Physics 631 Electrodynamics II at Purdue

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= Course homepage

> *From a long view of the history of mankind—seen from, say, ten thousand years from now—there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade.*
>
> — Richard P. Feynman, In The Feynman Lectures on Physics (1964), Vol. 2, page 1-11.

Begin with the speed of light, $c = 299792458 \ \frac{m}{s}$ EXACTLY

= a defined constant, the same in all inertial reference frames

MAXWELL EQUATIONS

SI units		Gaussian units
$\nabla\cdot\vec{B} = 0$ $\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$	Homogeneous, no sources	$\nabla\cdot\vec{B} = 0$ $\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$
$\nabla\cdot\vec{D} = \rho$ $\nabla\times\vec{H} - \frac{\partial\vec{D}}{\partial t} = \vec{J}$	Inhomogeneous, source eqns	$\nabla\cdot\vec{D} = 4\pi\rho$ $\nabla\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}$
$\vec{D} = \epsilon_0\vec{E} + \vec{P}$ $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$	in macroscopic media	$\vec{D} = \vec{E} + 4\pi\vec{P}$ $\vec{H} = \vec{B} - 4\pi\vec{M}$
$\frac{\vec{F}}{q} = \vec{E} + \vec{v}\times\vec{B}$	Lorentz force $\vec{F}$ on point charge q $\vec{v} = d\vec{x}/dt$	$\frac{\vec{F}}{q} = \vec{E} + \frac{\vec{v}}{c}\times\vec{B}$
$\frac{1}{2m}(\vec{p} - q\vec{A})^2$	Kinetic energy of point charge q with momentum $\vec{p}$ in a vector potential $\vec{A}$	$\frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2$
$\vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t}$ $\vec{B} = \nabla\times\vec{A}$	Potentials	$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ $\vec{B} = \nabla\times\vec{A}$

To interconvert equations or specific quantities, Gaussian ⟷ SI units see Jackson Appendix, p. 775

Chapter 8 Waveguides & resonant cavities

Here we treat EM waves propagating or resonating inside metal cavities or waveguides, beginning with a discussion of the behavior of the fields at or near a boundary.

First, consider a PERFECT conductor

(On your own, read Jackson Sec 5.18, pp 218 – 221)

Qualitative concept: we visualize that in a "perfect conductor," charges are so free to move that they can respond instantly to any fields inside the material.

$\Rightarrow$ $\vec{E} = 0$ inside a perfect conductor, just as we discussed for electrostatics.

$\Rightarrow$ instantaneously, the surface charges Σ will move to obey $\boxed{\vec{D} \cdot \hat{n} = \Sigma}$

where $\vec{D}$ = electric displacement just outside the surface,

and $\hat{n}$ = surface unit normal pointing outwards from conductor

and similarly, surface currents $\vec{K}$ can respond so freely and quickly that $\boxed{\hat{n} \times \vec{H} = \vec{K}}$

where $\vec{H}$ = magnetic field just outside

And the fields all vanish exactly inside for a perfect conductor,

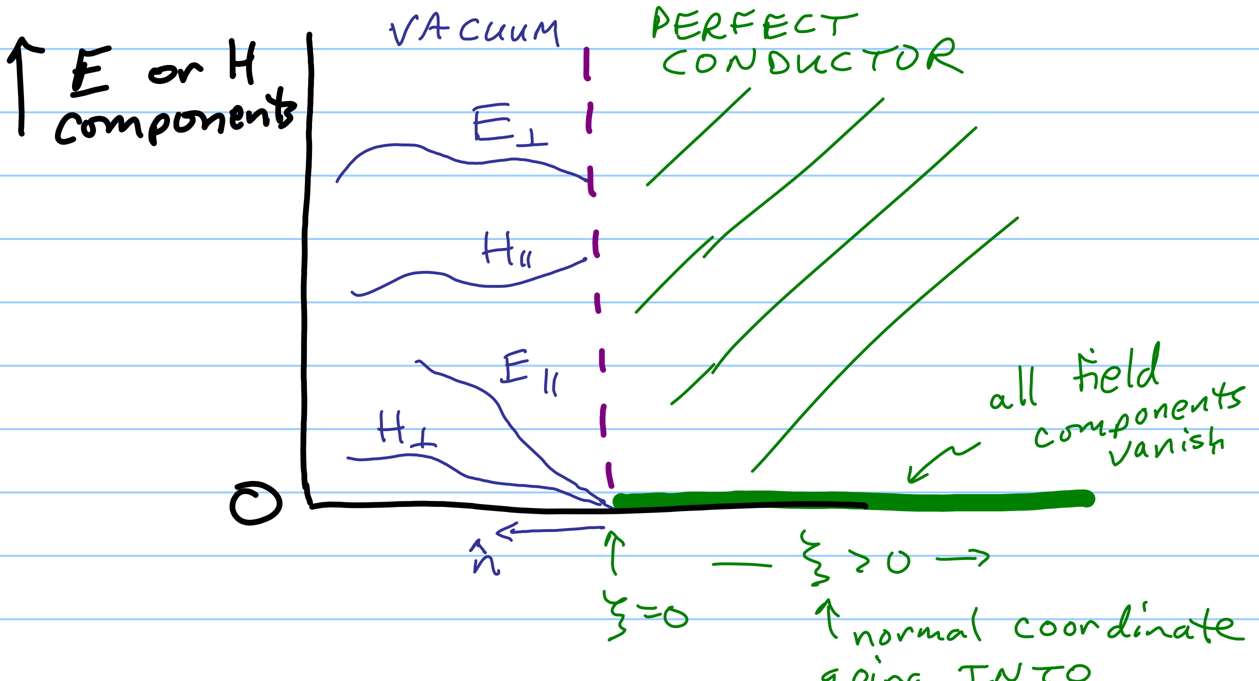
$$\Rightarrow \quad \vec{E}_c = 0 \qquad \vec{H}_c = 0$$
$$\vec{D}_c = 0 \qquad \vec{B}_c = 0$$

Then the other boundary conditions that hold at the surface are

$$\hat{n} \cdot (\vec{B} - \vec{B}_c) = 0 \quad \Rightarrow \vec{B} \cdot \hat{n} = 0 \text{ just outside}$$

and $\hat{n} \times (\vec{E} - \vec{E}_c) = 0 \quad \Rightarrow \vec{E} \times \hat{n} = 0 = \vec{E}_{\parallel}$ "

KEY CONCLUSION: just outside a perfectly conducting surface, the $\vec{E}$-field must be NORMAL and the $\vec{H}$-field must be TANGENTIAL



Good but imperfect conductors

Inside a real conductor, the fields are attenuated over a characteristic distance called the SKIN DEPTH,

$$\boxed{\delta \equiv \left(\frac{2}{\mu_c \omega \sigma} \right)^{1/2}}$$

To understand the fields near the surface of a real conductor more realistically, begin by assuming that to a good approximation there is a normal E_c-field and a tangential H_c-field just outside + inside.

Also, just treat monochromatic fields now, assuming t-dependences are all $e^{-i\omega t}$

$\Rightarrow$ Use Maxwell's equations inside the conductor to obtain the relationships among the fields in the transition zone, $0 \lesssim \xi \lesssim \delta$

$$\Rightarrow \quad \nabla \times \vec{H}_c = \vec{J} = \sigma \vec{E}_c \quad , \text{ neglecting } \frac{\partial D}{\partial t}$$

$$\Rightarrow \quad \vec{E}_c \simeq \frac{1}{\sigma} \nabla \times \vec{H}_c$$

and

$$\vec{H}_c = -\frac{i}{\mu_c \omega} \nabla \times \vec{E}_c$$

Again, $\hat{n} \equiv$ normal unit vec pointing out of conductor

ξ = distance into conductor from surface

$\hat{\xi} = -\hat{n}$

Then $\vec{\nabla} \simeq -\hat{n}\frac{\partial}{\partial \xi}$, neglecting derivatives in directions parallel to surface

$$\Rightarrow \quad \vec{E}_c = -\frac{1}{\sigma}\hat{n} \times \frac{\partial \vec{H}_c}{\partial \xi} \qquad (1)$$

$$\vec{H}_c = \frac{i}{\mu_c \omega}\hat{n} \times \frac{\partial \vec{E}_c}{\partial \xi} \qquad (2)$$

and so (2)

$$\Rightarrow \quad \hat{n} \times \vec{H}_c = \frac{i}{\mu_c \omega}\hat{n} \times \left(\hat{n} \times \frac{\partial \vec{E}_c}{\partial \xi}\right) = \frac{i}{\mu_c \omega}\left(\hat{n}\left(\hat{n} \cdot \frac{\partial \vec{E}_c}{\partial \xi}\right) - \frac{\partial \vec{E}_c}{\partial \xi}\right)$$

whereas (1) plugged into (2)

$$\Rightarrow \quad \vec{H}_c = \frac{i}{\mu_c \omega}\hat{n} \times \frac{\partial}{\partial \xi}\left(-\frac{1}{\sigma}\hat{n} \times \frac{\partial \vec{H}_c}{\partial \xi}\right) = -\frac{i}{\mu_c \omega}\sigma \hat{n} \times \frac{\partial^2}{\partial \xi^2}(\hat{n} \times \vec{H}_c)$$

or $\hat{n} \times \vec{H}_c = \frac{i}{\mu_c \omega \sigma}\frac{\partial^2}{\partial \xi^2}(\hat{n} \times H_c)$, (again using $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$)

Now multiply by $-i\mu_c \omega \sigma$

$$\Rightarrow \quad \frac{\partial^2}{\partial \xi^2}(\hat{n} \times \vec{H}_c) + \underbrace{i\mu_c \omega \sigma}_{\text{this is } 2i/\delta^2}(\hat{n} \times \vec{H}_c) = 0$$

(and $\hat{n} \cdot \vec{H}_c = 0$)

Then this simple differential equation can be solved, giving $\vec{H}_c = \vec{H}_{\parallel}\, e^{i(i\mu_c \omega \sigma)^{1/2}\xi}$

and recalling that $i^{1/2} = (e^{i\pi/2})^{1/2} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$,

we have $\vec{H}_c = \vec{H}_{\parallel}\, e^{i\frac{\xi}{\delta}}\, e^{-\xi/\delta}$

where $\vec{H}_{\parallel}$ = parallel $\vec{H}$-field in vacuum at surface

and $\delta = \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2} \Rightarrow \mu_c \omega = \frac{2}{\delta^2 \sigma}$

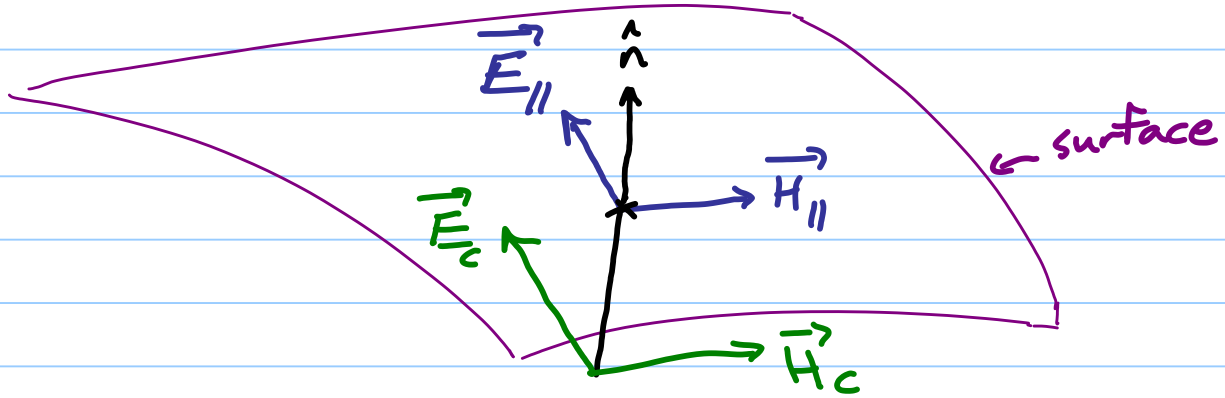
Then $\vec{E}_c = -\frac{1}{\sigma}\hat{n}\times\frac{\partial \vec{H}_c}{\partial \xi} = -\frac{1}{\sigma}\,\frac{i-1}{\delta}\,\hat{n}\times\vec{H}_{\|}\,e^{i\xi/\delta}e^{-\xi/\delta}$

which can be rewritten as:

$$\vec{E}_c = \left(\frac{\mu_c\,\omega}{2\sigma}\right)^{1/2}(1-i)\,\hat{n}\times\vec{H}_{\|}\,e^{i\xi/\delta}\,e^{-\xi/\delta}$$

= also tangential to surface (orthogonal to both $\hat{n}$ and $\vec{H}_{\|}$)

Aside: there is also a small component of $\vec{E}_c$ normal to the surface, $\vec{E}_c\cdot\hat{n}\simeq\frac{i\omega\epsilon}{\sigma}E_{\perp}$, but this is one order smaller than $\vec{E}_{c,\|}$. See footnote, p. 355. Neglect $\vec{E}_c\cdot\hat{n}$ here.



So just outside the surface, (rather than 0 as for a perfect conductor)

$$\vec{E}_{\|} = \left(\frac{\mu_c\,\omega}{2\sigma}\right)^{1/2}(1-i)\,\hat{n}\times\vec{H}_{\|}$$

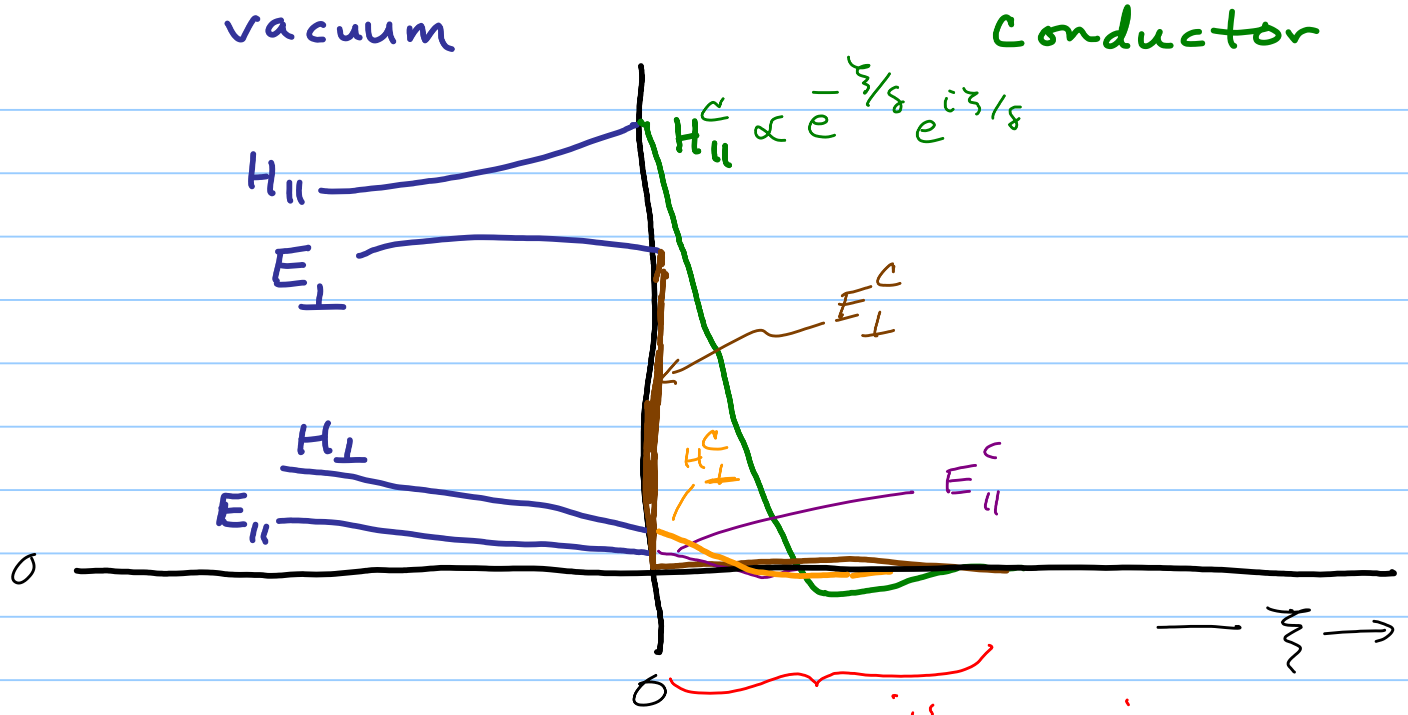
And so we have a time-averaged energy flow INTO the conductor, which per unit (area·time) equals

$$\frac{dP_{loss}}{da} = -\frac{1}{2}\mathrm{Re}\left(\hat{n}\cdot\left(\vec{E}_{\|}\times\vec{H}_{\|}^{*}\right)\right) = \frac{1}{2}\mathrm{Re}\left(\frac{\mu_c\,\omega}{2\sigma}\right)^{1/2}(1-i)\,|\vec{H}_{\|}|^2$$

$$= \frac{1}{2}\,\frac{1}{\delta\sigma}\,|\vec{H}_{\|}|^2 = \frac{1}{2}\,\frac{\mu_c\,\omega\,\delta^2}{2\delta}\,|\vec{H}_{\|}|^2$$

giving $\boxed{\frac{dP_{loss}}{da} = \frac{\mu_c\,\omega\,\delta}{4}\,|\vec{H}_{\|}|^2}$

"Real" Conductor, Finite σ



transition region near conductor surface, over which fields decrease to 0, whose depth is of order

$$\Delta\xi \sim \delta$$

Ohm's Law implies that a current exists near the surface of the conductor, namely

$$\vec{J} = \sigma \vec{E}_c = \frac{1-i}{\delta} \hat{n} \times \vec{H}_{\parallel} e^{i\xi/\delta} e^{-\xi/\delta}$$

and if you visualize this as a true surface current, it can be expressed in terms of $\vec{H}_{\parallel}$ outside:

$$\vec{K}_{\text{effective}} = \int_0^{\infty} \vec{J}\, d\xi = \hat{n} \times \vec{H}_{\parallel}$$

Then it is convenient to express the power loss per unit area in terms of only this $\vec{K}_{\text{eff}}$:

$$\frac{dP_{\text{loss}}}{da} = \frac{1}{2\sigma\delta} |\vec{K}_{\text{eff}}|^2$$

8.2 cylindrical waveguides + cavities

Goal: treat the propagation of EM waves in regions bounded by conductors

Here "cylindrical" means a constant cross section, either

1) cylinder with no end caps, which supports traveling waves

or

2) cylinder with end caps, which supports standing waves

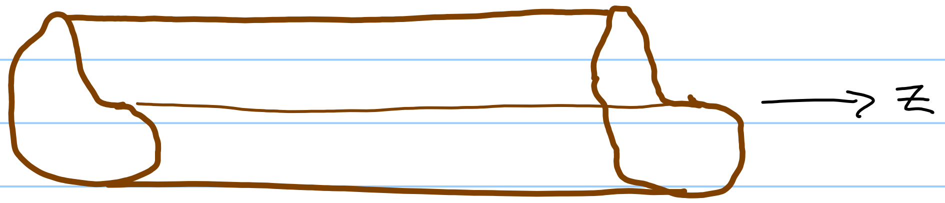
Inside, with no sources, Maxwell's equations for harmonic time dependences are:

$$\nabla \times \vec{E} = i\omega \vec{B} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = -i\mu\epsilon\omega \vec{E} \qquad \nabla \cdot \vec{E} = 0$$

So if we assume the interior is a uniform, nondissipative medium,

$$\Rightarrow \left(\nabla^2 + \mu\epsilon\omega^2\right)\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$



Look for travelling wave solutions:

i.e. try $\vec{E}(\vec{x},t) = \vec{E}(x,y)\, e^{\pm ikz - i\omega t}$

$$\vec{B}(\vec{x},t) = \vec{B}(x,y)\, e^{\pm ikz - i\omega t}$$

Plugging this in gives the following equation for the 2D amplitudes:

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right]\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$

where the transverse Laplacian is

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla^2 - \frac{\partial^2}{\partial z^2}$$

Next express the fields into a parallel component $\vec{E}_z$ and a transverse component $\vec{E}_t$

$\Rightarrow$ $\vec{E} = \vec{E}_z + \vec{E}_t$ where $\vec{E}_z = E_z \hat{z}$

and $\vec{E}_z = \hat{z}(\hat{z}\cdot\vec{E})$

$$\vec{E}_t = (\hat{z}\times\vec{E})\times\hat{z}$$

(and do this for $\vec{B}$ also)

Then in this notation Maxwell's equations are:

$$\boxed{\begin{aligned}\nabla_t\cdot\vec{E}_t + \frac{\partial E_z}{\partial z} = 0\\ \nabla_t\cdot\vec{B}_t + \frac{\partial B_z}{\partial z} = 0\end{aligned}}$$

and

$$\left(\nabla_t + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\vec{E}_z + \vec{E}_t\right) = i\omega\left(\vec{B}_z + \vec{B}_t\right)$$

Now break this up into longitudinal ($\hat{z}$) and transverse (t) components:

LONGITUDINAL – take $\hat{z}\cdot(\nabla\times\vec{E})$

$$\Rightarrow \boxed{\hat{z}\cdot(\nabla_t\times\vec{E}_t) = i\omega\,\hat{z}\cdot\vec{B}_z}$$

TRANSVERSE – take $\hat{z}\times(\nabla\times\vec{E}) = i\omega\hat{z}\times\vec{B}$

$$\Rightarrow \hat{z}\times(\nabla_t\times\vec{E}) + \hat{z}\times\left(\hat{z}\frac{\partial}{\partial z}\times\vec{E}\right)$$

$$= \hat{z}\times(\nabla_t\times\vec{E}_z) + \hat{z}\times\left(\hat{z}\times\frac{\partial\vec{E}_t}{\partial z}\right)$$

Next use $\nabla\times(\vec{a}\,\psi) = \nabla\psi\times\vec{a} + \psi\nabla\times\vec{a}$

$$\Rightarrow \nabla_t\times(\hat{z}E_z) = (\nabla_t E_z)\times\hat{z}$$

and

$$\hat{z}\times\left(\hat{z}\times\frac{\partial\vec{E}_t}{\partial z}\right) = \hat{z}\left(\cancelto{0}{\hat{z}\cdot\frac{\partial\vec{E}_t}{\partial z}}\right) - \frac{\partial\vec{E}_t}{\partial z}\left(\cancelto{1}{\hat{z}\cdot\hat{z}}\right)$$

whereby

$$\hat{z}\times(\nabla\times\vec{E}) = -\hat{z}\times(\nabla_t E_z) - \frac{\partial\vec{E}_t}{\partial z} = i\omega\hat{z}\times\vec{B}$$

which can be further simplified to

$$\boxed{\overrightarrow{\nabla_t E}_z - \frac{\partial}{\partial z}\vec{E}_t = i\omega\,\hat{z}\times\vec{B}_t} \quad ①$$

and similarly,

$$\boxed{\overrightarrow{\nabla_t B}_z - \frac{\partial}{\partial z}\vec{B}_t = -i\mu\epsilon\omega\,\hat{z}\times\vec{E}_t} \quad ②$$

and

$$\boxed{\hat{z}\cdot(\nabla_t\times\vec{B}_t) = -i\mu\epsilon\omega E_z} \quad ③$$

If now the entire z-dependence is taken to be e^{ikz}, as expected for propagation along the cylinder, we get

$$\vec{\nabla}_t E_z - ik\vec{E}_t = i\omega\hat{z}\times\vec{B}_t$$

plug in here

and $$\vec{\nabla}_t B_z - ik\vec{B}_t = -i\mu\epsilon\omega\,\hat{z}\times\vec{E}_t$$

or $$\vec{B}_t = \frac{1}{ik}\left\{\vec{\nabla}_t B_z + i\mu\epsilon\omega\,\hat{z}\times\vec{E}_t\right\}$$

$$\Rightarrow \vec{\nabla}_t E_z - ik\vec{E}_t = i\omega\hat{z}\times\frac{\vec{\nabla}_t B_z}{ik} + i\omega\left(\frac{\mu\epsilon\omega}{k}\right)\hat{z}\times(\hat{z}\times\vec{E}_t)$$

$$= \frac{\omega}{k}\hat{z}\times\vec{\nabla}_t B_z - i\frac{\omega^2\mu\epsilon}{k}\vec{E}_t$$

and so $$\vec{E}_t\left(-ik + i\frac{\omega^2\mu\epsilon}{k}\right) = -\vec{\nabla}_t E_z + \frac{\omega}{k}\hat{z}\times\vec{\nabla}_t B_z$$

or $$\boxed{\vec{E}_t = \frac{ik}{k^2-\omega^2\mu\epsilon}\left\{-\vec{\nabla}_t E_z + \frac{\omega}{k}\hat{z}\times\vec{\nabla}_t B_z\right\}} \quad ④$$

and $$\boxed{\vec{B}_t = \frac{ik}{k^2-\omega^2\mu\epsilon}\left\{-\vec{\nabla}_t B_z - \frac{\mu\epsilon\omega}{k}\hat{z}\times\vec{\nabla}_t E_z\right\}} \quad ⑤$$

The point is that knowledge of E_z, B_z will imply how to get $\vec{E}_t, \vec{B}_t$!

