e. set $$\frac{x^2}{\gamma^2} + i(k-k_0)x = \frac{1}{\gamma^2}\left[x + \frac{i(k-k_0)\gamma^2}{2}\right]^2 + \frac{(k-k_0)^2\gamma^2}{4}$$

and change variables to $x' = x + \frac{i(k-k_0)\gamma^2}{2}$ whereby

$$A(k) = \frac{1}{2\sqrt{2\pi}} e^{-(k-k_0)^2\frac{\gamma^2}{4}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{x'^2}{\gamma^2}} dx'}_{=\gamma\sqrt{\pi}}$$

$$+ (k \to -k_0)$$

Now plug into the time-dependent wavepacket integral, with $\omega(k) \simeq \omega_0 + v_g(k-k_0) + \frac{1}{2}\omega_0 a^2 (k-k_0)^2$

with $v_g \equiv \left.\frac{d\omega}{dk}\right|_{k_0}$, $a^2 \equiv \frac{1}{\omega_0} \left.\frac{d^2\omega}{dk^2}\right|_{k_0}$

$\Rightarrow$ the final wavepacket is

$$\boxed{u(x,t) = \frac{e^{ik_0x - i\omega_0 t}\gamma}{4} \frac{e^{-(x-v_g t)^2/(\gamma^2 + 4ia^2\omega_0 t)}}{\sqrt{\gamma^2 + 4ia^2\omega_0 t}}}$$

and $|u(x,t)|^2 = \frac{\gamma^2}{16} \frac{e^{-(x-v_g t)^2\left(\frac{2\gamma^2}{\gamma^4 + 16a^4\omega_0^2 t^2}\right)}}{(\gamma^4 + 16a^4 t^2 \omega_0^2)^{1/2}}$

And based on this expression, we conclude that:

① In media where μ, ϵ, and n are constants, independent of ω, k, the phase and group velocities are equal since

$\omega = \frac{kc}{n} \Rightarrow v_g = \frac{d\omega}{dk} = \frac{c}{n} = v_{phase}$

(2) In media where $n \to n(\omega)$, $k = \frac{n(\omega)\omega}{c}$

$$\Rightarrow \boxed{v_g = \frac{1}{\left.\frac{dk}{d\omega}\right|_{\omega_0}} = \frac{c}{n + \omega \frac{dn}{d\omega}}}$$

NORMAL DISPERSION occurs when $n \geq 1$ and $\frac{dn}{d\omega} > 0$

$$\Rightarrow v_g < c$$

ANOMALOUS DISPERSION has $\frac{dn}{d\omega} < 0$ so that $v_g > c$ could arise. This apparent violation of special relativity (causality) is traceable to a failure of our Taylor approximation to $\omega(k)$.

(3) The pulse width spreads in general, as time increases.

When the above approximations are valid, under conditions of normal dispersion, the energy propagates at speed $v_g \leq c$.