TEM modes Suppose we now look for transverse electromagnetic waves, for which $E_z = 0 = B_z$, i.e. $\vec{E}_{TEM} = \vec{E}_t$, $\vec{B}_{TEM} = \vec{B}_t$

$$\Rightarrow \boxed{\begin{aligned}\nabla_t\cdot\vec{E}_{TEM} &= 0\\ \nabla_t\times\vec{E}_{TEM} &= 0\end{aligned}}$$

← These are the same as the equations of a 2-dim'l problem in electrostatics, with $\rho = 0$!

Conclusions

(1) NO TEM modes can exist in a hollow waveguide bounded by a single conducting surface. This is because the uniqueness theorem for Laplace's equation which implies here that there is one solution having $\Phi = \text{constant}$, $\vec{E}_t = 0$, everywhere inside a conducting shell = waveguide boundary surface

(2) Referring to ① and ② above, and recalling that the z-dependence is e^{ikz}, we have:

$$\vec{E}_t = -\frac{\omega}{k}\,\hat{z}\times\vec{B}_t$$

$$\vec{B}_t = \underline{\frac{\mu\epsilon\omega}{k}\,\hat{z}\times\vec{E}_t}$$

plug in here!

So we see a consistency relation,

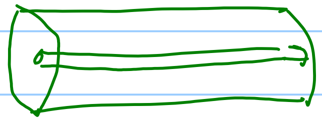
$$\vec{E}_t = -\frac{\omega}{k}\,\hat{z}\times(\hat{z}\times\vec{E}_t)\,\frac{\mu\epsilon\omega}{k} = \frac{\omega^2}{k^2}\mu\epsilon\,\vec{E}_t$$

$$\Rightarrow \boxed{\frac{\omega^2}{k^2} = \frac{1}{\mu\epsilon}}\,, \quad \text{and} \quad \boxed{\vec{B}_t = \sqrt{\mu\epsilon}\,\hat{k}\times\vec{E}_t}$$

$\Rightarrow$ These coincide with the equations we found before for propagation in a homogeneous medium!

i.e. $\vec{E}_t(\vec{x},t) = \vec{E}_t^{(0)}\,e^{\pm ikz - i\omega t}$, $k = \frac{\omega}{c}\left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)^{1/2}$

(3) The TEM mode is the simplest mode in some waveguides with 2 or more conductors, such as a coaxial cable



or a parallel transmission line,

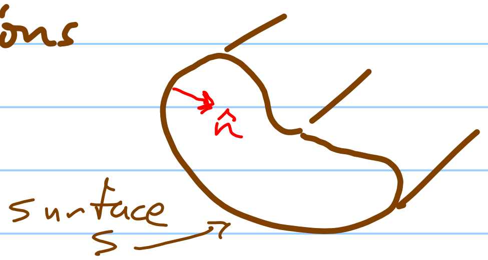
(see problems 8.2, 8.3)

Modes allowed for a perfect cylindrical conductor

recall the boundary conditions

$$\hat{n}\times\vec{E}=0$$

$$\hat{n}\cdot\vec{B}=0$$



These are the relevant BCs because the fields vanish in the conductor, and notice that $\hat{n}\cdot\vec{D}$ and $\hat{n}\times\vec{H}$ could be nonzero outside the conductor because there could be surface charges and/or currents.

$\Rightarrow$ $E_z\Big|_S = 0$, and another parallel component of $\vec{E}$ that must vanish is $(\hat{n}\times\hat{z})\cdot\vec{E}$
'' $\hat{\rho}\cdot\vec{E}$

and $\hat{n}\cdot\vec{B}=0 \Rightarrow$ consider Eq. 8.24(b),

$$\hat{z}\cdot(\nabla_t\times\vec{B}_t) = -i\mu\epsilon\omega E_z$$

, if we define $\hat{\rho}=\hat{n}\times\hat{z}$,

then $\vec{B}_t=(B_\rho, B_n, 0)\Rightarrow \nabla_t\times\vec{B}_t\Big|_S = \hat{z}\left(\frac{\partial B_n}{\partial \rho}-\frac{\partial B_\rho}{\partial n}\right)\Big|_S$

More to the point, consider Eq. 8.24(a), the $\hat{n}$-component:

$$\cancel{\frac{\partial B_n}{\partial z}}\overset{\to 0}{}\Big|_s - i\mu\epsilon\omega\left\{\hat{z}\times\left(E_\ell\big|_s, E_n\big|_s, 0\right)\right\}_n = \frac{\partial B_z}{\partial n}\Big|_s$$

"$E_\ell|_s = 0$"

$$\Rightarrow \boxed{\frac{\partial B_z}{\partial n}\Big|_s = 0, \quad E_z\big|_s = 0}$$

A KEY POINT: The boundary condition $\boxed{E_z|_s = 0}$ but $E_z \neq 0$ inside, plus the BC, $B_z = 0$ everywhere, in addition to the 2D wave equation above (see notes p. 9 above), is already enough information to specify the 2D eigenvalues (k^2) for transverse magnetic TM waves.

And similarly, the BC $\boxed{\frac{\partial B_z}{\partial n}\Big|_s = 0}$ and $E_z = 0$ everywhere, is sufficient to specify the eigenvalues allowed for k^2 in the case of transverse electric (TE) waves. Since the BCs are different, the eigenvalues will in general be different k^2 for TE versus TM waves.

These are the appropriate BCs to be imposed on solutions of the 2D Helmholtz equations:

$$(\nabla_t^2 + \gamma^2)E_z(x,y) = 0$$

$$(\nabla_t^2 + \gamma^2)B_z(x,y) = 0,$$

Subject to two different sets of BCs,

TRANSVERSE MAGNETIC WAVES (TM)

$B_z = 0$ everywhere inside

$E_z|_s = 0$ but $E_z \neq 0$ inside

OR

TRANSVERSE ELECTRIC WAVES (TE)

$E_z = 0$ everywhere inside

$\left.\frac{\partial B_z}{\partial n}\right|_s = 0$ but $B_z \neq 0$ inside

And one can show that the full set of all TM and TE modes (plus the TEM mode if it exists) are COMPLETE and can be superposed to describe any EM phenomena inside the waveguide.

A unified notation for both the TM and TE modes: Call the desired z-component to be calculated Ψ, i.e.

For TM WAVES where $B_z = 0$, recall 8.26a

$$\Rightarrow \quad \vec{E}_t = \frac{ik}{\gamma^2} \vec{\nabla}_t E_z \text{ , so define } \boxed{E_z = \Psi}$$

for TM waves

where $\gamma^2 = \mu\epsilon\omega^2 - k^2$

and then $\vec{B}_t = \frac{\mu\epsilon\omega}{k} \hat{z} \times \vec{E}_t = \mu \vec{H}_t$

or $\vec{H}_t = \frac{\epsilon\omega}{k} \hat{z} \times \vec{E}_t \equiv \frac{\hat{z}}{Z_t} \times \vec{E}_t$

while for TE WAVES where $E_z = 0$, (8.26b)

$\Rightarrow \vec{B}_t = \frac{ik}{\gamma^2}\vec{\nabla}_t B_z$, so define $\boxed{\begin{array}{c}\Psi = H_z\\ \text{for TE waves}\end{array}}$

and (8.23a) $\Rightarrow \vec{E}_t = -\frac{\omega}{k}\hat{z}\times\vec{B}_t$

$$\Rightarrow \hat{z}\times\vec{E}_t = -\frac{\omega}{k}\hat{z}\times(\hat{z}\times\vec{B}_t) = \frac{\omega}{k}\vec{B}_t = \frac{\mu\omega}{k}\vec{H}_t$$

whereby $\vec{H}_t = \frac{k}{\mu\omega}\hat{z}\times\vec{E}_t \equiv \frac{\hat{z}}{Z_\lambda}\times\vec{E}_t$

where $Z_\lambda \equiv \begin{cases}\frac{k}{\epsilon\omega}, & \text{TM waves}\\ \frac{\mu\omega}{k}, & \text{TE waves}\end{cases}$ = "wave impedance"

And in this notation, both TM and TE waves obey $(\nabla_t^2 + \gamma^2)\Psi = 0$, $\gamma^2 = \mu\epsilon\omega^2 - k^2$

subject to either BC:

$\Psi|_S = 0$ (TM) OR $\frac{\partial\Psi}{\partial n}\Big|_S = 0$ (TE)

From previous experience, we expect that Ψ must be oscillatory to satisfy BCs on opposite sides of a hollow cylinder

$\Rightarrow \gamma^2 \geq 0$

$\Rightarrow$ This will define a differential eigenvalue problem, with eigenvalues γ_λ^2 and corresponding eigenfunctions Ψ_λ,

$\lambda = 1, 2, 3, \ldots$

These eigensolutions are called the MODES of the waveguide

And recall that, for any frequency ω, the z-propagation wavenumber in mode λ is:

$$k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2$$

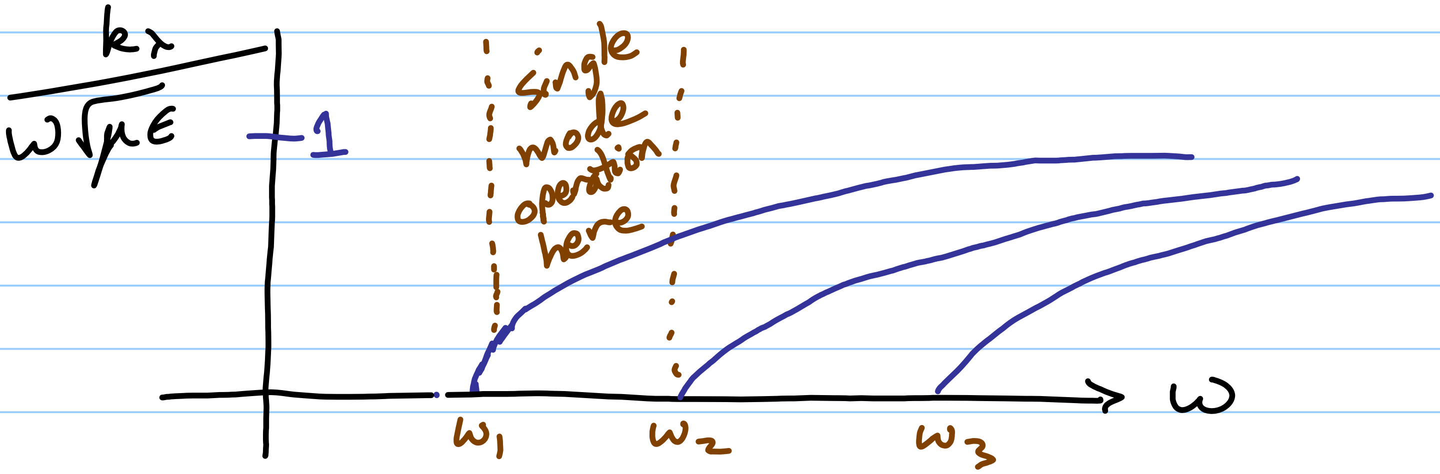
and k_λ^2 should be nonnegative to allow propagation of mode λ.

$\Rightarrow$ Thus there is a CUTOFF FREQUENCY, (for mode λ)

$$\omega_\lambda \equiv \frac{\gamma_\lambda}{\sqrt{\mu\epsilon}}$$

and we can write $k_\lambda = (\mu\epsilon)^{1/2}(\omega^2 - \omega_\lambda^2)^{1/2}$

Then for $\omega > \omega_\lambda$, k_λ = real and waves in mode λ can propagate along z, while for $\omega < \omega_\lambda$, k_λ = imaginary, and mode λ supports only evanescent modes (or cutoff modes) that decay exponentially in z.



We can find the phase velocity along the waveguide, v_p, which is mode-dependent,

$$e^{ik_\lambda x - i\omega t} = e^{ik_\lambda(x - v_p t)}$$

where $$v_p = \frac{\omega}{k_\lambda} = \frac{1}{\sqrt{\mu\epsilon}} \frac{1}{\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}} > \frac{1}{\sqrt{\mu\epsilon}}$$

Notice that the phase velocity can exceed the speed of light, c, since

$$v_p \xrightarrow[\omega \to \omega_\lambda]{} \infty$$

and $$\omega(k_\lambda) = (k_\lambda^2 + \gamma_\lambda^2)^{1/2} \frac{1}{\sqrt{\mu\epsilon}}$$

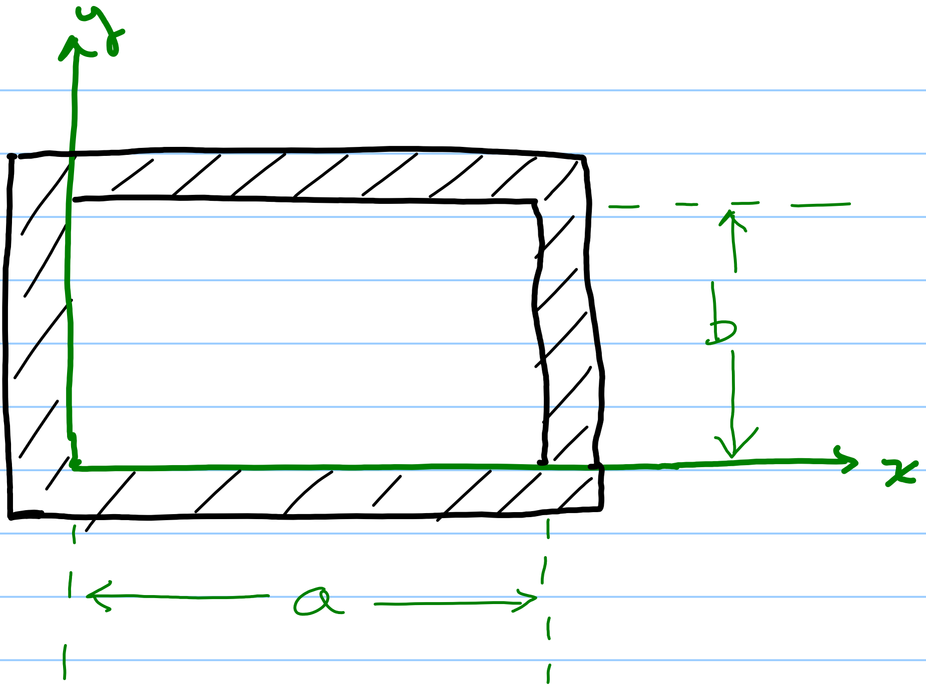
so the group velocity of a wavepacket is

$$v_g = \frac{d\omega}{dk_\lambda} = \frac{d}{dk_\lambda}\left[\frac{1}{\sqrt{\mu\epsilon}}(k_\lambda^2 + \gamma_\lambda^2)^{1/2}\right]$$

$$\Rightarrow \quad v_g = \frac{1}{\sqrt{\mu\epsilon}} \frac{k_\lambda}{\sqrt{k_\lambda^2 + \gamma_\lambda^2}} = \frac{c}{n} \frac{k_\lambda}{(k_\lambda^2 + \gamma_\lambda^2)^{1/2}}$$

Notice that in the usual case where $n > 1$, this translates into $\boxed{v_g < c}$ for all real γ_λ, k_λ.

Rectangular wave guide



Solution for the TE modes

$\psi = H_z$ obeys

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2 \right) \psi(x,y) = 0$$

subject to the boundary conditions

$$\frac{\partial \psi}{\partial n} = 0 \quad , \text{ i.e. } \begin{cases} \dfrac{\partial \psi}{\partial x} = 0 & \text{at } x=0 \text{ and } x=a \\ \dfrac{\partial \psi}{\partial y} = 0 & \text{at } y=0 \text{ and } y=b \end{cases}$$

This is a separable PDE with separable BCs and its solutions are

$$(H_z)_{m,n} = \psi_{mn}(x,y) = H_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

and $\gamma_{m,n}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$, $m,n = 0,1,2,\ldots$

The lowest frequency has $m = n = 0$

$\Rightarrow \Psi_{0,0}(x,y) =$ constant,

which gives zero fields inside

Accordingly, this "trivial solution" is of no physical relevance, so it is omitted from the list of solutions

The cutoff frequencies are

$$\omega_{m,n} = \frac{\pi}{\sqrt{\mu\epsilon}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{1/2}$$

For definiteness, suppose $a > b$.
Then the lowest mode has $m = 1$, $n = 0$, and the cutoff frequency is

$$\omega_{1,0} = \frac{\pi}{a\sqrt{\mu\epsilon}} \text{ , since } \gamma_{1,0} = \frac{\pi}{a}$$

and $k_{1,0} = \sqrt{\mu\epsilon}\,(\omega^2 - \omega_{1,0}^2)^{1/2}$

and $\vec{H}_t = \frac{ik}{\gamma^2} \vec{\nabla}_t H_z$

$$= -\frac{i k_{m,n} H_0}{\gamma_{m,n}^2} \left\{ \hat{x} \frac{m\pi}{a} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} + \hat{y} \frac{n\pi}{b} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \right\}$$

and $$\vec{H}_z \xrightarrow[n=0]{m=1} H_0 \cos\frac{\pi x}{a}\,\hat{z}$$

$$\vec{H}_t^{(1,0)} \xrightarrow[n=0]{m=1} \frac{-ik_{1,0}}{\gamma_{1,0}^2} H_0 \frac{\pi}{a} \sin\frac{\pi x}{a}\,\hat{x}$$

and $$\vec{E}_t^{(1,0)} = -\frac{\omega\mu}{k_{1,0}}\,\hat{z}\times\vec{H}_t^{(1,0)}$$

$$= H_0 \frac{ik_{1,0}}{\gamma_{1,0}^2}\frac{\omega\mu}{k_{1,0}}\,\hat{y}\,\sin\frac{\pi x}{a}$$

and the final component, of course, is $E_z^{(1,0)} = 0$, since this is a TE mode.

This is a linearly-polarized E-field for the lowest TE-mode.

Observations

(1) All fields have an additional factor $e^{i(k_{1,0}z-\omega t)}$

(2) The factor of i in $\vec{H}_t$ and $\vec{E}_t$ implies a 90° phase difference between H_z and $\vec{H}_t$ (or $\vec{E}_t$)

(3) The formulas for the TM mode are obtained by setting $\psi \to E_z$, and $\cos() \to \sin()$

and the lowest nontrivial mode has

$$m=1 \;,\; n=1$$

observe that the LOWEST TM mode has a higher cutoff frequency than does the lowest TE mode.

Energy Flow and attenuation

Next, determine the time-averaged rate of energy flow down the cavity. We are interested in $\mathrm{Re}\left(\frac{1}{2}\vec{E}\times\vec{H}^*\right)$ for complex fields, as seen in Chap. 6.

For now, just consider $\vec{S}=\frac{1}{2}\vec{E}\times\vec{H}^*$

Then for TM,

$$\vec{S}=\frac{1}{2}\left(\vec{E}_t+\vec{E}_z\right)\times\left[\left(\frac{\epsilon\omega}{k}\right)^*\left(\hat{z}\times\vec{E}_t^*\right)\right]$$

$$=\underbrace{\frac{1}{2}\left(\frac{\epsilon\omega}{k}\right)^*\vec{E}_t\times\left(\hat{z}\times\vec{E}_t^*\right)}_{\frac{1}{2}\left(\frac{\epsilon\omega}{k}\right)^*|\vec{E}_t|^2}+\underbrace{\frac{1}{2}\left(\frac{\epsilon\omega}{k}\right)^*\vec{E}_z\times\left(\hat{z}\times\vec{E}_t^*\right)}_{-\frac{1}{2}\left(\frac{\epsilon\omega}{k}\right)^*E_z\vec{E}_t^*}$$

or putting it together,

$$\vec{S}=\frac{1}{2}\frac{\epsilon^*\omega^*}{k^*}\left\{\hat{z}\left|\frac{k^2}{\gamma^4}\right|\left|\vec{\nabla}_tE_z\right|^2-\left(\frac{-ik^*}{\gamma^{*2}}\right)E_z\nabla_tE_z^*\right\}$$

or for ϵ,ω,k all real,

$$\boxed{\vec{S}_{TM}=\frac{1}{2}\frac{\omega k\epsilon}{\gamma^4}\left\{\hat{z}\left|\vec{\nabla}_tE_z\right|^2+\frac{i\gamma^2}{k}E_z\nabla_tE_z^*\right\}}$$

Summary of the Key Chap. 8 Formulas

suggestion: print this page out for reference

Maxwell's equation solutions: $\vec{H}(x,y,z,t)=\vec{H}(x,y)e^{i(kz-\omega t)}$

$$(\nabla_t^2+\gamma^2)\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}=0 \qquad \vec{E}(x,y,z,t)=\vec{E}(x,y)e^{i(kz-\omega t)}$$

$$\nabla_t\cdot\vec{E}_t+ikE_z=0 \qquad \vec{\nabla}_t E_z-ik\vec{E}_t=i\omega\mu\hat{z}\times\vec{H}_t$$

$$\nabla_t\cdot\vec{H}_t+ikH_z=0 \qquad \vec{\nabla}_t H_z-ik\vec{H}_t=-i\epsilon\omega\hat{z}\times\vec{E}_t$$

$$\gamma^2=\mu\epsilon\omega^2-k^2 \qquad k^2=\mu\epsilon\omega^2-\gamma^2$$

$$\gamma_\lambda^2=\mu\epsilon\omega_\lambda^2 \qquad k_\lambda^2=\mu\epsilon(\omega^2-\omega_\lambda^2)$$

$$\vec{E}_t=\frac{i}{\gamma^2}\left(k\vec{\nabla}_t E_z-\mu\omega\hat{z}\times\vec{\nabla}_t H_z\right)$$

$$\vec{H}_t=\frac{i}{\gamma^2}\left(k\vec{\nabla}_t H_z+\epsilon\omega\hat{z}\times\vec{\nabla}_t E_z\right)$$

$$\vec{H}_t=\frac{\epsilon\omega}{k}\hat{z}\times\vec{E}_t=\frac{\hat{z}}{Z}\times\vec{E}_t \quad (TM);\ Z=\frac{k}{\epsilon\omega}$$

$$\vec{H}_t=\frac{k}{\mu\omega}\hat{z}\times\vec{E}_t=\frac{\hat{z}}{Z}\times\vec{E}_t \quad (TE);\ Z=\frac{\mu\omega}{k}$$

$$\vec{E}_t=\frac{ik}{\gamma^2}\vec{\nabla}_t E_z=\frac{ik}{\gamma^2}\vec{\nabla}_t\psi \quad (TM)$$

$$\vec{H}_t=\frac{ik}{\gamma^2}\vec{\nabla}_t H_z=\frac{ik}{\gamma^2}\vec{\nabla}_t\psi \quad (TE)$$

$$v_p v_g=\frac{1}{\mu\epsilon} \qquad v_g=\frac{1}{\sqrt{\mu\epsilon}}\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}$$

Next derive $\vec{S}_{TE}$:

$\boxed{TE}$

$$\vec{S} = \frac{1}{2}\vec{E}_t \times \left(\vec{H}_t^{*} + \vec{H}_z^{*}\right)$$

$$= \frac{1}{2}\vec{E}_t \times \left(\hat{z}\times\vec{E}_t^{*}\right)\left(\frac{k}{\mu\omega}\right)^{*}$$

which we write in terms of $H_z = \Psi$, by using $\vec{H}_t = \frac{ik}{\gamma^2}\nabla_t\Psi$, and 8.31,

$$\vec{H}_t = \frac{k}{\mu\omega}\hat{z}\times\vec{E}_t$$

Then taking $\hat{z}\times(8.31)$,

$$\Rightarrow \hat{z}\times\vec{H}_t = \frac{k}{\mu\omega}\hat{z}\times\left(\hat{z}\times\vec{E}_t\right) = -\frac{k}{\mu\omega}\vec{E}_t$$

or

$$\boxed{\vec{E}_t = -\frac{\mu\omega}{k}\hat{z}\times\vec{H}_t}$$

whereby

$$\vec{S} = \frac{1}{2}\left(-\frac{\mu\omega}{k}\right)\left(\hat{z}\times\vec{H}_t\right)\times\left(\vec{H}_t^{*} + \vec{H}_z^{*}\right)$$

$$= \frac{\mu\omega}{2k}\left\{\hat{z}\,|\vec{H}_t|^2 - H_z^{*}\vec{H}_t\right\}$$

$$= \frac{\mu\omega}{2k}\left\{\hat{z}\left|\frac{k^2}{\gamma^4}\right|\,|\vec{\nabla}_t H_z|^2 - \frac{ik}{\gamma^2}H_z^{*}\vec{\nabla}_t H_z\right\}$$

and if k, γ, μ are all real,

$$\Rightarrow \boxed{\vec{S}^{(TE)} = \frac{\omega k}{2\gamma^4}\left\{\hat{z}\mu\,|\vec{\nabla}_t H_z|^2 - \frac{i\gamma^2\mu}{k}H_z^{*}\nabla_t H_z\right\}}$$

Observe, moreover, that since $\Psi = \begin{cases} E_z\,, & \text{TM modes} \\ H_z\,, & \text{TE modes,} \end{cases}$

obeys a REAL PDE with REAL BCs, we may take Ψ to be REAL without any loss of generality.

Then the terms $iE_z\nabla_t E_z^*$ and $iH_z^*\nabla_t H_z$ do not contribute to the energy flow along the guide, after we take $\mathrm{Re}\,\vec{S}$.

And the total time-averaged power flow is

$$P = \int_A \vec{S}\cdot\hat{z}\,da = \frac{\omega k}{2\gamma^4}\left\{\begin{matrix}\epsilon\\ \text{or}\\ \mu\end{matrix}\right\}\int_A |\nabla_t\Psi|^2\,da$$

for $\left\{\begin{matrix}\text{TM}\\ \text{or}\\ \text{TE}\end{matrix}\right\}$ modes

Next, application of Green's 1st identity in 2D gives:

0 for TM and TE modes

$$P = \frac{\omega k}{2\gamma^4}\left\{\begin{matrix}\epsilon\\ \text{or}\\ \mu\end{matrix}\right\}\left(\oint_C \Psi^*\cancelto{0}{\frac{\partial\Psi}{\partial n}}\,dl - \int_A \Psi^*\underbrace{\nabla_t^2\Psi}_{\overset{\parallel}{-\gamma^2\Psi}}\,da\right)$$

or $P = \frac{\omega k}{2\gamma^2}\left\{\begin{matrix}\epsilon\\ \text{or}\\ \mu\end{matrix}\right\}\int_A |\Psi|^2\,da$, and using

$k \equiv \sqrt{\mu\epsilon}\,(\omega^2-\omega_\lambda^2)^{1/2}$, $\gamma^2 \equiv \mu\epsilon\omega_\lambda^2$,

$$\Rightarrow \boxed{P = \frac{\omega}{2\sqrt{\mu\epsilon}\,\omega_\lambda^2}(\omega^2-\omega_\lambda^2)^{1/2}\left\{\begin{matrix}\epsilon\\ \text{or}\\ \mu\end{matrix}\right\}\int_A |\Psi|^2\,da}$$

Similarly, the time-averaged electric and magnetic energies per unit volume stored inside is (6.133):

$$W_e = \frac{1}{4}\vec{E}\cdot\vec{D}^* \quad , \quad W_m = \frac{1}{4}\vec{B}\cdot\vec{H}^*$$

and thus the energy stored per unit length inside the guide is, for real ϵ, μ:

$$\frac{dU}{dl} = \int da \left(\frac{\epsilon}{4}|\vec{E}|^2 + \frac{\mu}{4}|\vec{H}|^2 \right)$$

and we can recast this in terms of ψ:

TM case $\vec{H}_t = \frac{\epsilon\omega}{k}\hat{z}\times\vec{E}_t$ and $\omega_\lambda^2 = \frac{\gamma_\lambda^2}{\mu\epsilon}$

and $k_\lambda^2 = \mu\epsilon(\omega^2 - \omega_\lambda^2) = \mu\epsilon\omega^2 - \gamma_\lambda^2$

$$\Rightarrow \frac{dU}{dl} = \int da \left\{ \frac{\epsilon}{4}|\vec{E}_t|^2 + \frac{\epsilon}{4}|E_z|^2 + \frac{\mu}{4}|\vec{H}_t|^2 \right\}$$

where $\vec{E}_t = \frac{ik}{\gamma^2}\nabla_t\psi$, $\vec{H}_t = \frac{i\epsilon\omega}{\gamma^2}\hat{z}\times\nabla_t\psi$

and $E_z = \psi$

$$\Rightarrow \frac{dU}{dl} = \int da \left\{ \frac{\epsilon}{4}|\psi|^2 + \frac{\epsilon}{4\gamma^4}|\nabla_t\psi|^2(2k^2+\gamma^2) \right\}$$

and after again employing Green's identity,

$$\Rightarrow \int da |\nabla_t\psi|^2 = \cancelto{0}{\oint dl\, \psi^* \frac{\partial\psi}{\partial n}} - \int da\, \psi^* \nabla_t^2 \psi$$

$$= \gamma^2 \int |\psi|^2 da$$

So $$\frac{dU}{d\ell} = \int da\, \frac{\epsilon}{4} |\psi|^2 \left(1 + \frac{2k^2\gamma^2 + \gamma^4}{\gamma^4}\right)$$

$$(\;) = \frac{2\gamma^2 + 2k^2}{\gamma^2} = \frac{2\mu\epsilon\omega^2}{\gamma^2} = 2\frac{\omega^2}{\omega_\lambda^2}$$

$$\Rightarrow \boxed{\frac{dU}{d\ell} = \int da\, \frac{\epsilon}{2} \frac{\omega^2}{\omega_\lambda^2} |\psi|^2 \text{ for TM modes}}$$

And a similar derivation for TE modes gives:

$$\boxed{\frac{dU}{d\ell} = \frac{\mu}{2} \frac{\omega^2}{\omega_\lambda^2} \int da\, |\psi|^2 \text{ for TE modes}}$$

The ratio between power flow rate and energy density is interpreted as the velocity at which energy flows down the cavity, i.e.

$$\frac{P}{\frac{dU}{d\ell}} = \frac{\omega(\omega^2 - \omega_\lambda^2)^{1/2}}{\omega^2 \sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}}\left(1 - \frac{\omega^2}{\omega_\lambda^2}\right)^{1/2} = v_g$$

$\Rightarrow$ as expected, $P = \left(\frac{dU}{d\ell}\right) v_g$, confirming that energy flows at the group velocity, and $v_g \le c$

Generalization – walls of finite but high conductivity

We will see that for σ large but $< \infty$, now

$$k_\lambda = \underbrace{k_\lambda^{(0)}}_{\substack{\text{wavenumber}\\ \text{for perfectly}\\ \text{conducting mode}}} + \underbrace{\alpha_\lambda}_{\substack{\text{small shift}\\ \text{in real part,}\\ \text{mainly important}\\ \text{when } k_\lambda^{(0)} \to 0}} + \underbrace{i\beta_\lambda}_{\substack{\text{acquires a}\\ \text{small imaginary}\\ \text{part}\\ i\beta_\lambda}}$$

Energy conservation argument

Expectation: the power flow down the waveguide should be attenuated exponentially,

$$\Longrightarrow \quad P(z) = P_0 e^{-2\beta_\lambda z}$$

i.e.

$$\beta_\lambda = \frac{-\frac{dP(z)}{dz}}{2P(z)}$$

but in Sec. 8.1 we derived the rate of energy loss into a wall of finite conductivity as

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_C |\hat{n}\times\vec{H}|^2 dl = -\oint_C \frac{dP}{da}\, dl$$

Let's work this out explicitly for a TM mode:
↳ $H_z = 0$

$$|\hat{n} \times \vec{H}|^2 = |\vec{H}_t|^2 = |\hat{n} \times \vec{H}_t|^2$$

$$= |\hat{n} \times (\hat{z} \times \nabla_t \psi)|^2 \frac{k^2}{\gamma^4} \frac{\epsilon^2 \omega^2}{k^2}$$

$$= |\hat{z}(\hat{n} \cdot \nabla_t \psi)|^2 \frac{\epsilon^2 \omega^2}{\gamma^4} = \left|\frac{\partial \psi}{\partial n}\right|^2 \frac{\omega^2}{\mu^2 \omega_\lambda^4}$$

and so

$$\boxed{-\frac{dP^{TM}}{dz} = \frac{1}{2\sigma\delta} \frac{\omega^2}{\mu^2 \omega_\lambda^4} \oint_C \left|\frac{\partial \psi}{\partial n}\right|^2 dl \quad \text{for TM}}$$

and similarly

$$\boxed{-\frac{dP^{TE}}{dz} = \frac{1}{2\sigma\delta} \frac{\omega^2}{\omega_\lambda^2} \oint_C \left\{ \frac{1}{\mu\epsilon\omega_\lambda^2}\left(1 - \frac{\omega_\lambda^2}{\omega^2}\right) |\hat{n} \times \nabla_t \psi|^2 + \frac{\omega_\lambda^2}{\omega^2} |\psi|^2 \right\} dl}$$

Order-of-magnitude estimates

Idea: estimate the approximate magnitude of the transverse derivatives, very roughly, using:

$$(\nabla_t^2 + \mu\epsilon\omega_\lambda^2)\psi = 0$$

$\Rightarrow$ take $\left\langle \left|\frac{\partial \psi}{\partial n}\right|^2 \right\rangle \simeq \mu\epsilon\omega_\lambda^2 \langle |\psi|^2 \rangle$

$$= \langle |\hat{n} \times \nabla_t \psi|^2 \rangle$$

whereby

$$\oint_C \frac{1}{\omega_\lambda^2} \left|\frac{\partial \psi}{\partial n}\right|^2 dl \simeq \mu\epsilon C \langle |\psi|^2 \rangle$$

$$\simeq \frac{\mu\epsilon C}{A} \int_A |\psi|^2 da$$

circumference of the cross-section area

Or to be more precise, we can introduce a dimensionless constant ξ_λ of order unity, defined by:

$$\xi_\lambda \equiv \frac{\oint_C \frac{1}{\omega_\lambda^2}\left|\frac{\partial\psi}{\partial n}\right|^2 dl}{\frac{\mu\epsilon C}{A}\int_A |\psi|^2 da}$$

Next recall the following relations for TM modes, from above:

$$\boxed{-\frac{dP^{TM}}{dz} = \frac{1}{2\sigma\delta}\frac{\omega^2}{\mu^2\omega_\lambda^4}\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl \quad \text{for TM}}$$

and

$$\boxed{P = \frac{\omega}{2\sqrt{\mu\epsilon}\,\omega_\lambda^2}\left(\omega^2-\omega_\lambda^2\right)^{1/2}\left\{\begin{matrix}\epsilon\\ \text{or}\\ \mu\end{matrix}\right\}\int_A |\psi|^2 da}$$

which yields:

$$\boxed{\beta_\lambda^{TM} = -\frac{1}{2P}\frac{dP}{dz} = \left(\frac{\epsilon}{\mu}\right)^{1/2}\frac{1}{\sigma\delta_\lambda}\frac{C}{2A}\frac{(\omega/\omega_\lambda)^{1/2}}{\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}}\xi_\lambda}$$

where δ_λ = skin depth at the cutoff frequency ω_λ

If this derivation is repeated for TE modes, a similar expression results, except that

$$\xi_\lambda \longrightarrow \xi_\lambda + \frac{\omega_\lambda^2}{\omega^2}\eta_\lambda$$

where η_λ is also dimensionless and of order unity

Example take the microwave oven frequency,

$$\omega = (2\pi)\,2.5\,\text{GHz}$$

and conductivity $\sigma \simeq 6\times10^{7}\,\Omega^{-1}\,m^{-1}$

$$\mu_c \simeq \mu_0 = 4\pi\times10^{-7}\,\frac{N}{A^2}$$

Then the skin depth is $\delta = \left(\frac{2}{\mu_c\omega\sigma}\right)^{1/2} \simeq 1.3\,\mu m$

and take the cross-section radius to be

$$R = 10^{-2}\,m$$

$$\Rightarrow \quad \omega \simeq \omega_0\sqrt{3}$$

This leads to $\beta \simeq \left(\frac{\epsilon}{\mu}\right)^{1/2}\frac{1}{\sigma\delta}\,\frac{2\pi R}{2(\pi R^2)}\,\frac{3^{1/4}}{(2/3)^{1/2}} \simeq 0.005\,m^{-1}$

and consequently $P(z) = P(0)e^{-2\beta z}$ decays by a factor $\frac{1}{e}$ over a distance of

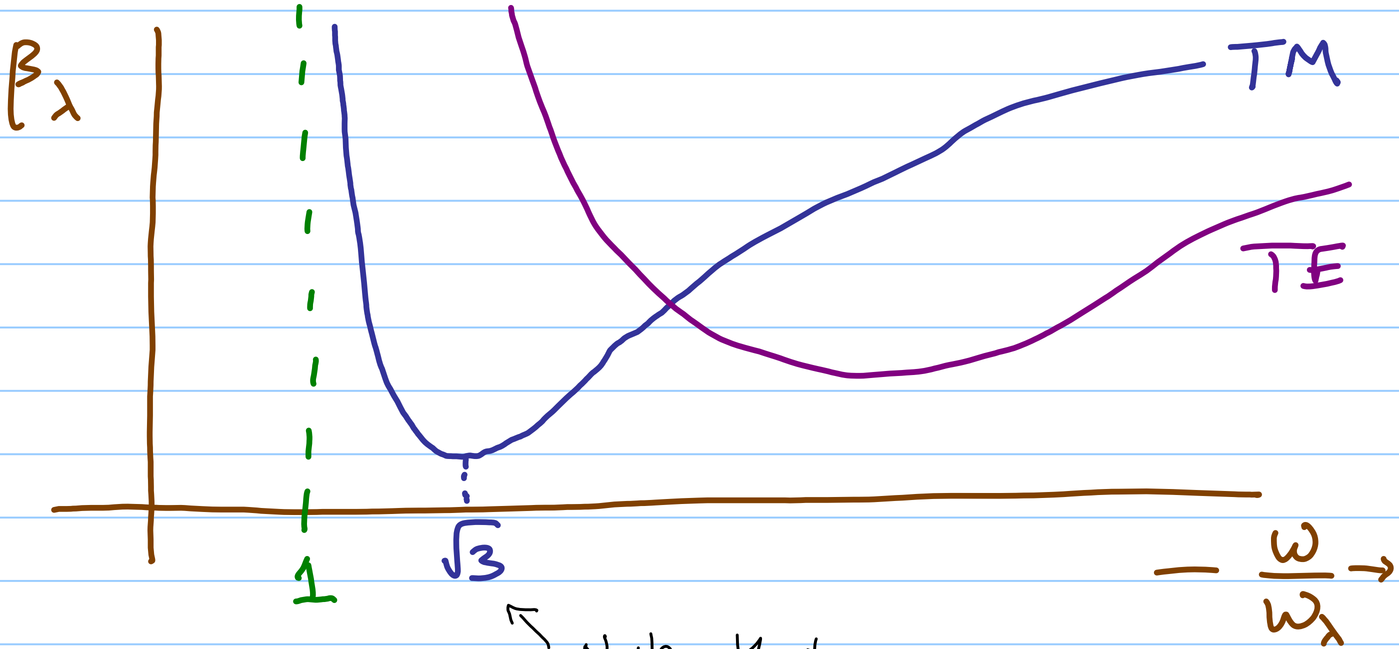
$$\boxed{\Delta z \simeq 100\,m}$$

Jackson says that 200–400 m is typical

Limiting Behavior

For high frequency,
$\beta \propto \omega^{1/2}$

Also, for ω near the cutoff frequency ω_λ, β_λ gets large and our approximate treatment fails



Note that
$\omega_{min} = \sqrt{3}\,\omega_\lambda$ always for TM, but for TE modes there is no simple, general formula for ω_{min}.

On your own, read Sec. 8.6 to learn about a more general and more accurate treatment.

Resonant Cavity = a waveguide with conducting endcaps.

$\Rightarrow$ Where we had travelling waves for a waveguide, now we will obtain STANDING WAVES.

i.e. $\psi(x,y)e^{\pm ikz} \longrightarrow \psi(x,y)\begin{cases}\cos kz\\ \sin kz\end{cases}$

A resonant cavity could have any shape, but we will only treat cylinders of constant cross-sectional area

e.g. Consider the TM mode and take the ends at $z=0$, $z=d$.

$\Rightarrow \psi = E_z$, and the BCs are:

(i) $E_z = 0$ on the sides, just as for the waveguides

and (ii) $\vec{E}_t = 0$ on the ends, so that $\vec{E}_{\parallel} = 0$

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We already implemented BC (i) in our solution above for the TM modes. To implement BC(ii), consider a solution of the form

$$E_z = \psi(x,y)\cos kz\, e^{-i\omega t}$$

$$= \psi(x,y)\,\frac{e^{ikz}+e^{-ikz}}{2}\,e^{-i\omega t}$$

To see why the relevant solution is $\cos kz$, recall Eq.(8.23 a), $\frac{\partial \vec{E}_t}{\partial z} + i\mu\omega\,\hat{z}\times\vec{H}_t = \vec{\nabla}_t E_z$

and 8.26 b says that

$$\vec{H}_t = \frac{i}{\gamma^2}\left(\cancelto{0 \text{ for TM}}{\pm k\nabla_t H_z} + \epsilon\omega\,\hat{z}\times\nabla_t E_z\right)$$

So, if $E_z = \psi(x,y)\cos kz$, then

$$\frac{\partial \vec{E}_t}{\partial z} - i\mu\omega\left(\frac{i\epsilon\omega}{\gamma^2}\right)\nabla_t E_z = \nabla_t E_z$$

or $$\frac{\partial \vec{E}_t}{\partial z} = \frac{\gamma^2-\mu\epsilon\omega^2}{\gamma^2}\nabla_t E_z = \frac{-k^2}{\gamma^2}\nabla_t E_z$$

and integrating this over z gives

$$\vec{E}_t = -\frac{k^2}{\gamma^2}\nabla_t\psi(x,y)\,\frac{\sin kz}{k}$$

So we see that the endcap BCs will be satisfied if $k = p\frac{\pi}{d}$, $p = 0, 1, 2, \ldots$ and then

$$\boxed{\begin{aligned} \vec{E}_t &= \frac{-p\pi}{d\gamma^2}\sin\left(\frac{p\pi z}{d}\right)\nabla_t\psi \\ \vec{H}_t &= \frac{i\epsilon\omega}{\gamma^2}\cos\left(\frac{p\pi z}{d}\right)\hat{z}\times\nabla_t\psi \end{aligned}}$$

for TM modes

$p = 0, 1, 2 \ldots$

And by a similar argument (derive on your own)

$$\boxed{\begin{aligned} \vec{E}_t &= -\frac{i\omega\mu}{\gamma^2}\sin\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi \\ \vec{H}_t &= \frac{p\pi}{d\gamma^2}\cos\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi \end{aligned}}$$

for TE modes

$p = 1, 2, \ldots$

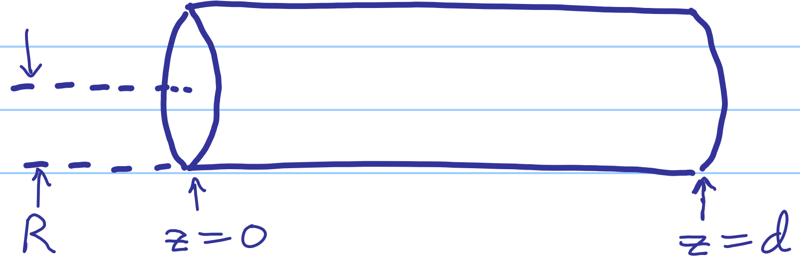
($p=0$ would give the trivial 0 solution only for TE)

and $\gamma^2 = \mu\epsilon\omega^2 - \left(\frac{p\pi}{d}\right)^2$

so the allowed eigenfrequencies of the perfectly conducting cavity are ONLY the following discrete values:

$$\omega_{\lambda p}^2 = \frac{1}{\mu\epsilon}\left(\gamma_\lambda^2 + \left(\frac{p\pi}{d}\right)^2\right)$$

Example right-circular cylinder cavity



Ψ must obey:

$$\left(\nabla_t^2 + \gamma^2\right)\Psi(\rho,\phi) = 0$$

$$\Rightarrow \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\underbrace{\frac{\partial^2}{\partial\phi^2}}_{-m^2,\ m=0,1,2,\ldots} + \gamma^2\right)\Psi(\rho,\phi) = 0$$

and this is solved by separable solutions,

$$\Psi(\rho,\phi) = R(\rho)\left\{\begin{matrix}\cos m\phi \\ \text{or} \\ \sin m\phi\end{matrix}\right\}$$

and the ρ solutions are Bessel functions, namely

$$R(\rho) = J_m(\gamma\rho) \leftarrow \text{must be } J_m \text{ to be regular at } 0.$$

The appropriate BC for a TM mode is

$$\Psi\Big|_s = 0 \Rightarrow J_m(\gamma R) = 0$$

$$\Rightarrow \gamma = \frac{x_{mn}}{R} \quad \text{where } x_{mn} = n^{th} \text{ root of } J_m(x_{mn}) = 0$$

and then

$$\Psi_{m,n}^{\binom{s}{c}}(\rho,\phi) = J_m\left(x_{mn}\frac{\rho}{R}\right)\left\{\begin{matrix}\sin m\phi \\ \text{or} \\ \cos m\phi\end{matrix}\right.$$

and

$$\gamma_{mn}^2 = \frac{x_{mn}^2}{R^2} = \mu\epsilon\omega^2 - \left(\frac{p\pi}{d}\right)^2$$

Finally, the cavity eigenfrequencies are

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\left[\left(\frac{x_{mn}}{R}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right]^{1/2}$$

The lowest TM mode is for

$$(m, n, p) = (0, 1, 0),$$

for which $\omega_{010} = \frac{1}{\sqrt{\mu\epsilon}}\frac{x_{01}}{R} = (2.405\ldots)\frac{c}{n}\frac{1}{R}$

and for this mode,

$$E_z(\rho, \phi, z) = E_0 J_0\left(\frac{x_{01}\rho}{R}\right)e^{-i\omega_{010}t}$$

and $\vec{E}_t(\rho, \phi, z) = 0$ since $p = 0$

but $\vec{H}_t(\rho, \phi, z) = -i\left(\frac{\epsilon}{\mu}\right)^{1/2} E_0 J_1\left(\frac{x_{01}\rho}{R}\right)e^{-i\omega_{010}t}$

Energy dissipation in a cavity and the cavity Q-value

Ohmic heating of the cavity wall conductor and/or dissipation in the dielectric media inside the cavity, modifies the δ-function frequency response predicted by the analysis above.

It is conventional to characterize cavity losses by:

$$Q \equiv \omega_0 \frac{\text{Stored energy}}{\text{Power Loss Rate}} =$$ dimensionless "QUALITY FACTOR"

where ω_0 is the resonance frequency in the absence of losses.

Derivation Let U = "time-averaged" energy stored in a cavity, i.e. averaged over 1 period, $\Delta t = \frac{2\pi}{\omega_0}$

Then write the power loss rate as:

$$\frac{dU}{dt} = -\xi U$$, where clearly ξ has units of $\frac{1}{\text{time}}$, as does ω_0

$\Rightarrow$ Write $\xi \equiv \frac{\omega_0}{Q}$

whereby $$\boxed{Q = -\frac{\omega_0 U}{dU/dt}}$$

and note that $Q \to \infty$ as losses $\to 0$

$$\Rightarrow U(t) = U_0 e^{-\omega_0 t/Q}$$

and recall that if the stored energy dissipates at the rate $e^{-\omega_0 t/Q} = e^{-\xi t}$, then the FIELDS inside decay as $e^{-\xi t/2}$

i.e. $$\vec{E}(t) = \vec{E}_0 \, e^{-\omega_0 t/2Q} \, e^{-i(\omega_0 + \underset{\uparrow}{\Delta\omega})t}$$

There can be a possible small shift of ω_0 due to losses, also

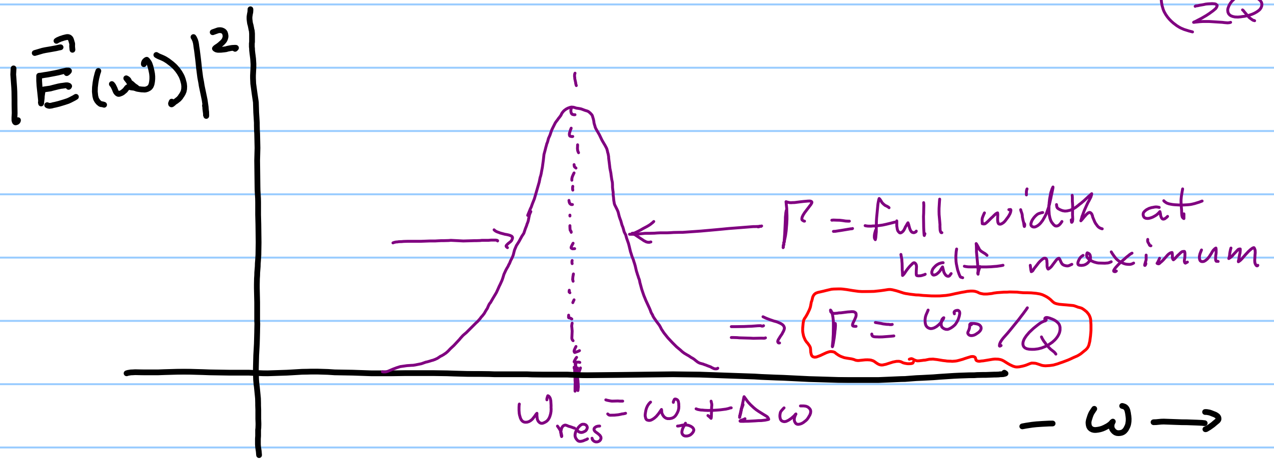
We can Fourier-analyze this, starting from

$$\vec{E}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(\omega) \, e^{-i\omega t} \, d\omega$$

$$\Rightarrow \vec{E}(\omega) = \frac{\vec{E}_0}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{\omega_0 t}{2Q}} \, e^{+i(\omega - \omega_0 - \Delta\omega)t} \, dt$$

$$= \frac{-\vec{E}_0}{\sqrt{2\pi}} \; \frac{1}{i(\omega - \omega_0 - \Delta\omega) - \frac{\omega_0}{2Q}}$$

and thus $$|\vec{E}(\omega)|^2 = \frac{|\vec{E}_0|^2}{2\pi} \; \frac{1}{(\omega - \omega_0 - \Delta\omega)^2 + \left(\frac{\omega_0}{2Q}\right)^2}$$



Note that typically $\frac{\Delta\omega}{\Gamma} \lesssim 1$

Calculation of the cavity Q

We utilize the main result from Sec. 8.1, namely that the fields just inside the conductor are determined by the values of $\vec{H}_{\parallel}$ just outside:

$$\Longrightarrow \quad \vec{H}_c = \vec{H}_{\parallel}\, e^{-\xi/\delta}\, e^{i\xi/\delta}$$

$$\vec{E}_c = \left(\frac{\mu_c \omega}{2\sigma}\right)^{1/2}(1-i)(\hat{n}\times\vec{H}_{\parallel})\, e^{-\frac{\xi}{\delta}}\, e^{\frac{i\xi}{\delta}}$$

and the time-averaged flow of power per unit area is (8.12),

$$\frac{dP_{loss}}{da} = \frac{\mu_c \omega \delta}{4}\,|\vec{H}_{\parallel}|^2 = \frac{1}{2\sigma\delta}\,|\vec{H}_{\parallel}|^2$$

So we integrate this over the area of the surface enclosing the cavity,

assumes both ends have equal losses

$$P_{loss} = \frac{1}{2\sigma\delta}\oint_C dl \int_0^d dz\, |\hat{n}\times\vec{H}|^2_{sides} + \frac{2}{2\sigma\delta}\int_A da\, |\hat{n}\times\vec{H}|^2_{end}$$

and we need the stored energy,

$$U = \int_A da \int_0^d dz \left(\frac{1}{4}\vec{E}\cdot\vec{D}^* + \frac{1}{4}\vec{B}\cdot\vec{H}^*\right)$$

which simplified in our derivation above to

$$U = \int_0^d dz \int_A da \begin{Bmatrix} \epsilon \\ \text{or} \\ \mu \end{Bmatrix} \frac{1}{2}\left(1 + \frac{k_{\lambda p}^2}{\gamma_\lambda^2}\right)|\psi|^2 \begin{Bmatrix} \cos^2\left(\frac{p\pi z}{d}\right) \\ \text{or} \\ \sin^2\left(\frac{p\pi z}{d}\right) \end{Bmatrix}$$

for $\begin{Bmatrix} TM \\ \text{or} \\ TE \end{Bmatrix}$ modes (You might want to review the derivation of 8.52!)

$$\Rightarrow U = \frac{d}{4}\begin{Bmatrix} \epsilon \\ \text{or} \\ \mu \end{Bmatrix}\left[1 + \left(\frac{p\pi}{\gamma_\lambda d}\right)^2\right]\left(\int |\psi|^2 da\right)\left(1 + \delta_{p,0}\right)$$

Aside: this Kronecker-delta arises since the average of $\cos^2()$ is $\frac{1}{2}$ if $p \neq 0$, but 1 if $p = 0$

so specializing now to TM modes,

$$P_{loss} = \frac{\epsilon}{\sigma\delta\mu}\left[1 + \left(\frac{p\pi}{\gamma_\lambda d}\right)^2\right]\left\{1 + \xi_\lambda \frac{Cd}{4A}\left(1 + \delta_{p,0}\right)\right\}\int_A |\psi|^2 da$$

and as we've seen previously,

$$\xi_\lambda = \frac{A}{C}\frac{1}{\mu\epsilon\omega_\lambda^2}\frac{\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl}{\int_A |\psi|^2 da}$$

(see 8.62)

Putting this all together, we obtain (for TM-modes)

$$Q^{TM} = \frac{\mu}{\mu_c}\frac{d}{\delta}\frac{1 + \delta_{p,0}}{2\left[1 + \xi_\lambda \frac{Cd}{4A}\left(1 + \delta_{p,0}\right)\right]}$$

or

$$Q^{TM} = \left(\frac{\mu}{\mu_c}\right)\left(\frac{V}{S\delta}\right) \times \left(\begin{matrix}\text{dimensionless geometrical} \\ \text{factor of order unity}\end{matrix}\right)$$

where $S = 2A + Cd = S =$ total surface area

and $Ad = V =$ cavity volume

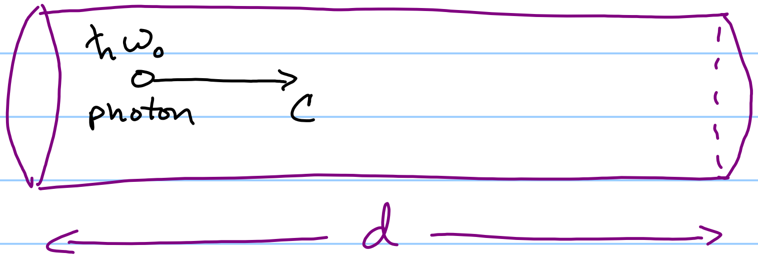
Order of Magnitude Estimate

Copper cavity with $R = d = 10\,\text{cm}$, at $1\,\text{GHz}$ has $Q \simeq 2 \times 10^4$

But a superconducting cavity could have

$$Q^{SC} \simeq 10^{10}$$

Photon interpretation of Q



Suppose an $\hbar\omega_0$-energy photon bounces back and forth, with a probability p to escape each time it hits either end cap

$\Rightarrow$ The average energy loss per second is

$P_{loss} = p \dfrac{\hbar\omega_0}{(d/c)}$, whereas the energy stored is $\hbar\omega_0$

$$\Rightarrow Q = \omega_0 \frac{\text{energy stored}}{\text{energy loss per second}}$$

or

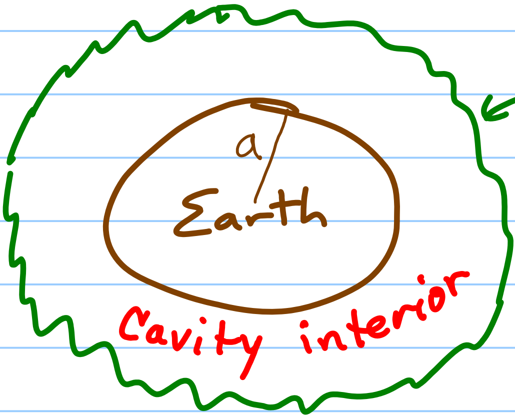
$$Q = \omega_0 \frac{\hbar\omega_0}{p \frac{c}{d} \hbar\omega_0} = \frac{\omega_0 d}{pc} = \frac{2\pi}{T} \frac{\Delta t}{p}$$

But $\frac{\Delta t}{P} \simeq$ average time for a photon to escape, call this τ

$$\Rightarrow Q \overset{\sim}{=} 2\pi \frac{\tau}{T}$$

or $\boxed{Q \simeq 2\pi \left(\text{Number of radiation cycles during the "photon lifetime"}\right)}$

Further Reading Sec. 8.9 Schumann resonances of the Earth-Ionosphere "EM-cavity"

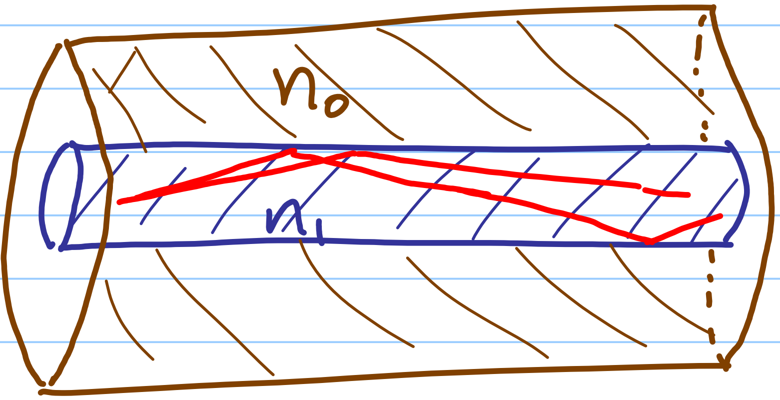


$\Rightarrow$ text derives TE mode frequencies for l^{th} spherical waves as

$$\omega_l \simeq \frac{c}{a}\sqrt{l(l+1)}$$

$$\simeq 2\pi\,(8, 14, 20, \ldots \text{Hz})$$

8.10 EM wave propagation in an optical fiber



$\Leftarrow$ light rays undergo total internal reflection at interface

Chapter 9 Radiating Systems

Let's consider monochromatic sources and fields, i.e.

$$\boxed{\begin{aligned} \rho(\vec{x},t) &= \rho(\vec{x})\, e^{-i\omega t} \\ \vec{J}(\vec{x},t) &= \vec{J}(\vec{x})\, e^{-i\omega t} \end{aligned}} \longleftarrow \text{only real parts are relevant}$$

Then in the Lorentz gauge, the potentials obey:

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

Then formally, these equations can be solved using a Green's function, e.g.

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \int dt'\, G(\vec{x},t;\vec{x}',t')\,\vec{J}(\vec{x}',t')$$

and if there are no boundary surfaces, we normally will want the RETARDED Green function,

$$G^{(+)}(\vec{x},t;\vec{x}',t') = \frac{\delta\left[t' - \left(t - \frac{R}{c}\right)\right]}{R}$$

where $R = |\vec{x} - \vec{x}'|$

and t' (source) is earlier than t by $\frac{R}{c}$

$$\Rightarrow \vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\iint d^3x'\,dt'\;\frac{\delta[t'-(t-\frac{R}{c})]}{R}\,J(\vec{x}',t')$$

or

$$\boxed{\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\;\frac{\vec{J}\left(\vec{x}',\,t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}}$$

In words: $\vec{A}(\vec{x},t)$ depends on $\vec{J}$ at $\vec{x}'$, but at a time <u>earlier</u> than t by $\frac{|\vec{x}-\vec{x}'|}{c}$, i.e. the light propagation time.

This can be evaluated further if we consider a monochromatic source,

i.e. $\vec{J}(\vec{x}',t') = \vec{J}(\vec{x}')\,e^{-i\omega t'}$

$$\Rightarrow \boxed{\vec{A}(\vec{x},t) = e^{-i\omega t}\left(\frac{\mu_0}{4\pi}\int d^3x'\;\vec{J}(\vec{x}')\,\frac{e^{\frac{i\omega}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\right)}$$

"$\vec{A}(\vec{x})$"

and similarly $\Phi(\vec{x},t) = \Phi(\vec{x})\,e^{-i\omega t}$ where

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\;\frac{e^{ikR}}{R}\,\rho(\vec{x}')}$$

But in fact, in the context of radiated fields, <u>we will only need $\vec{A}$</u>, since $\vec{H} = \frac{1}{\mu_0}\nabla\times\vec{A}$

and $\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$, so in regions where $\vec{J}=0$,

$$-\frac{i\omega}{c^2}\vec{E} = \mu_0 \nabla \times \vec{H}$$

or

$$\boxed{\vec{E}(\vec{x}) = \frac{i Z_0}{k} \nabla \times \vec{H}}$$

holds for harmonic time-dependence away from sources, where $\vec{J}(\vec{x}) \to 0$

where $Z_0 \equiv \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}$ = impedance of free space.

The only Catch (22) here is that:

the integral for $\vec{A}(\vec{x})$ is difficult to evaluate

$\Rightarrow$ One systematic way to approach this is to use the partial wave expansion (9.98), namely

$$G = \frac{e^{ikR}}{4\pi R} = ik \sum_{l=0}^{\infty} j_l(kr_<)\, h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\hat{x}')\, Y_{lm}(\hat{x})$$

But we can begin by treating some limits by more elementary means

First note the three relevant distance scales:

$d \simeq$ physical size/extent of sources

$\lambda = \frac{2\pi c}{\omega}$ = light wavelength

r = observation distance

There are 3 different limits of interest,

Near Zone (static): $d \ll r \ll \lambda$

Intermediate Zone (induction): $d \ll r \simeq \lambda$

Far Zone (radiation): $d \ll \lambda \ll r$

Consider the <u>NEAR ZONE</u>: $d \ll r \ll \lambda$

$$\Rightarrow e^{ik|\vec{x}-\vec{x}'|} \simeq 1$$

$$\Rightarrow \vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

just as in <u>magnetostatics</u>! (Eq. 5.32)

$\Rightarrow \vec{A}(\vec{x})$ is the <u>same</u> as for a static, steady current source, except for an overall $e^{-i\omega t}$ time-dependence.

When helpful, we can perform a multipole expansion.

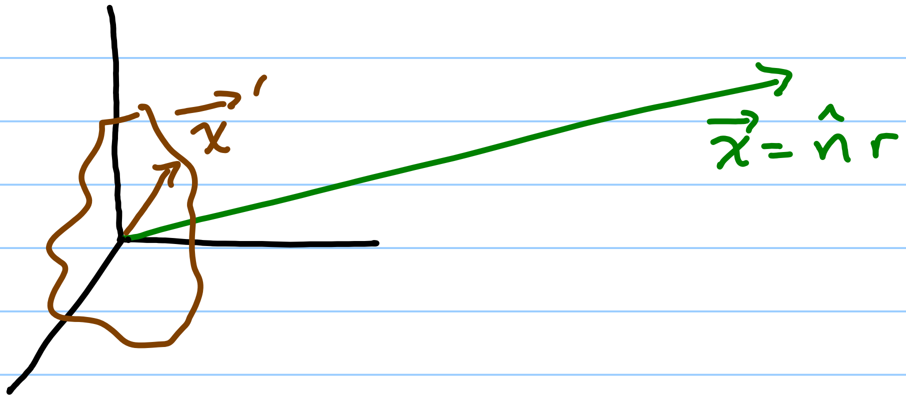
Next, the <u>FAR ZONE</u>, $d \ll \lambda \ll r$

Observe that in all 3 zones, $r \gg d$,

$$\Rightarrow |\vec{x}-\vec{x}'| = |\vec{x}| \left(1 + \frac{x'^2}{x^2} - 2\frac{\vec{x}\cdot\vec{x}'}{x^2}\right)^{1/2}$$

or $$|\vec{x}-\vec{x}'| = r\left(1 - \frac{\vec{x}}{r}\cdot\frac{\vec{x}'}{r} + \ldots\right.$$

or $$\boxed{|\vec{x}-\vec{x}'| \simeq r - \hat{n}\cdot\vec{x}'}$$



Then far from the source,

$$\vec{A}(\vec{x}) \xrightarrow[r\gg d]{} \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')\,e^{-ik\hat{n}\cdot\vec{x}'}\,d^3x'$$

and we see that $\vec{A}(\vec{x})$ far from the source is an <u>OUTGOING</u> spherical wave, decaying like $\frac{1}{r}$.

The angle dependence comes from $\hat{n} = (\theta,\phi)$, and we can write asymptotically,

$$\vec{A}(\vec{x}) = \vec{F}^{(k)}(\theta,\phi)\,\frac{e^{ikr}}{r}$$

From this, we obtain the fields:

$$\vec{H} = \frac{1}{\mu_0}\nabla\times\vec{A} \xrightarrow[kr\gg 1]{} \hat{\theta}\,\frac{e^{ikr}}{r}(-ik)\,F_\phi^{(k)}(\theta,\phi)$$

$$+\,\hat{\phi}\,\frac{e^{ikr}}{r}\,(ik)\,F_\theta^{(k)}(\theta,\phi) + O(r^{-2})$$

and

$$\vec{E} = \frac{iZ_0}{k}\nabla\times\vec{H} \xrightarrow[kr\gg 1]{} \frac{iZ_0 k}{\mu_0}\,\frac{e^{ikr}}{r}\left(\hat{\theta}\,F_\theta^{(k)}(\theta,\phi)+\hat{\phi}\,F_\phi^{(k)}(\theta,\phi)\right)$$

$$+\,O(r^{-2})$$

NOTES These fields obey:

$$\vec{H}\cdot\hat{n} = 0$$

$$\vec{E}\cdot\hat{n} = 0$$

$$\vec{H}\cdot\vec{E} = 0$$

$\Rightarrow$ The electric and magnetic fields are *transverse* to the propagation axis, and to each other, as expected.

Moreover, they fall off like $\frac{1}{r}$.

If $kx' \simeq kd \ll 1 \Rightarrow e^{X} = 1 + X + \frac{X^2}{2!} + \cdots$

and

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\sum_{l=0}^{\infty}\frac{1}{l!}\int d^3x'\,\vec{J}(\vec{x}')\,(-ik\hat{n}\cdot\vec{x}')^{l}$$

successive terms are of order $(kd)^{l}$.

Typically, the dominant radiated term is the lowest-ℓ term that is nonzero

Explicit treatment of the lowest few multipoles:

ELECTRIC MONOPOLE

Consider $\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}', t-\frac{R}{c})}{R}$

$$\xrightarrow[R\to r\to\infty]{} \frac{1}{4\pi\epsilon_0 r}\int d^3x' \rho(\vec{x}', t-\frac{R}{c})$$

$$= \frac{Q(t-\frac{R}{c})}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r},$$

since the total charge is t-independent

=> We conclude that:

- a charge monopole produces static fields only
- Fields with an $e^{-i\omega t}$ time-dependence NEVER have a monopole term
- the $\frac{1}{r}$ potential generates a $\frac{1}{r^2}$ E-field, not a $\frac{1}{r}$ field expected for radiation.

ELECTRIC DIPOLE RADIATION

The leading term in our expansion of the vector potential is the $l=0$ term:

$$\vec{A}(\vec{x}) \xrightarrow[l=0]{} \vec{A}_{dipole}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\vec{J}(\vec{x}')$$

At this point we derive a useful identity:

Consider
$$\int d^3x'\,J_i(\vec{x}') = \int d^3x'\,(\nabla' x_i')\cdot\vec{J}(\vec{x}')$$

and using $\nabla'\cdot(\psi\vec{a}) = \nabla\psi\cdot\vec{a} + \psi\nabla'\cdot\vec{a}$

with $\psi = x_i'$, $\vec{a} = \vec{J}(\vec{x}')$

$$\Rightarrow \int d^3x'\,J_i(\vec{x}') = \int d^3x'\Big[\underbrace{\nabla'\cdot\big(x_i'\vec{J}(\vec{x}')\big)}_{\text{use divergence theorem}} - x_i'\underbrace{\nabla'\cdot\vec{J}(\vec{x}')}_{= -\frac{\partial\rho}{\partial t} = +i\omega\rho}\Big]$$

use divergence theorem

$= \oint da'\, x_i'\,\hat{n}'\cdot\vec{J}(\vec{x}')$

$= 0$ normally, though not necessarily, e.g. for an infinite waveguide

$-\dfrac{\partial\rho}{\partial t} = +i\omega\rho$ by the charge continuity equation

So we have the following identity for a localized, monochromatic source,

$$\boxed{\int d^3x'\,\vec{J}(\vec{x}') = -i\omega\int \vec{x}'\rho(\vec{x}')\,d^3x'}$$

Hence we can rewrite the vector potential for electric dipole radiation as

$$\vec{A}_{dipole}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\frac{e^{ikr}}{r}\underbrace{\int d^3x'\,\vec{x}'\rho(\vec{x}')}_{``\vec{p}\text{, the electric dipole moment!}}$$

and we have the key result:

$$\boxed{\vec{A}_{dipole}(\vec{x}) = -\frac{i\omega\mu_0}{4\pi}\vec{p}\,\frac{e^{ikr}}{r}}$$

Now we calculate the fields, using

$$\nabla\times(\psi\vec{p}) = \nabla\psi\times\vec{p} + \psi\,\cancel{\nabla\times\vec{p}} \rightarrow 0 \text{ since } \vec{p} = \text{constant}$$

$$\Rightarrow \vec{H} = \frac{1}{\mu_0}\nabla\times\vec{A} = -\frac{ikc}{4\pi}\left(\nabla\frac{e^{ikr}}{r}\right)\times\vec{p}$$

where $\nabla\frac{e^{ikr}}{r} = \hat{n}\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}$, $\hat{n} = \hat{r}$

$$\Rightarrow \boxed{\vec{H}\xrightarrow[r\to\infty]{}\frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}} + O\left(\frac{1}{r^2}\right)$$

Notice that the dominant term at $r\to\infty$ came here from $\nabla e^{ikr} \rightarrow ik\hat{r}e^{ikr}$ and we will use to simplify subsequent derivations

e.g. the E-field in the <u>far zone</u> is

$$\vec{E} = \frac{iZ_0}{k}\nabla\times\vec{H} \longrightarrow -Z_0\,\hat{n}\times\vec{H} \quad + O\left(\frac{1}{r^2}\right)$$

$$\Rightarrow \boxed{\vec{E} \xrightarrow[r\to\infty]{} \frac{cZ_0k^2}{4\pi}(\hat{n}\times\vec{p})\times\hat{n}\;\frac{e^{ikr}}{r}}$$

(note that $cZ_0 = \frac{1}{\epsilon_0}$)

And a particularly important quantity is the time-averaged radiated power:

$$\langle\vec{S}\rangle = \frac{1}{2}\,\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right) \propto \frac{1}{r^2} \text{ at } r\to\infty$$

And the <u>ANGULAR DISTRIBUTION</u> of power radiated is defined as the quantity

$$\frac{dP}{d\Omega} \equiv r^2\,\hat{n}\cdot\langle\vec{S}\rangle = \frac{\mathrm{Re}}{2}\,\frac{c^2k^4Z_0}{16\pi^2}\,\underbrace{[(\hat{n}\times\hat{p})\times\hat{n}]\times(\hat{n}\times\vec{p}^*)}\cdot\hat{n}$$

$$= -(\hat{n}\times\vec{p}^*)\times[(\hat{n}\times\vec{p})\times\hat{n}]$$

$$= -(\hat{n}\times\vec{p})\,\cancel{(\hat{n}\times\vec{p}^*)\cdot\hat{n}}\to 0 + \hat{n}\,|\hat{n}\times\vec{p}|^2 \qquad (\text{BAC}-\text{CAB rule})$$

$$= \hat{n}\,|\hat{n}\times(\hat{n}\times\vec{p})|^2$$

and as a special case when $\hat{n}\cdot\vec{p} = \cos\theta = \text{real}$, then $|\hat{n}\times\vec{p}|^2 = \sin^2\theta$

and finally

$$\boxed{\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2} |\vec{p}|^2 \sin^2\theta}$$

this is an important formula, but remember, it is valid ONLY if $\vec{p}$ = real electric dipole moment vector, to within an OVERALL phase factor

i.e. this is not applicable to a ROTATING dipole

Another key quantity is the TOTAL POWER RADIATED, namely

$$P = \int \frac{dP}{d\Omega} d\Omega = 2\pi \int_0^\pi \sin\theta \, d\theta \, \frac{dP}{d\Omega}$$

$$\Rightarrow \text{need } \int_0^\pi \sin^3\theta \, d\theta = \frac{4}{3}$$

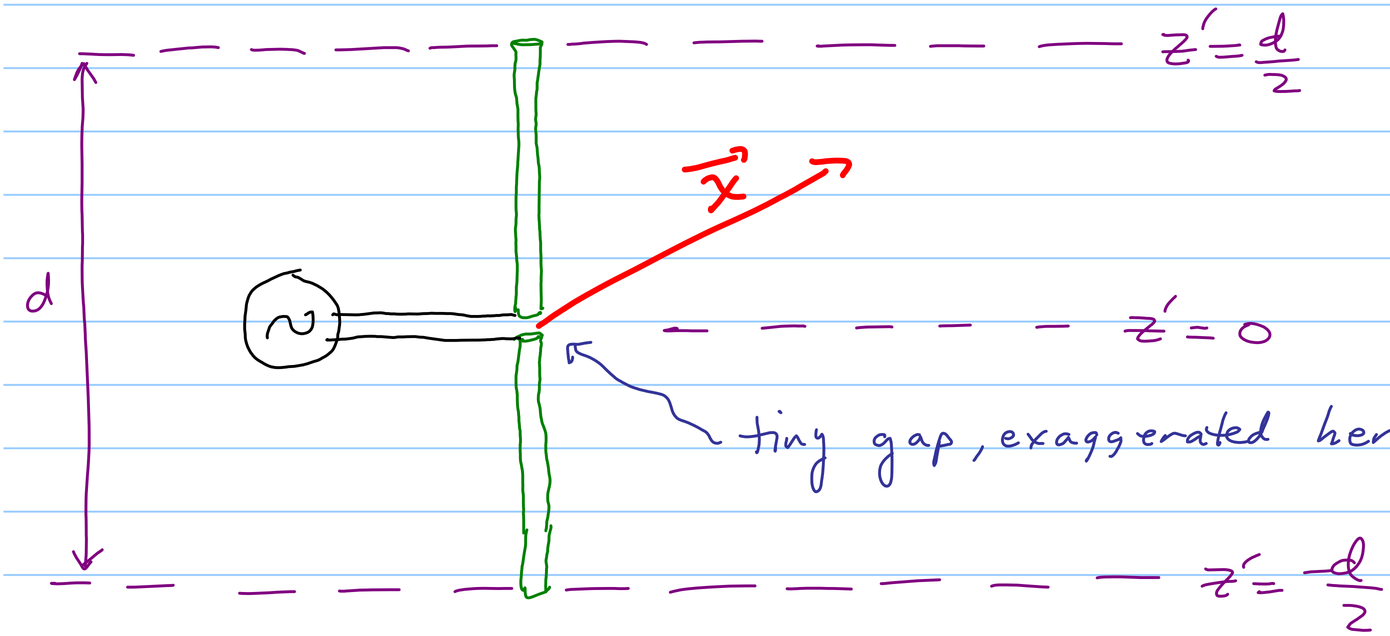
So finally

$$\boxed{P = \frac{Z_0 c^2 k^4}{12\pi} |\vec{p}|^2}$$

again for real $\vec{p}$

Example of an electric dipole radiator

– the linear, center-fed antenna with $d \ll \lambda$



Assumptions we will make:

(a) The sources and fields have a harmonic time-dependence $e^{-i\omega t}$

(b) The instantaneous current I drops linearly to zero at the ends of the antenna, and is in the same direction in each half

i.e. $I(z)e^{-i\omega t} = I_0\left(1 - \frac{2|z|}{d}\right)e^{-i\omega t}$

or the spatial current density is

$$\vec{J}(x,y,z) = \delta(x)\,\delta(y)\,I_0\left(1 - \frac{2|z|}{d}\right)\hat{z}$$

for $|z| \le d/2$

From the identity derived above (bottom of notes p. 51) we could compute the electric dipole moment using either of the following forms:

$$\vec{p} = \int \vec{x}'\rho(\vec{x}')d^3x' = -\frac{1}{i\omega}\int d^3x'\,\vec{J}(\vec{x}')$$

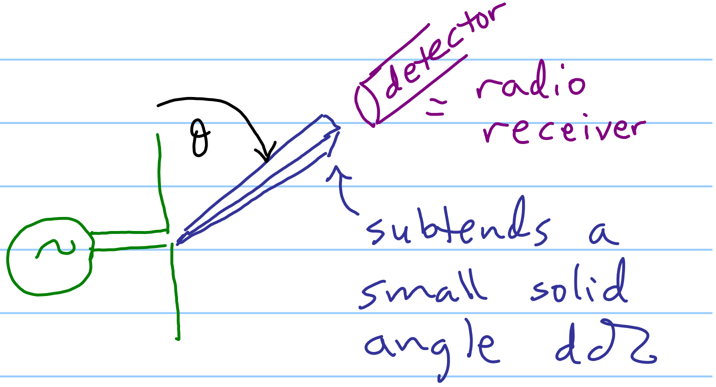
where $\rho = \frac{1}{i\omega}\nabla\cdot\vec{J} = \frac{I_0}{i\omega}\delta(x)\delta(y)\frac{\partial}{\partial z}\left(1-\frac{2|z|}{d}\right)$

or we could write this as a charge per unit length as

$$\tilde{\rho}(z) = \int dx\,dy\,\rho = \pm\frac{2i}{wd}I_0 \begin{cases} \text{at } 0<z<\frac{d}{2} \\ \text{or} \\ -\frac{d}{2}<z<0 \\ \text{resp.}\end{cases}$$

$$\Rightarrow \vec{p} = \hat{z}\int_{-\frac{d}{2}}^{0} dz'\,z'\left(-\frac{2iI_0}{wd}\right) + \hat{z}\int_{0}^{\frac{d}{2}} dz'\,z'\left(\frac{2iI_0}{wd}\right)$$

or $\vec{p} = \hat{z}\,\frac{iI_0 d}{2\omega}$



Since the dipole moment $\vec{p}$ is real to within an overall phase factor (i), we can simply plug this in to determine the angular distribution of radiated power, namely

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4\sin^2\theta\,|\vec{p}|^2 = \frac{\cancel{k^2}\cancel{c^2}\,I_0 Z_0^{2}(kd)^2}{(32\pi^2)(4\cancel{\omega^2})}\sin^2\theta$$

or $$\boxed{\frac{dP}{d\Omega} = \frac{I_o^2 Z_o}{128\pi^2}(kd)^2 \sin^2\theta}$$

and the total average power radiated is

$$P = \frac{Z_o c^2 k^4}{12\pi} |\vec{p}|^2 \quad \text{or} \quad \boxed{P = \frac{Z_o I_o^2 (kd)^2}{48\pi}}$$

note: $P \propto \omega^2$ or

$P \propto \left(\frac{d}{\lambda}\right)^2$ for $\lambda >> d$

If we view the antenna as an AC-circuit component with ohmic losses, i.e. with an effective resistance, we can equate the radiated power P to

$$P = \frac{1}{2} |I_o|^2 R_{rad}$$

where we denote R_{rad} = radiation resistance

Then for this example,

$$R_{rad} = \frac{Z_o (kd)^2}{24\pi} \simeq 5 (kd)^2 \text{ ohms}$$

(assuming the antenna is a perfect conductor)

Magnetic Dipole and Electric Quadrupole Radiation

Both contributions are included in the $l=1$ term of Eq. 9.11:

$$\vec{A}(\vec{x}) = \mu_0 ik \sum_{l,m} h_l^{(1)}(kr) Y_{lm}(\hat{x}) \int \vec{J}(\vec{x}') j_l(kr') Y^*_{lm}(\hat{x}') d^3x'$$

keeping $l=1$ only

$$\longrightarrow \mu_0 ik\, h_1^{(1)}(kr) \sum_{m=-1}^{1} Y_{1m}(\hat{n}) \int \vec{J}(\vec{x}') j_1(kr') Y^*_{1m}(\hat{x}') d^3x'$$

Any standard reference can be consulted to see that:

$$h_1^{(1)}(kr) = -\frac{e^{ikr}}{kr}\left(1+\frac{i}{kr}\right)$$

and

$$j_1(kr) \xrightarrow[kr<<1]{} \frac{kr}{3} + O(kr)^3$$

and the spherical harmonic addition implies that

$$\sum_m Y_{lm}(\hat{n}) Y^*_{lm}(\hat{x}') = \frac{2l+1}{4\pi} P_l(\hat{n}\cdot\hat{x}') \xrightarrow[l\to 1]{} \frac{3}{4\pi}\hat{n}\cdot\hat{x}'$$

$$\Rightarrow \vec{A}(\vec{x}) = -\frac{3\mu_0}{4\pi} ik \frac{e^{ikr}}{kr}\left(1+\frac{i}{kr}\right) \int d^3x' \vec{J}(\vec{x}') \frac{kr'}{3} \hat{n}\cdot\hat{x}'$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right) \int d^3x' \vec{J}(\vec{x}') \hat{n}\cdot\vec{x}'$$

We will see that this expression includes both the magnetic dipole AND electric quadrupole bits.

To separate them requires a bit of study. The CONCEPT of this derivation is the same as before, namely to use $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$ to replace $\vec{J}$ by ρ.

However, recall that any vector field such as $\vec{J}$ can be written as the SUM of a longitudinal ($\vec{J}_\ell$) and a transverse ($\vec{J}_t$) part, such that $\nabla \times \vec{J}_\ell = 0$ and $\nabla \cdot \vec{J}_t = 0$ (see pp. 241-2)

Our previous logic led to terms like $\nabla \cdot \vec{J} = i\omega\rho$ thus works ONLY for the longitudinal $\vec{J}_\ell$ and NOT for $\vec{J}_t$.

$\Rightarrow$ We must treat $\vec{J}_t$ explicitly (this will turn out to be the magnetic dipole part)

** WARNING ** This development may seem messy and ad hoc. It is more elegant + systematic to do this using vector spherical harmonics, Sec. 9.7.

But we will tackle it here using identities, e.g:

$$\hat{n} \times (\vec{x}\,' \times \vec{J}) = \vec{x}\,'(\hat{n} \cdot \vec{J}) - \vec{J}(\hat{n} \cdot \vec{x}\,')$$

or

$$(\hat{n} \cdot \vec{x}\,')\vec{J} = \vec{x}\,'(\hat{n} \cdot \vec{J}) - \hat{n} \times (\vec{x}\,' \times \vec{J})$$

add $(\hat{n} \cdot \vec{x}\,')\vec{J}$ to both sides

$$\Rightarrow 2(\hat{n} \cdot \vec{x}\,')\vec{J} = \vec{x}\,'(\hat{n} \cdot \vec{J}) + (\hat{n} \cdot \vec{x}\,')\vec{J} - \hat{n} \times (\vec{x}\,' \times \vec{J})$$

$$\Rightarrow (\hat{n}\cdot\vec{x}\,')\vec{J} = \frac{1}{2}\left[(\hat{n}\cdot\vec{x}\,')\vec{J} + \vec{x}\,'(\hat{n}\cdot\vec{J})\right]$$
$$- \frac{1}{2}\hat{n}\times(\vec{x}\,'\times\vec{J})$$

observe: this term is symmetric under interchange of $\vec{x}\,'$ and $\vec{J}$

this term is antisymmetric under $\vec{x}\,' \longleftrightarrow \vec{J}$

Initially, just consider these 1st two symmetric terms
$\Rightarrow$ ELECTRIC QUADRUPOLE

Recall the identity we proved,

$$\int d^3x'\, J_i(\vec{x}\,') = -\int d^3x'\, x_i'\, \nabla'\cdot\vec{J}(\vec{x}\,')$$

but replace $J_i \longrightarrow (\hat{n}\cdot\vec{x}\,')J_i$

$$\Rightarrow \int d^3x'\,(\hat{n}\cdot\vec{x}\,')J_i(\vec{x}\,') = -\int d^3x'\, x_i'\, \nabla'\cdot\left[(\hat{n}\cdot\vec{x}\,')\vec{J}\right]$$
$$= -\int d^3x'\, x_i'\left\{\nabla'(\hat{n}\cdot\vec{x}\,')\cdot\vec{J} + (\hat{n}\cdot\vec{x}\,')\nabla'\cdot\vec{J}\right\}$$

and

$$\left[\nabla'(\hat{n}\cdot\vec{x}\,')\right]\cdot\vec{J} = \hat{n}\cdot\vec{J}$$

$$\longrightarrow = (\hat{n}\cdot\nabla')\vec{x}\,' = \hat{x}'(\hat{n}\cdot\hat{x}) + \hat{y}'(\hat{n}\cdot\hat{y}') + \hat{z}'(\hat{n}\cdot\hat{z}')$$
$$= \hat{n}$$

$$\Rightarrow \int d^3x'\,(\hat{n}\cdot\vec{x}\,')\vec{J}(\vec{x}\,')$$
$$= -\int d^3x'\,\vec{x}\,'\left[\hat{n}\cdot\vec{J} + (\hat{n}\cdot\vec{x}\,')(\nabla'\cdot\vec{J})\right]$$

iwp

or finally we obtain

$$\boxed{\int d^3x' \left[(\hat{n}\cdot\vec{x}')\vec{J} + \vec{x}'(\hat{n}\cdot\vec{J}) \right] = -i\omega \int d^3x' \, \vec{x}'(\hat{n}\cdot\vec{x}')\rho(\vec{x}')}$$

Now, plugging these identities into our $l=1$ expression for $\vec{A}$ gives

$$\vec{A}_{l=1}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\left(1+\frac{i}{kr}\right)$$

$$\times\left\{ -\frac{i\omega}{2}\int d^3x' \, \vec{x}'(\hat{n}\cdot\vec{x}')\rho(\vec{x}') \right.$$

$$\left. - \hat{n}\times\int \frac{\vec{x}'\times\vec{J}}{2} d^3x' \right\}$$

components of electric quadrupole

magnetic dipole

Now consider these two contributions separately:

(1) MAGNETIC DIPOLE

$\vec{m} = \frac{1}{2}\int \vec{x}'\times\vec{J} \, d^3x'$ as in Eq. 5.54

$$\vec{A}_{M1} = \frac{\mu_0}{4\pi} ik \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\hat{n}\times\vec{m}$$

So in the radiation zone, $\left|\frac{i}{kr}\right| << 1$, we obtain

$$\boxed{\begin{aligned} \vec{H}_{M1} &= \frac{1}{\mu_0}\nabla\times\vec{A}_{M1} = \frac{k^2}{4\pi}(\hat{n}\times\vec{m})\times\hat{n}\,\frac{e^{ikr}}{r} \\ &\text{and} \\ \vec{E}_{M1} &= Z_0\vec{H}_{M1}\times\hat{n} = -\frac{Z_0 k^2}{4\pi}\hat{n}\times\vec{m}\,\frac{e^{ikr}}{r} \end{aligned}}$$

There is a close parallel between the **E1** and **M1** expressions:

	E1	M1
$\vec{H}$	$\frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}$	$\frac{k^2}{4\pi}(\hat{n}\times\vec{m})\times\hat{n}\,\frac{e^{ikr}}{r}$
$\vec{E}$	$\frac{cZ_0k^2}{4\pi}(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r}$	$-\frac{Z_0k^2}{4\pi}\hat{n}\times\vec{m}\,\frac{e^{ikr}}{r}$
$\frac{dP}{d\Omega}$	$\frac{c^2Z_0k^4}{32\pi^2}\lvert(\hat{n}\times\vec{p})\times\hat{n}\rvert^2$	$\frac{Z_0k^4}{32\pi^2}\lvert(\hat{n}\times\vec{m})\times\hat{n}\rvert^2$

and there are close parallels with:

$$\vec{p}\longrightarrow\vec{m}$$
$$\vec{E}\longrightarrow\vec{H}$$
$$\vec{H}\longrightarrow\vec{E}$$

(2) Electric quadrupole radiation

$$\vec{A}_{E2}(\vec{x}) = -\frac{\mu_0}{4\pi}ck^2\frac{e^{ikr}}{r}\int\vec{x}\,'(\hat{n}\cdot\vec{x}\,')\rho(\vec{x}\,')\,d^3x' + O(r^{-2})$$

whereby $\vec{H}_{E2} = \frac{1}{\mu_0}ik\hat{n}\times\vec{A}_{E2} + O(r^{-2})$

$$= \frac{-ick^3}{8\pi}\frac{e^{ikr}}{r}\int(\hat{n}\times\vec{x}\,')(\hat{n}\cdot\vec{x}\,')\rho(\vec{x}\,')\,d^3x'$$

constants, can be factored out

Let's express this in terms of the quadrupole moment tensor (e.g., see Eq. 4.9):

$$Q_{\alpha\beta} = \int \left(3x'_\alpha x'_\beta - r^2\delta_{\alpha\beta}\right)\rho(\vec{x}')\,d^3x'$$

$$\Rightarrow \quad \int \rho\,\vec{x}'(\vec{x}'\cdot\hat{n})\,d^3x' = \frac{1}{3}\overleftrightarrow{Q}\cdot\hat{n} + \left(\frac{1}{3}\int \rho r'^2 d^3x'\right)\overleftrightarrow{I}\cdot\hat{n}$$

$$\left(\begin{array}{l} \text{where } \overleftrightarrow{I} = 3\times 3 \text{ identity matrix} \\ \quad \Rightarrow \overleftrightarrow{I}\cdot\hat{n} = \hat{n} \quad , \text{ since } \sum_j \delta_{ij}n_j = n_i \end{array}\right)$$

and thus

$$\hat{n}\times\int \rho\,\vec{x}'(\vec{x}'\cdot\hat{n})\,d^3x' = \frac{1}{3}\hat{n}\times(\overleftrightarrow{Q}\cdot\hat{n})$$

which allows us to write more simply,

$$\boxed{\vec{H}_{E2}(\vec{x}) = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\,\hat{n}\times\underbrace{(\overleftrightarrow{Q}\cdot\hat{n})}_{\text{Jackson calls this vector } \vec{Q}(\hat{n})}}$$

and

$$\boxed{\begin{aligned} \vec{E}_{E2}(\vec{x}) &= -Z_0\,\hat{n}\times\vec{H}_{E2} \\ &= -\frac{iZ_0ck^3}{24\pi}\frac{e^{ikr}}{r}\left[\hat{n}\times(\overleftrightarrow{Q}\cdot\hat{n})\right]\times\hat{n} \end{aligned}}$$

and

$$\boxed{\frac{dP}{d\Omega} = \frac{c^2Z_0}{1152\pi^2}k^6\left|\left[\hat{n}\times(\overleftrightarrow{Q}\cdot\hat{n})\right]\times\hat{n}\right|^2}$$

Aside: we can simplify the E2 angular distribution of power, e.g. consider

$$\left[\hat{n}\times(\overleftrightarrow{Q}\cdot\hat{n})\right]\cdot\left[\hat{n}\times(\overleftrightarrow{Q}^{*}\cdot\hat{n})\right]$$

$$= \epsilon_{ijk}\, n_j\, Q_{kl}\, n_l\, \epsilon_{ij'k'}\, n_{j'}\, Q_{k'l'}\, n_{l'}$$

(using repeated index sum convention)

but

$$\epsilon_{ijk}\epsilon_{ij'k'} = \delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}$$

thus $[\ ]\cdot[\ ]$

$$= \overset{\text{“}\hat{n}\cdot\hat{n}=1\text{”}}{(n_j n_{j'}\delta_{jj'})}\, Q_{kl}\, Q^{*}_{k'l'}\,\delta_{kk'}\, n_l\, n_{l'}$$

$$- n_j\, Q^{*}_{k'l'}\,\delta_{jk'}\, n_{j'}\, Q_{kl}\,\delta_{kj'}\, n_l\, n_{l'}$$

$$= n_l (Q^{\dagger}Q)_{ll'}\, n_{l'} - (n_j Q^{*}_{jl'} n_{l'})(n_k Q_{kl} n_l)$$

or finally

$$\boxed{\left|\hat{n}\times(\overleftrightarrow{Q}\cdot\hat{n})\right|^2 = \hat{n}\cdot\overleftrightarrow{Q^{\dagger}Q}\cdot\hat{n} - \left|\hat{n}\cdot\overleftrightarrow{Q}\cdot\hat{n}\right|^2}$$

We can next derive the total power radiated, as follows, using 2 identities:

$$\int d\Omega\; n_\alpha n_\beta = \frac{4\pi}{3}\delta_{\alpha\beta}$$

$$\int d\Omega\; n_\alpha n_\beta n_\gamma n_\delta = \frac{4\pi}{15}\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)$$

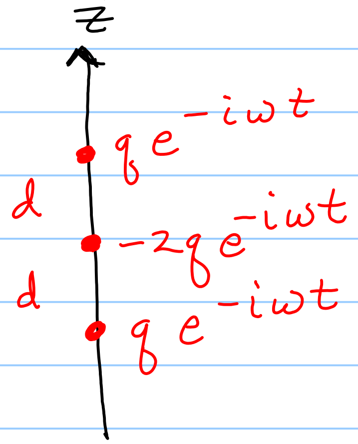
which gives finally

$$\boxed{P = \int \frac{dP}{d\Omega}\, d\Omega = \frac{c^2 Z_0 k^6}{1440\pi} \sum_{\alpha\beta} |Q_{\alpha\beta}|^2}$$

Example - electric quadrupole radiation

A simple quadrupole is 3 charges on a line:

$q_1 = q$ at $(0, 0, d)$
$q_2 = -2q$ at $(0,0,0)$
$q_3 = q$ at $(0,0,-d)$



$$\Rightarrow Q_{ij} = \sum_k q_k \left(3 x_i^{(k)} x_j^{(k)} - r^{(k)2} \delta_{ij}\right)$$

$$\Rightarrow Q_{zz} = q(3d^2 - d^2)(2) = 4qd^2 \equiv Q_0$$

$$Q_{xx} = Q_{yy} = q(0 - d^2)(2) = -2qd^2 = -\tfrac{1}{2} Q_0$$

and all off-diagonal terms vanish

$$\Rightarrow \overleftrightarrow{Q} \cdot \hat{n} \equiv \vec{Q}(\hat{n}) = -\frac{1}{2} Q_0 (\hat{x} n_x + \hat{y} n_y) + Q_0 \hat{z} n_z$$

$$= Q_0 \left(-\frac{1}{2} \hat{x} \sin\theta \cos\phi - \frac{1}{2} \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta\right)$$

and $\hat{n} \cdot \overleftrightarrow{Q} \cdot \hat{n} = Q_0 \left(-\frac{1}{2} \sin^2\theta + \cos^2\theta\right) = \frac{Q_0}{4}(1 + 3\cos 2\theta)$

$$= \frac{Q_0}{2} P_2(\cos\theta)$$

and $\overleftrightarrow{Q^{+}Q}\cdot\hat{n} = |Q_0|^2\left(\frac{1}{4}\hat{x}\sin\theta\cos\phi + \frac{1}{4}\hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta\right)$

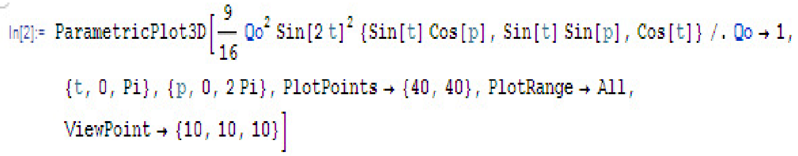
so
$$\hat{n}\cdot\overleftrightarrow{Q^{+}Q}\cdot\hat{n} = Q_0^2\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right) = \frac{Q_0^2}{8}\left(5+3\cos 2\theta\right)$$

giving
$$\hat{n}\cdot\overleftrightarrow{Q^{+}Q}\cdot\hat{n} - |\hat{n}\cdot\overleftrightarrow{Q}\cdot\hat{n}|^2 = \frac{9}{4}Q_0^2\sin^2\theta\cos^2\theta = \frac{9}{16}Q_0^2\sin^2 2\theta$$

and putting this together gives

$$\boxed{\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{1152\pi^2}\,\frac{9}{16}\,Q_0^2\sin^2 2\theta}$$

↖ interestingly, this vanishes at $\theta = 0, \frac{\pi}{2}$, and π



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In[2]:= ParametricPlot3D[9/16 Qo^2 Sin[2 t]^2 {Sin[t] Cos[p], Sin[t] Sin[p], Cos[t]} /. Qo → 1,
    {t, 0, Pi}, {p, 0, 2 Pi}, PlotPoints → {40, 40}, PlotRange → All,
    ViewPoint → {10, 10, 10}]
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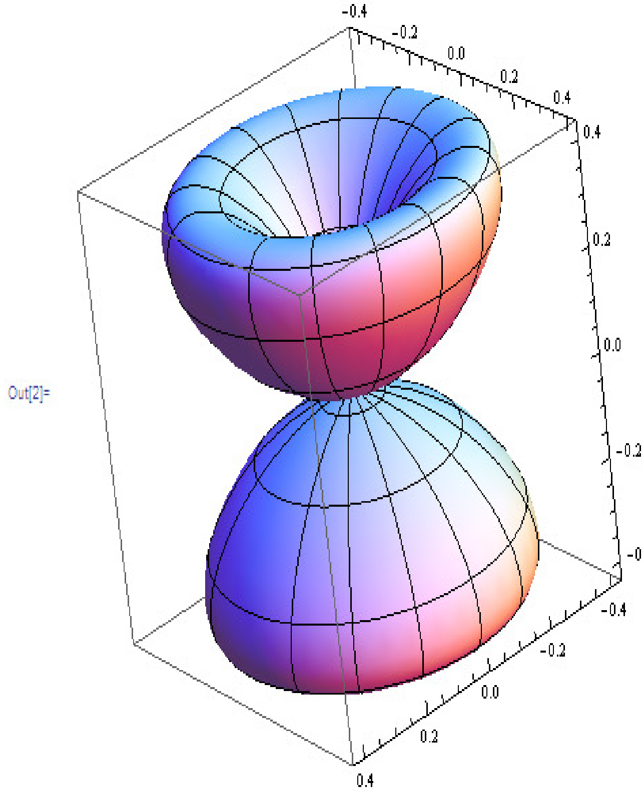


← angular distribution of radiated power from a linear quadrupole aligned with the z-axis.

Note that this is a spherical polar plot, which means that for each θ, ϕ, what is plotted is the value of $\frac{dP}{d\Omega}$ as the "distance" from (0,0,0)

In the long-wavelength limit, the multipole expansion converges rapidly.

i.e. consider $\left|\frac{A_{E2}}{A_{E1}}\right| \simeq \frac{kQ}{P} \simeq \frac{kqd^2}{qd} \simeq kd \ll 1$

and similarly

$$\frac{1}{2}\int d^3x'\, \vec{x}' \times \vec{J} \xrightarrow[\vec{J}\to\rho\vec{v}]{} \simeq qvd$$

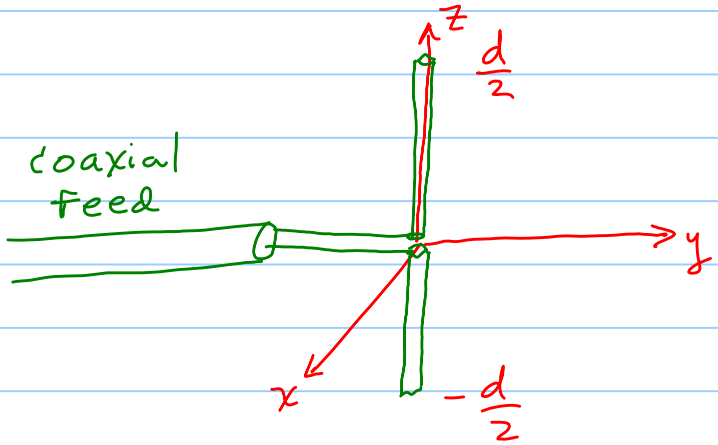
$$\left|\frac{A_{M1}}{A_{E1}}\right| \simeq \frac{m}{Pc} \simeq \frac{qvd}{qdc} \simeq \frac{v}{c} \simeq \frac{\omega d}{c}$$

$= kd \ll 1$ also.

Sec. 9.4 Explicit treatment of the center-fed antenna, without a multipole expansion

For this case, the full integral for $\vec{A}(\vec{x})$ can be evaluated in the radiation zone

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, e^{ik\hat{n}\cdot\vec{x}'} \vec{J}(\vec{x}')$$



We will assume that the current is given by

$$\vec{I}(z') = \hat{z}\, I_0 \sin\left(\frac{kd}{2} - k|z|\right) \times \theta\left(\frac{d}{2} - |z|\right)$$

for an infinitely-thin antenna

As in Chap. 5, we can make the usual replacement, $\int \vec{J}\, d^3x' (\cdots) \rightarrow \int \vec{I}\, d\vec{l}' (\cdots)$

Note 1 This procedure is only approximately valid, to pre-specify the currents and then find the radiation fields produced. In reality the fields can act back and influence the currents, but this effect is usually small and we neglect it here

Note 2 This is basically the same problem we treated before, except there we solved it in the limit $\lambda >> d$

Here we write the exact solution in the far zone as

$$\vec{A}(\vec{x}) = \hat{z}\,\frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\,I_0 \int_{-\frac{d}{2}}^{\frac{d}{2}} dz'\, \sin\left(\frac{kd}{2} - k|z'|\right) e^{-ikz'\cos\theta}$$

(using $k\hat{n}\cdot\vec{x}' = kz'\cos\theta$)

or

$$\vec{A}(\vec{x}) = \frac{\hat{z}\,\mu_0 I_0}{4\pi}\,\frac{e^{ikr}}{r}\,\frac{1}{2i}\left\{ \int_0^{d/2} e^{ik\frac{d}{2} - ikz'(1+\cos\theta)}\,dz' - \int_0^{\frac{d}{2}} e^{-ikd/2 + ikz'(1-\cos\theta)}\,dz' + \int_{-\frac{d}{2}}^{0} e^{ik\frac{d}{2} + ikz'(1-\cos\theta)}\,dz' - \int_{-\frac{d}{2}}^{0} e^{-ik\frac{d}{2} - ikz'(1+\cos\theta)}\,dz' \right\}$$

These integrals are all simple, giving:

$$\vec{A}(\vec{x}) = \hat{z}\,\frac{\mu_0}{4\pi}\,I_0\,\frac{e^{ikr}}{r}\,\frac{2}{k}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin^2\theta}\right]$$

and the fields derived from:

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A} \xrightarrow[\text{zone}]{\text{rad.}} \frac{1}{\mu_0} ik\hat{n} \times \vec{A}$$

$$\vec{E} = Z_0 \vec{H} \times \hat{n}$$

and the angular distribution of radiated power is

$$\frac{dP}{d\Omega} = \lim_{r \to \infty} \frac{1}{2} \mathrm{Re}\left[r^2 \hat{n} \cdot \vec{E} \times \vec{H}^* \right]$$

The vector structure is:

$$\hat{n} \cdot \left\{ -\hat{n} \times (\hat{n} \times \hat{z}) \right\} \times (\hat{n} \times \hat{z})^* = |\hat{n} \times (\hat{n} \times \hat{z})|^2$$

$$= \sin^2\theta$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{Z_0 I_0^2}{8\pi^2} \left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta} \right]^2}$$

This contains ALL multipole contributions

⇒ An important special case is the long-wavelength limit $kd = \frac{2\pi d}{\lambda} \ll 1$

$$\Rightarrow \frac{dP}{d\Omega} \xrightarrow[kd \ll 1]{} \frac{Z_0 I_0^2}{512\pi^2} (kd)^4 \sin^2\theta \, ,$$ which agrees with 9.28,

although we must observe that comparing 9.25 and 9.183 requires the currents to be related by

$$I_0^{\text{linear model}} = \frac{kd}{2} I_0(\text{sinusoidal model})$$

Note also that the ang. distrib. pattern from a half-wave antenna is <u>more</u> <u>directional</u> than the dipole pattern.

Sec. 9.5 Multipole expansion for a source in a waveguide

For starters, I give an overview of results from Sec 8.12B. Specifically, we can expand $\vec{E}, \vec{H}$ inside, in terms of separately propagating waves along $\pm\hat{z}$, namely

$$\vec{E} = \vec{E}^{(+)} + \vec{E}^{(-)}$$
$$\vec{H} = \vec{H}^{(+)} + \vec{H}^{(-)}$$

where the individual pieces have a mode expansion:

$$\vec{E}^{(\pm)} = \sum_\lambda A_\lambda^{(\pm)} \vec{E}_\lambda^{(\pm)}$$

$$\vec{H}^{(\pm)} = \sum_\lambda A_\lambda^{(\pm)} H_\lambda^{(\pm)}$$

For any given problem, the expansion coefficients $A_\lambda^{(\pm)}$ can often be found using the mode orthonormality relations, e.g. 8.131 - 134. Note that the TRANSVERSE FIELDS obey an orthonormality relation. I simply state these relations here, without proof.

The full fields have the following form for each mode:

$$\vec{E}_\lambda^{(\pm)}(x,y,z) = \left[\vec{E}_\lambda(x,y) + \vec{E}_{z\lambda}(x,y)\right] e^{\pm i k_\lambda z}$$

and

$$\vec{H}_\lambda^{(\pm)}(x,y,z) = \left[\pm \vec{H}_\lambda(x,y) + \vec{H}_{z\lambda}(x,y)\right] e^{\pm i k_\lambda z}$$

and the transverse component orthonormality relations are simply, normalizing the field modes to be <u>REAL</u>:

$$\int \vec{E}_\lambda \cdot \vec{E}_\mu \, da = \delta_{\lambda\mu}$$

and

$$\int \vec{H}_\lambda \cdot \vec{H}_\mu \, da = \frac{1}{Z_\lambda^2} \delta_{\lambda\mu}$$

and the time-averaged power flow obeys

$$\frac{1}{2} \int (\vec{E}_\lambda \times \vec{H}_\mu) \cdot \hat{z} \, da = \frac{1}{2 Z_\lambda} \delta_{\lambda\mu}$$

The longitudinal components also obey orthonormality relations,

$$\int E_{z\lambda} E_{z\mu} \, da = \frac{-\gamma_\lambda^2}{k_\lambda^2} \delta_{\lambda\mu} \qquad \text{TM modes}$$

$$\int H_{z\lambda} H_{z\mu} \, da = \frac{-\gamma_\lambda^2}{k_\lambda^2 Z_\lambda^2} \delta_{\lambda\mu} \qquad \text{TE modes}$$

The explicit normalized modes for a rectangular wave guide are in Eqs 8,135 – 8,136

Another aside from Chap. 8, pp 392 – 4 derives two key formulas needed in Chap. 9:

(1) A localized current source $\vec{J}(\vec{x})$ in the waveguide produces fields whose expansion coefficients are

note '$\mp$' order!

$$A_\lambda^{(\pm)} = -\frac{Z_\lambda}{2} \int_V \vec{J} \cdot \vec{E}_\lambda^{(\mp)} \, d^3x \qquad 8.146$$

AND

(2) A localized aperture in a waveguide generates additional fields in addition to an unperturbed propagating field $\vec{E}$, whose expansion coefficients are:

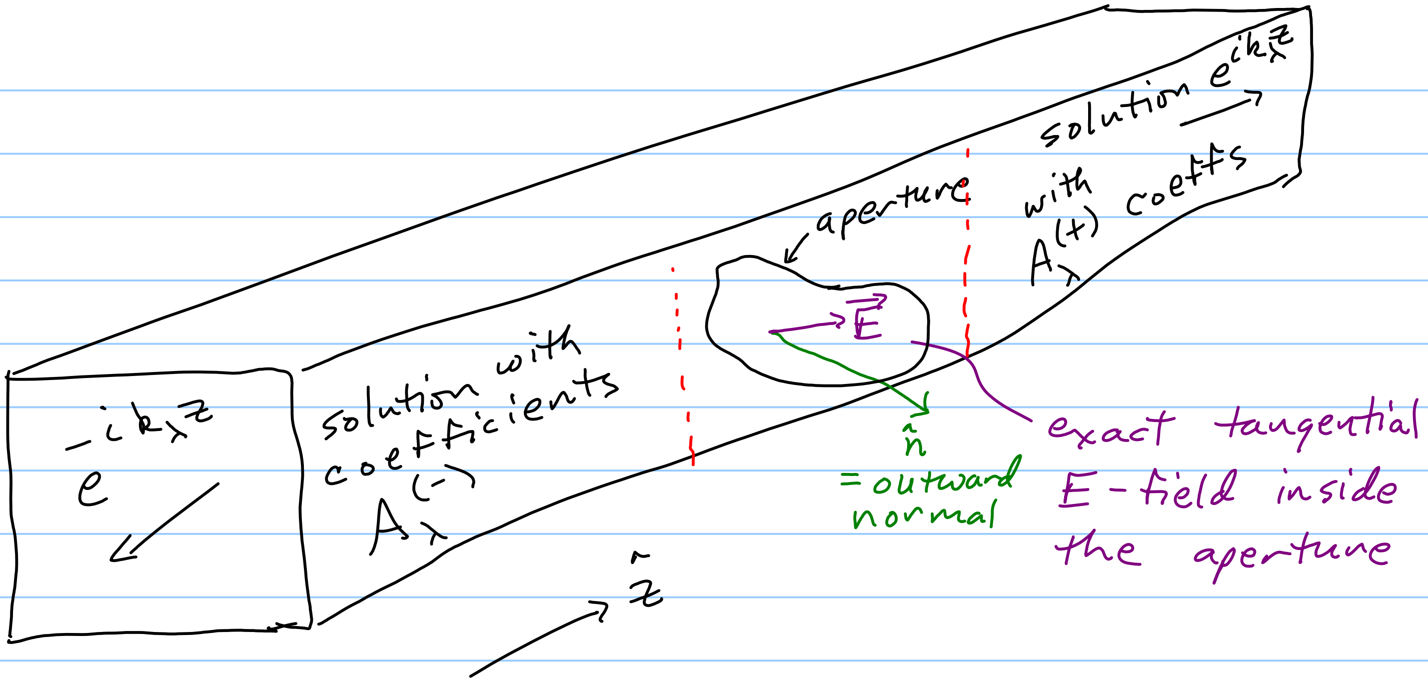
$$A_\lambda^{(\pm)} = \frac{Z_\lambda}{2} \int_{\text{aperture}} (\vec{E} \times \vec{H}_\lambda^{(\mp)}) \cdot \hat{n} \, da \qquad 8.147$$

where $\vec{E}$ = exact tangential $\vec{E}$-field in the aperture,

and

$\hat{n}$ = OUTWARDLY-directed normal unit vector

073



Now return to Sec 9.5, the multipole exp. for a small source inside a waveguide

↖ means small compared to the distance over which the fields vary

This "smallness" suggests that we can expand the $\vec{E}_\lambda^{(\pm)}$ into a Taylor series.

- use repeated index summation convention
- omit the labels λ and $(\pm)$ for simplicity:

$$\Rightarrow \int \vec{J}\cdot\vec{E}\,d^3x = \int \overbrace{\underbrace{J_\alpha(\vec{x})\,[\,E_\alpha(0)}_{\text{term 1}} + (\vec{x}\cdot\nabla)}^{\text{term 2}} E_\alpha(0) + \dots\,]\,d^3x$$

$$\text{term 1} = \vec{E}(0)\cdot\int \vec{J}\,d^3x = -i\omega\,\vec{p}\cdot\vec{E}(0) \quad \text{where}$$

$$\vec{p} = \frac{i}{\omega}\int \vec{J}(\vec{x})\,d^3x = \text{source electric dipole moment}$$
Eq. 9.14

$$\text{term 2} = \int d^3x \left\{ J_\alpha x_\beta \frac{\partial E_\alpha(0)}{\partial x_\beta} \right\} \Longleftarrow$$ now analyze this as we did Eq. 9.30

$$= \int d^3x \left\{ \frac{1}{4} \left(J_\alpha x_\beta - J_\beta x_\alpha \right) \left(\frac{\partial E_\alpha(0)}{\partial x_\beta} - \frac{\partial E_\beta(0)}{\partial x_\alpha} \right) \right.$$

$$\left. + \frac{1}{2} \left(J_\alpha x_\beta + J_\beta x_\alpha \right) \frac{\partial E_\alpha(0)}{\partial x_\beta} \right\}$$

gives a magnetic dipole term

gives an electric quadrupole contribution

or $$\text{term 2} = i\omega \, \vec{m} \cdot \vec{B}(0) - \frac{i\omega}{6} \left[\nabla \cdot \left(\overleftrightarrow{Q} \cdot \vec{E} \right) \right]_{\vec{x} \to 0}$$

So finally, this gives

$$A_\lambda^{(\pm)} = \frac{i\omega Z_\lambda}{2} \left\{ \vec{p} \cdot \vec{E}_\lambda^{(\mp)}(0) - \vec{m} \cdot \vec{B}_\lambda^{(\mp)}(0) + \frac{1}{6} \left[\nabla \cdot \left(\overleftrightarrow{Q} \cdot \vec{E}_\lambda^{(\mp)} \right) \right]_{\vec{x} \to 0} + \dots \right\} \quad (9.69)$$

Similarly, if there is a localized aperture in a waveguide carrying EM-fields, we can expand in terms of the effective dipole moments associated with the aperture, (9.71)

$$A_\lambda^{(\pm)} = \frac{i\omega Z_\lambda}{4} \left[\vec{P}_{eff} \cdot \vec{E}_\lambda^{(\mp)}(0) - \vec{m}_{eff} \cdot \vec{B}_\lambda^{(\mp)}(0) + \dots \right]$$

where the "effective dipole moments" are

$$\vec{P}_{eff} = \epsilon \hat{n} \int \vec{x} \cdot \vec{E}_{tangential} \, da$$

and

$$\vec{m}_{eff} = \frac{2}{i\mu\omega} \int \hat{n} \times \vec{E}_{tangential} \, da$$

and $\vec{E}_{tangential}$ = exact tangential $\vec{E}$-field in the aperture

Sec. 9.6 spherical waves

Goal - develop some machinery that can systematically describe arbitrary multipoles

Analogy - the electrostatics PDEs treated last semester, e.g. $\nabla^2 \Phi = -\rho/\epsilon_0$, solved using

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}\,')\,d^3x'}{|\vec{x}-\vec{x}\,'|},$$

often simplifies with a spherical GF expansion,

$$G(\vec{x},\vec{x}\,') = \frac{1}{4\pi R} = \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\hat{x}')\,Y_{lm}(\hat{x})$$

with G chosen to obey $\nabla^2 G = -\delta(\vec{x}-\vec{x}\,')$

(Note 4π - factor difference compared to Chap.3)

Similarly, here we have in Chap. 9, equations like

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho/\epsilon_0 = \left(\nabla^2 + k^2\right)\Phi$$

for harmonic $e^{-i\omega t}$ dependence

And we tackle this by finding a GF:

$$\left(\nabla^2 + k^2\right) G(\vec{x},\vec{x}\,') = -\delta(\vec{x}-\vec{x}\,')$$

whose solution we found in Chap. 6, if no boundaries, to be

$$G(\vec{x},\vec{x}') = \frac{e^{ikR}}{4\pi R} \quad \text{, with } \vec{R} = \vec{x} - \vec{x}'$$

But it will be desirable to derive a form for G that factorizes the dependences on $\vec{x}$ and $\vec{x}'$, in the form:

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} = ik\sum_{l=0}^{\infty} j_l(kr_<)\,h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y^*_{lm}(\hat{x}')\,Y_{lm}(\hat{x})$$

Sketch of the derivation - use the completeness expansion in the angles, with an unknown radial GF $g_l(r,r')$:

$$\Rightarrow G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty} g_l(r,r')\sum_m Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

use some standard properties next:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\vec{L}^2(\theta,\phi)}{r^2}$$

where $\vec{L} = \frac{\vec{x}\times\vec{P}}{\hbar} = \vec{x}\times(-i\vec{\nabla})$ = QM angular mom operator divided by $\hbar$, making it dimensionless

and the Y_{lm} are simultaneous eigenfunctions of L_z and $\vec{L}^2$, namely:

$$L_z Y_{\ell m}(\theta,\phi) = m Y_{\ell m}(\theta,\phi)$$

$$\vec{L}^2 Y_{\ell m}(\theta,\phi) = \ell(\ell+1) Y_{\ell m}(\theta,\phi)$$

Returning to what we must solve,

$$\sum_{\ell m}\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\vec{L}^2}{r^2} + k^2\right] G_\ell(r,r') Y^*_{\ell m}(\hat{x}') Y_{\ell m}(\hat{x}) = -\frac{\delta(r-r')}{r^2}\delta(\hat{x}-\hat{x}')$$

and using $\delta(\hat{x}-\hat{x}') = \sum_{\ell m} Y^*_{\ell m}(\hat{x}') Y_{\ell m}(\hat{x})$

$$\Rightarrow \sum_{\ell m}\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2\right) G_\ell(r,r') Y^*_{\ell m}(\hat{x}') Y_{\ell m}(\hat{x}) = -\frac{\delta(r-r')}{r^2}\sum_{\ell m} Y^*_{\ell m}(\hat{x}') Y_{\ell m}(\hat{x})$$

and since this holds for all $\hat{x}, \hat{x}'$, we must have the radial GF obeying:

$$\boxed{\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} G_\ell(r,r') + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) G_\ell(r,r') = -\frac{\delta(r-r')}{r^2}}$$

This can be solved using the Sturm-Liouville GF that we worked out last semester, where we found that to solve

$$\hat{\mathcal{L}}_x G = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) G(x,x') + q(x) G(x,x') = -\frac{\delta(x-x')}{p(x')},$$

over an interval $a \leq x \leq b$, $a \leq x' \leq b$, we should first find 2 solutions of $\mathcal{L}u = 0$, obeying: $u_1(x)$ obeys desired BCs at $x = a$
$u_2(x)$ " " " " $x = b$

and then we have

$$\boxed{G(x,x') = -\frac{u_1(x_<)\,u_2(x_>)}{\beta(x')\,p(x')\,W(u_1,u_2)\big|_{x'}}}$$

This fits exactly our present problem, with

$$p(r) \longrightarrow r^2 \qquad \beta(r') \longrightarrow 1$$

and the homogeneous solutions are formed using $u_\ell(r) = j_\ell(kr), n_\ell(kr), h_\ell^{(1,2)}(kr)$

– any 2 of these can be superposed to match BCs.

For the infinite space problem, we choose

$u_1(r) = j_\ell(kr)$, regular at $r \to 0$

$u_2(r) = h_\ell^{(1)}(kr)$, outgoing wave at $r \to \infty$

and $p(r')W(u_1,u_2) =$ constant and can be evaluated at <u>any</u> r'-value.

e.g, use $r \to 0$ forms, where

$$h_\ell^{(1)}(kr) \xrightarrow[r\to 0]{} -i\frac{(2\ell-1)!!}{(kr)^{\ell+1}}$$

$$j_\ell(kr) \xrightarrow[r\to 0]{} \frac{(kr)^\ell}{(2\ell+1)!!}$$

and $p(r)W(u_1,u_2) = p(r)(u_1 u_2' - u_1' u_2)$

$$= r^2\left[\frac{-i(2\ell-1)!!}{(2\ell+1)!!}\right] W\left((kr)^\ell, (kr)^{-\ell-1}\right)$$

$$= -i/k$$

$$\Longrightarrow g_\ell(r,r') = ik\, j_\ell(kr_<)\, h_\ell^{(1)}(kr_>)$$

giving the final free space GF as

$$\boxed{G(\vec{x},\vec{x}\,') = ik\sum_{\ell m} j_\ell(kr_<)\, h_\ell^{(1)}(kr_>)\, Y^*_{\ell m}(\hat{x}')\, Y_{\ell m}(\hat{x})}$$

Sec. 9.7 Multipole expansion of the EM Fields

Recall For harmonic fields and sources, in a source-free region of space, Maxwell's equations take the form:

$$\boxed{\begin{array}{ll} \nabla\cdot\vec{E}=0 & \nabla\times\vec{E}=ikZ_0\vec{H} \\ \nabla\cdot\vec{H}=0 & \nabla\times\vec{H}=-\frac{ik}{Z_0}\vec{E} \end{array}}$$

These equations can be solved in two ways that connect with our ideas in Chap. 8 about "TM" versus "TE" modes:

Case (a) TM or "electric" multipoles,

First solve for $\vec{H}$, then compute $\vec{E}$:

(1) $(\nabla^2+k^2)\vec{H}=0$

(2) $\nabla\cdot\vec{H}=0$

(3) $\vec{E}=\frac{iZ_0}{k}\nabla\times\vec{H}$ ($\nabla\cdot\vec{E}$ automatically 0)

Case (b) TE or "magnetic multipoles",

First solve for $\vec{E}$, then compute $\vec{H}$:

(1) $(\nabla^2+k^2)\vec{E}=0$

(2) $\nabla\cdot\vec{E}=0$

(3) $\vec{H}=-\frac{i}{kZ_0}\nabla\times\vec{E}$

Our strategy: ① Solve the Helmholtz eqn

② Use zero divergence

③ Carry out a multipole expansion

Now, the general solution to the Helmholtz equation (homogeneous) in terms of constants $\vec{A}_{\ell m}^{(1,2)}$ is

$$\vec{H} = \sum_{\ell m} \left[\vec{A}_{\ell m}^{(1)} h_\ell^{(1)}(kr) + \vec{A}_{\ell m}^{(2)} h_\ell^{(2)}(kr) \right] Y_{\ell m}(\hat{x})$$

could choose any pair of sph. Bessel functions

But we now need to determine the $\vec{A}_{\ell m}^{(1,2)}$ such that $\nabla \cdot \vec{H} = 0$.

How to do this?

One way is to start from identities like

$$\nabla = \frac{\vec{r}}{r} \frac{\partial}{\partial r} - \frac{i}{r^2} \vec{r} \times \vec{L}$$

and then work out expressions of the form

$$\nabla \cdot \left(\sum_{\ell m} h_\ell(r)\, Y_{\ell m}(\theta, \phi)\, A_{\ell m} \right), \text{ etc.}$$

and this is essentially how Jackson proceeds.

I find it useful to develop vector spherical harmonics systematically, deviating from the text at this point.

References on VECTOR SPHERICAL HARMONICS:

(1) Varshalovich, Moskalev, Khersonskii
Quantum Theory of Angular Momentum, Sec. 7.3, p. 208

(2) Edmonds, Angular momentum in QM, Sec. 5.9, p. 81

(3) Blatt & Weisskopf, Theoretical Nuclear Physics,
Appendix B, p. 796

Background - Rotation of a vector field

Consider a classical vector field $\vec{A}(\vec{x})$ whose governing equations are invariant under coordinate rotations and reflections (parity)

$\Rightarrow$ Then if $\vec{A}(\vec{x})$ is a solution, so is another one $\vec{A}'(\vec{x})$ obtained by "grabbing" $\vec{A}(\vec{x})$ and rotating it as a whole through angle α about $\hat{z}$. The explicit expressions for the rotated components are

$$A'_x(x,y,z) = \cos\alpha\, A_x(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha, z)$$
$$- \sin\alpha\, A_y(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha, z)$$

$$A'_y(x,y,z) = \sin\alpha\, A_x(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha, z)$$
$$+ \cos\alpha\, A_y(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha, z)$$

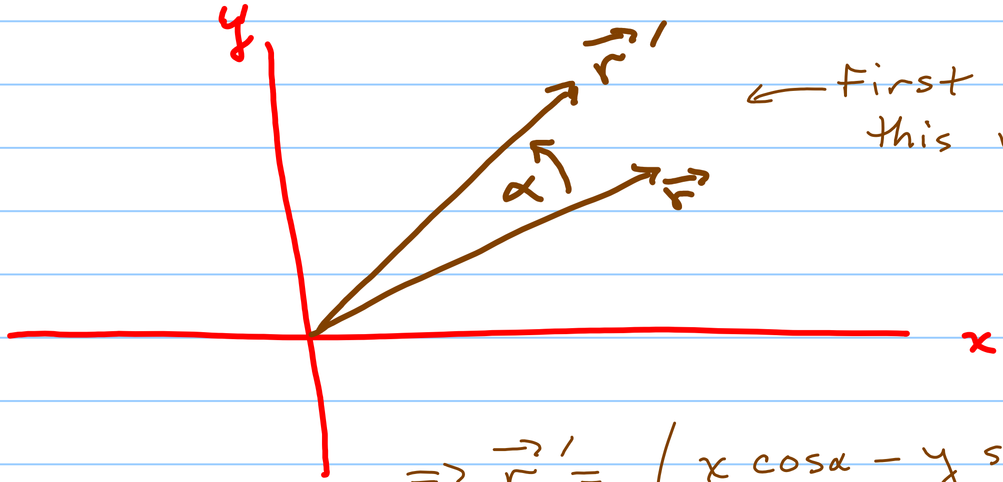
$$A'_z(x,y,z) = A_z(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha, z)$$

In these expressions, the "components" or directions of $\vec{A}$ appear to have been rotated in the positive sense by α, whereas the vector argument of $\vec{A}$ has been rotated by $-\alpha$. WHY?

To understand this consider a 2D vector field, as a concrete example,

$$\vec{A}(\phi) = \hat{\phi}\sin\phi = (-\hat{x}\sin\phi + \hat{y}\cos\phi)\sin\phi$$

and consider its rotation through a small, positive α:



← First just rotate this vector $\vec{r}$

$$\Rightarrow \vec{r}\,' = \begin{pmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \end{pmatrix}$$

or in matrix form,

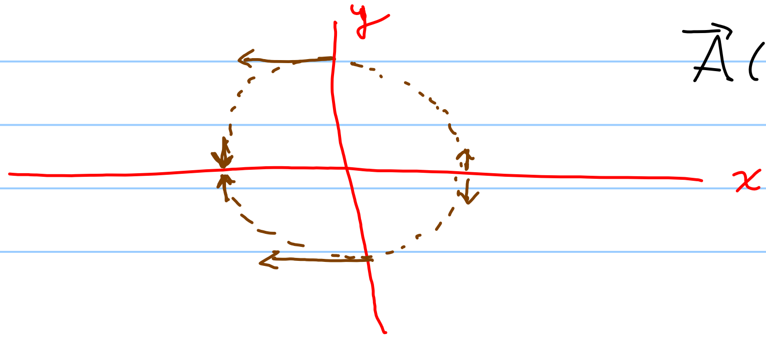
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

↑ call this rotation matrix, R for a simple CCW rotation through α

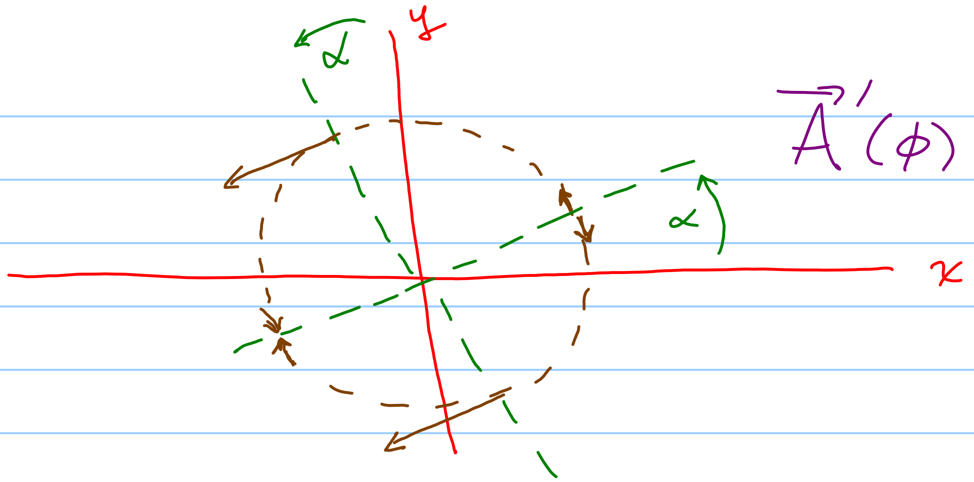
$$\Rightarrow \vec{r}\,' = R\,\vec{r}$$

Now look at the UNROTATED vector field

$\vec{A}(\phi)$



and after an active rotation through α, it looks like



To visualize this active rotation, think of "grabbing" the vector field and physically twisting it through α.

By inspection you should now be able to convince yourself that the NEW vector field is related to the old one by

$$\vec{A}'(\phi) = R\vec{A}(\phi-\alpha)$$

which is more generally expressed as:

$$\boxed{\vec{A}'(\vec{x}) = R\left[\vec{A}(R^{-1}\vec{x})\right]}$$

It is instructive to look at this rotation in the infinitesimal limit, working to 1st order, $\alpha \ll 1$,

$$\Rightarrow \vec{A}'(\vec{x}) \simeq \vec{A}(\vec{x}) - i\alpha J_z \vec{A}(\vec{x})$$

$$= \left[1 - i\alpha(L_z + S_z)\right]\vec{A}(\vec{x}),$$

where $$L_z = -i\frac{\partial}{\partial\phi}$$

and S_z is a 3x3 matrix defined by how the components or directions of the vector field rotate,

$$\Rightarrow S_z\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix} = \begin{pmatrix}-iA_y\\iA_x\\0\end{pmatrix} = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix}$$

Interpretation The operator $J_z = L_z + S_z$ can be viewed as analogous to the (QM) decomposition of the total angular momentum operator $\vec{J}$ into "orbital" $\vec{L}$ and "spin" $\vec{S}$ parts.

Observe - the eigenvectors and eigenvalues of S_z are

$$\boxed{\begin{array}{ll} & \underline{\text{eigenvalue}} \\ \hat{e}_{+1} = -\frac{1}{\sqrt{2}}(\hat{x}+i\hat{y}) & +1 \\ \hat{e}_0 = \hat{z} & 0 \\ \hat{e}_{-1} = \frac{1}{\sqrt{2}}(\hat{x}-i\hat{y}) & -1 \end{array}}$$

these are just the orthonormal spherical components of the ordinary unit vectors!

i.e. the S_z operator handles the reshuffling of the x,y - components of the vector field.

Some key notes and details:

(1) J_z commutes with the differential operators that define the vector field $\vec{A}$

(2) J_z commutes with the curl operator, i.e.

$$\nabla \times (J_z \vec{A}) = J_z \nabla \times \vec{A}$$

(3) J_x, J_y, J_z obey the standard angular momentum commutation relations, i.e.

$$[J_x, J_y] = J_x J_y - J_y J_x = i J_z, \text{ et cycl.}$$

(4) $L_z = -i \dfrac{\partial}{\partial \phi}$, $S_z \vec{A} = i \hat{z} \times \vec{A}$

$S_x \vec{A} = i \hat{x} \times \vec{A}$, etc.

(5) $$\boxed{\begin{aligned} \vec{S}^2 \hat{e}_m &= 2 \hat{e}_m \\ S_z \hat{e}_m &= m \hat{e}_m \end{aligned}}$$

i.e. $2 = s(s+1)$ with $s=1$ "spin 1"

(6) $\hat{e}_m^* \cdot \hat{e}_{m'} = \delta_{mm'}$

Based on the connections between $\vec{S}$ and $\vec{L}$ and the spin and orbital angular momenta in quantum mechanics, we can find the total angular momentum J eigenstates using CLEBSCH-GORDAN coefficients!

Definition of the vector spherical harmonics

tensor coupling notation: $\vec{Y}^{\ell(J)} \equiv \left[Y^{(\ell)} \otimes \hat{e}^{(1)} \right]^{(J)}$

whose explicit components are given by

$$\boxed{\vec{Y}^{(\ell)}_{J,M}(\theta,\phi) = \sum_{m=-\ell}^{+\ell} \sum_{\sigma=-1}^{1} Y_{\ell m}(\theta,\phi)\, \hat{e}_\sigma \langle \ell m, 1\sigma | JM \rangle}$$

and the allowed J-values follow from the usual rules of angular momentum addition, namely

$$|\ell-1| \leq J \leq \ell+1 \quad \text{and} \quad J \geq 0$$

i.e. here, $J = \underset{\text{absent if } \ell=0}{\ell-1}, \ \ell, \text{ or } \ell+1$

It turns out that the most important of the vector spherical harmonics, for E&M applications are those with $\boxed{J=\ell}$

and we denote them by $\boxed{\vec{X}_{\ell M}(\theta,\phi) = \vec{Y}^{(\ell)}_{\ell M}(\theta,\phi)}$

Also, it turns out that this can be written as

$$\boxed{\vec{X}_{\ell m}(\theta,\phi) = \frac{\vec{L}}{\sqrt{\ell(\ell+1)}} Y_{\ell m}(\theta,\phi)}$$

which coincides with Jackson Eq. 9.119

If desired, that preceding result can be checked by using:

$$L_z = -i\frac{\partial}{\partial\phi}, \quad L_\pm = L_x \pm iL_y = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

plus results known from QM, such as:

$$L_\pm Y_{\ell m} = \left[(\ell \mp m)(\ell \pm m + 1)\right]^{1/2} Y_{\ell, m\pm 1}$$

$$[\vec{L}^2, \vec{L}] = 0 \qquad \vec{L}^2 Y_{\ell m} = \ell(\ell+1) Y_{\ell m}$$

$$\vec{L} \times \vec{L} = i\vec{L}$$

$$[\vec{L}, \nabla^2] = 0 \qquad L_z Y_{\ell m} = m Y_{\ell m}$$

$$\vec{x} \cdot \vec{L} = 0$$

and

$$\vec{J}^2 \vec{Y}^\ell_{JM} = J(J+1)\vec{Y}^\ell_{JM} \qquad \vec{L}^2 \vec{Y}^\ell_{JM} = \ell(\ell+1)\vec{Y}^\ell_{JM}$$

$$J_z \vec{Y}^\ell_{JM} = M\vec{Y}^\ell_{JM} \qquad \vec{S}^2 \vec{Y}^\ell_{JM} = 2\vec{Y}^\ell_{JM}$$

and a key orthonormality relation is:

$$\int \vec{Y}^{\ell\,*}_{JM}(\theta,\phi) \cdot \vec{Y}^{\ell'}_{J'M'}(\theta,\phi)\, d\Omega = \delta_{JJ'}\delta_{\ell\ell'}\delta_{MM'}$$

And these $\vec{Y}^\ell_{JM}$ are complete, so given ANY 3D vector field $\vec{A}(\vec{x})$, we can write:

$$\vec{A}(\vec{x}) = \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \Big\{ F_{JM}(r)\, \vec{X}_{JM}(\hat{x}) + g_{JM}(r)\, \vec{Y}^{J+1}_{JM}(\hat{x}) + h_{JM}(r)\, \vec{Y}^{J-1}_{JM}(\hat{x}) \Big\}$$

where

$$F_{JM}(r) = \int d\Omega \left[\vec{X}_{JM}(\theta,\phi)\right]^{*} \cdot \vec{A}(r,\theta,\phi)$$

$$g_{JM}(r) = \int d\Omega \left[\vec{Y}^{J+1}_{JM}(\theta,\phi)\right]^{*} \cdot \vec{A}(r,\theta,\phi)$$

$$h_{JM}(r) = \int d\Omega \left[\vec{Y}^{J-1}_{JM}(\theta,\phi)\right]^{*} \cdot \vec{A}(r,\theta,\phi)$$

and where (Varshalovich 7.3.53):

$$\nabla \cdot \left[F(r)\, \vec{X}_{JM}(\theta,\phi)\right] = 0$$

$$\nabla \cdot \left[F(r)\, \vec{Y}^{J+1}_{JM}(\theta,\phi)\right] = -\left(\frac{J+1}{2J+1}\right)^{1/2} \left(\frac{d}{dr} + \frac{J+2}{r}\right) F(r)\, Y_{JM}(\theta,\phi)$$

$$\nabla \cdot \left[F(r)\, \vec{Y}^{J-1}_{JM}(\theta,\phi)\right] = \left(\frac{J}{2J+1}\right)^{1/2} \left(\frac{d}{dr} - \frac{J-1}{r}\right) F(r)\, Y_{JM}(\theta,\phi)$$

$$\hat{n}\, Y_{JM}(\hat{n}) = -\left(\frac{J+1}{2J+1}\right)^{1/2} \vec{Y}^{J+1}_{J,m}(\hat{n}) + \left(\frac{J}{2J+1}\right)^{1/2} \vec{Y}^{J-1}_{J,m}(\hat{n})$$

$$\nabla \left[F(r)\, Y_{\ell m}\right] = -\left(\frac{\ell+1}{2\ell+1}\right)^{1/2} \left(\frac{d}{dr} - \frac{\ell}{r}\right) F(r)\, \vec{Y}^{\ell+1}_{\ell m} + \left(\frac{\ell}{2\ell+1}\right)^{1/2} \left(\frac{d}{dr} + \frac{\ell+1}{r}\right) F(r)\, \vec{Y}^{\ell-1}_{\ell m}$$

$$\boxed{\begin{aligned}\nabla\times\left[F(r)\,\vec{X}_{JM}\right] &= i\left(\frac{J}{2J+1}\right)^{1/2}\left(\frac{d}{dr}-\frac{J}{r}\right)F(r)\,\vec{Y}^{J+1}_{JM}\\ &\quad + i\left(\frac{J+1}{2J+1}\right)^{1/2}\left(\frac{d}{dr}+\frac{J+1}{r}\right)F(r)\,\vec{Y}^{J-1}_{JM}\end{aligned}}$$

And for the important special case where

$F(r) \longrightarrow Z_J(kr)$ = <u>ANY</u> spherical Bessel solution, $j_J(kr)$, $h^{(1)}_J(kr)$, etc..,

$$\boxed{\begin{aligned}\frac{1}{k}\nabla\times\left[Z_\ell(kr)\,\vec{X}_{\ell m}\right] &= -i\left(\frac{\ell}{2\ell+1}\right)^{1/2} Z_{\ell+1}(kr)\,\vec{Y}^{\ell+1}_{\ell m}\\ &\quad + i\left(\frac{\ell+1}{2\ell+1}\right)^{1/2} Z_{\ell-1}(kr)\,\vec{Y}^{\ell-1}_{\ell m}\\[1ex] \frac{1}{k}\nabla\times\left[Z_{\ell+1}(kr)\,\vec{Y}^{\ell+1}_{\ell m}\right] &= i\left(\frac{\ell}{2\ell+1}\right)^{1/2} Z_\ell(kr)\,\vec{X}_{\ell m}\\[1ex] \frac{1}{k}\nabla\times\left[Z_{\ell-1}(kr)\,\vec{Y}^{\ell-1}_{\ell m}\right] &= -i\left(\frac{\ell+1}{2\ell+1}\right)^{1/2} Z_\ell(kr)\,\vec{X}_{\ell m}\end{aligned}}$$

Application to Physics!

Eq. V 7.3.53 above, i.e. $\nabla\cdot\left[F(r)\,\vec{X}_{\ell m}\right] = 0$, shows that we can enforce the <u>transversality</u> condition on either $\vec{E}$ or $\vec{H}$ simply by choosing to expand that field in terms of $\vec{X}_{\ell m}$ <u>only</u>.

e.g. for TM = "electric" multipole solutions

these obey $(\nabla^2+k^2)\vec{H}^{(E)}_{\ell m}=0$, $\nabla\cdot\vec{H}^{(E)}_{\ell m}=0$,

we can write $\vec{H}^{(E)}_{\ell m}=F_\ell(kr)\vec{X}_{\ell m}(\theta,\phi)$

and $\vec{E}^{(E)}_{\ell m}=\dfrac{iZ_0}{k}\nabla\times\vec{H}^{(E)}_{\ell m}$

where

$$F_\ell(kr)=A^{(1)}_\ell h^{(1)}_\ell(kr)+A^{(2)}_\ell h^{(2)}_\ell(kr)$$

could choose ANY two linearly-independent solutions of the spherical Bessel equation

Similarly, for TE = "magnetic" multipole solutions,

$$(\nabla^2+k^2)\vec{E}^{(M)}_{\ell m}=0 \quad \text{and} \quad \nabla\cdot\vec{E}^{(M)}_{\ell m}=0$$

are automatically satisfied by choosing

$$\vec{E}^{(M)}_{\ell m}=G_\ell(kr)\vec{X}_{\ell m}(\theta,\phi)$$

and then $\vec{H}^{(M)}_{\ell m}=\dfrac{-i}{kZ_0}\nabla\times\vec{E}^{(M)}_{\ell m}$

and $G_\ell(kr)=B^{(1)}_\ell h^{(1)}_\ell(kr)+B^{(2)}_\ell h^{(2)}_\ell(kr)$

GENERAL SOLUTIONS

The two sets just described form a complete set of solutions, so using them we can expand ANY solution of Maxwell's Equations in source-free space (for monochromatic waves), as

$$\vec{H} = \sum_{lm} \left\{ a_E^{(lm)} F_\ell(kr) \vec{X}_{lm} - \frac{i}{k} a_M^{(lm)} \nabla \times \left[G_\ell(kr) \vec{X}_{lm} \right] \right\}$$

$$\vec{E} = \sum_{lm} \left\{ \frac{i Z_0}{k} a_E^{(lm)} \nabla \times F_\ell(kr) \vec{X}_{lm} + Z_0 G_\ell(kr) \vec{X}_{lm} a_M^{(lm)} \right\}$$

=> "Fourier's Trick" can be used to find the coefficients. For a particularly simple form, see Jackson Eq. 9.123, namely:

$$\boxed{\begin{aligned} a_M^{(lm)} G_\ell(kr) &= \frac{k}{\sqrt{\ell(\ell+1)}} \int Y_{lm}^{*} \, \vec{r} \cdot \vec{H} \, d\Omega \\ Z_0 \, a_E^{(lm)} F_\ell(kr) &= \frac{-k}{\sqrt{\ell(\ell+1)}} \int Y_{lm}^{*} \, \vec{r} \cdot \vec{E} \, d\Omega \end{aligned}}$$

and this is a rather amazing result! Knowledge of $\vec{r} \cdot \vec{E}$ gives us the full electric multipole, and knowledge of $\vec{r} \cdot \vec{H}$ gives the full magnetic multipole!

See the course Lecture Notes directory for a Mathematica notebook that calculates vector spherical harmonics, and verifies that $\nabla\cdot[f(r)\vec{X}_{lm}] = 0$, etc.

Secs. 9.8, 9.9 – energy, angular momentum, and the angular distribution of radiation

Using these field expansions, it is straightforward to find various properties

ANGULAR MOMENTUM

From Chapter 6, problem 6.10, the instantaneous angular momentum density = $\frac{\text{angular momentum}}{\text{unit volume}}$ is

$$\vec{\mathcal{L}}^{\,inst.} = \mu\epsilon\, \vec{x}\times(\vec{E}\times\vec{H})$$

whereby the time-averaged $\vec{\mathcal{L}}$ for monochromatic fields is (in vacuum):

$$\vec{\mathcal{L}} = \frac{1}{2c^2}\,\mathrm{Re}\left[\vec{x}\times(\vec{E}\times\vec{H}^*)\right]$$

Jackson calls this $\vec{m}$ in Eq. 9.137.

We work this out now for an electric (TM) multipole, i.e. the ℓ-th, where

$$\vec{H}_\ell = \sum_{m=-\ell}^{\ell} a_E^{(\ell m)} \vec{X}_{\ell m} h_\ell^{(1)}(kr)$$

and $\vec{E}_\ell = \frac{i}{k} Z_0 \nabla \times \vec{H}_\ell$

$$\Rightarrow \vec{\mathcal{L}} = \frac{1}{2c^2} \mathrm{Re}\left\{ \underbrace{\vec{E}(\vec{x}\cdot\vec{H}^*)}_{\to\, 0 \text{ for TM!}} - \vec{H}^*(\vec{x}\cdot\vec{E}) \right\}$$

and note that $\vec{x}\cdot\vec{E} = \vec{x}\cdot\left(\frac{iZ_0}{k}\right)\nabla\times\vec{H}_\ell = \frac{iZ_0}{k}(\vec{x}\times\nabla)\cdot\vec{H}_\ell$

but $\vec{x}\times\nabla = -\frac{\vec{L}}{i}$

$$\Rightarrow \vec{\mathcal{L}}_\ell = \frac{1}{2c^2}\frac{Z_0}{k}\mathrm{Re}\left[\vec{H}_\ell^*(\vec{L}\cdot\vec{H}_\ell)\right] = \frac{\mu_0}{2\omega}\mathrm{Re}\left[\vec{H}^*(\vec{L}\cdot\vec{H})\right]$$

Then the angular momentum contained in a thin spherical shell between r and $r+dr$ is

$$d\vec{M}_\ell = \frac{\mu_0}{2\omega}\frac{dr}{k^2}\mathrm{Re}\left[\sum_{m,m'} a_E^{(\ell m')*} a_E^{(\ell m)} \int (\vec{L}\cdot\vec{X}_{\ell m'})^* \vec{X}_{\ell m}\, d\Omega\right]$$

now recall that $\vec{X}_{\ell m} = \frac{\vec{L}\, Y_{\ell m}}{\sqrt{\ell(\ell+1)}}$

so $(\vec{L}\cdot\vec{X}_{\ell m'})^*\vec{X}_{\ell m} = \cancel{\frac{\ell(\ell+1)}{\ell(\ell+1)}} \; Y_{\ell m'}^* \vec{L}\, Y_{\ell m}$

giving

$$\boxed{\frac{d\vec{M}_\ell}{dr} = \frac{\mu_0}{2\omega k^2}\,\mathrm{Re}\left[\sum_{mm'} a_E^{(\ell m')^*}\, a_E^{(\ell m)} \int Y^*_{\ell m'}\,\vec{L}\,Y_{\ell m}\,d\Omega\right]}$$

which can be simplified further if desired - see Jackson 9.141-143

For the special case where only one m-value is nonvanishing, i.e. a single (ℓ, m)-multipole, then the x,y-components vanish and

$$\boxed{\frac{dM_z}{dr} = \frac{\mu_0}{2\omega k^2}\, m \left|a_E^{(\ell m)}\right|^2}$$

We can similarly find the energy stored in this electric multipole between r, $r+dr$. For harmonic fields, the time-averaged energy per unit volume is

$$u = \frac{\epsilon_0}{4}\left(\vec{E}\cdot\vec{E}^* + Z_0^2\,\vec{H}\cdot\vec{H}^*\right)$$

these two terms are equal in the radiation zone

So the energy stored per unit dr is

$$\frac{dU}{dr} = \frac{\mu_0}{2k^2}\sum_{m,m'} a_E^{(\ell m')^*}\, a_E^{(\ell m)} \cancel{\int \vec{X}^*_{\ell m'}\cdot\vec{X}_{\ell m}\,d\Omega} \rightarrow \delta_{mm'}$$

and $$\frac{dU}{dr} = \frac{\mu_0}{2k^2} \sum_m \left| a_E^{(l,m)} \right|^2$$

Then if we compare this with the $\frac{dM_z}{dr}$ for a single (l,m)-mode, this ratio is:

$$\boxed{\frac{\frac{dM_z}{dr}}{\frac{dU}{dr}} = \frac{\text{angular momentum density}}{\text{energy density}} = \frac{m}{\omega}}$$

This result is consistent with the photon interpretation, where N photons in the (lm)-multipole carry angular momentum $N(m\hbar)$ and energy equal to $N(\hbar\omega)$, giving a ratio

$$\frac{N m\hbar}{N\hbar\omega} = \frac{m}{\omega} \quad \checkmark$$

Side comment – The above discussion only considers the photons' ORBITAL angular momentum ($\vec{L}$) but they can in general have SPIN angular momentum too, e.g. if circularly polarized.

Angular distribution of radiated power

⇒ go to large distances, retain the leading order terms only, use the usual short-cut that

$$\boxed{\nabla \longrightarrow ik\hat{n}}$$ in the radiation zone,

Also, for fields radiating outward from a localized source, the usual case, all radial solutions are $h_\ell^{(1)}(kr)$:

$$\Rightarrow \vec{H} = \sum_{\ell m}\left\{a_E^{(\ell,m)}\, h_\ell^{(1)}(kr)\,\vec{X}_{\ell m} - \frac{i}{k}a_M^{(\ell,m)}\,\nabla\times\left[h_\ell^{(1)}(kr)\vec{X}_{\ell m}\right]\right\}$$

and

$$\vec{E} = \sum_{\ell,m}\left\{\frac{iZ_0}{k}a_E^{(\ell,m)}\,\nabla\times h_\ell^{(1)}(kr)\,\vec{X}_{\ell m} + Z_0 h_\ell^{(1)}(kr)\vec{X}_{\ell m}a_M^{(\ell,m)}\right\}$$

and in the far zone,

$$h_\ell^{(1)}(kr) \xrightarrow[kr \gg \ell]{} (-i)^{\ell+1}\frac{e^{ikr}}{kr}$$

$$\boxed{\begin{aligned} &\Rightarrow \vec{H} \longrightarrow \frac{e^{ikr}}{kr}\sum_{\ell,m}(-i)^{\ell+1}\left[a_E^{(\ell,m)}\vec{X}_{\ell,m} + a_M^{(\ell m)}\,\hat{n}\times\vec{X}_{\ell m}\right] \\ &\vec{E} \longrightarrow Z_0\,\vec{H}\times\hat{n} \end{aligned}}$$

And the time-averaged power radiated per unit solid angle is

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{\ell,m}(-i)^{\ell+1}\left[a_E^{(\ell m)}\vec{X}_{\ell m}\times\hat{n} + a_M^{(\ell m)}\vec{X}_{\ell m}\right]\right|^2}$$

- Notice that interference between the different multipoles is generally complicated, but there is NO interference between the electric and magnetic multipoles, simply because

$$(\hat{n}\times\vec{L})\cdot\vec{L} = 0$$

- Note also that for a single multipole of a definite (ℓ, m), the angular dependence is identical for the E, M - multipoles, because $\hat{n}\cdot\hat{\theta} = 0 = \hat{n}\cdot\hat{\phi}$

$$\Rightarrow \quad |\hat{n}\times\vec{X}_{\ell m}|^2 = |\vec{X}_{\ell m}|^2$$

$$\Rightarrow \frac{dP^{(\ell m)}}{d\Omega} = \frac{Z_0}{2k^2}|a^{(\ell m)}|^2 \, |\vec{X}_{\ell m}|^2 \text{, or:}$$

$$\frac{dP^{(\ell m)}}{d\Omega} = \frac{Z_0|a^{(\ell m)}|^2}{2k^2\ell(\ell+1)}\left\{\frac{1}{2}(\ell-m)(\ell+m+1)|Y_{\ell,m+1}|^2 + m^2|Y_{\ell m}|^2 + \frac{1}{2}(\ell+m)(\ell-m+1)|Y_{\ell,m-1}|^2\right\}$$

⇒ experimentally, one can in principle deduce the multipolarity of the radiation just from the angular distribution of radiated power.

- But to distinguish electric from magnetic multipoles also requires a measurement of polarization.

- We can also easily determine the TOTAL RADIATED POWER, using

$$\int \vec{X}^{*}_{l'm'} \cdot \vec{X}_{lm} = \delta_{ll'} \, \delta_{mm'},$$

i.e.,

$$\boxed{P = \frac{Z_0}{2k^2} \sum_{l,m} \left(\left| a_E^{(l,m)} \right|^2 + \left| a_M^{(l,m)} \right|^2 \right)}$$

⇒ there is no interference in the total power P.

e.g. – see Table 9.1 on p. 437 for several simple cases.

For many physical systems, such as those in thermodynamic equilibrium, with no direction in space singled out, the source radiates an INCOHERENT sum of all m-components of the l-th multipole

We express this mathematically as

$$\frac{dP^{(\ell)}}{d\Omega} \longrightarrow \frac{Z_0}{2k^2} \sum_m |\vec{X}_{\ell,m}|^2 \, |a^{(\ell)}|^2$$

where $a^{(\ell m)} \longrightarrow a^{(\ell)}$, independent of m

but since $\sum_m |\vec{X}_{\ell m}|^2 = \frac{2\ell+1}{4\pi}$ = independent of θ, ϕ

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \underline{\text{isotropic}} \text{ in this case}}$$

Sec. 9.10 Sources of multipole radiation

Let's continue to assume purely harmonic $e^{-i\omega t}$ time-dependence for all sources and fields, and write down Maxwell's equations for $\vec{E}$ and $\vec{H}' \equiv \frac{\vec{B}}{\mu_0}$ including the sources:

$$\nabla \cdot \vec{H}' = 0 \qquad \nabla \times \vec{E} - ikZ_0 \vec{H}' = 0$$

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{H}' + \frac{ik\vec{E}}{Z_0} = \vec{J} + \nabla \times \vec{M}$$

where $\vec{M}$ = magnetization

and we assume that there are specified sources, namely $\rho(\vec{x})e^{-i\omega t}$, $\vec{J}(\vec{x})e^{-i\omega t}$, $\vec{M}(\vec{x})e^{-i\omega t}$

but omitting an explicit POLARIZATION source $\cancel{\vec{P}(\vec{x})e^{-i\omega t}}$

As always, we have the continuity equation too,

$$i\omega\rho = \nabla\cdot\vec{J}$$

Jackson's trick: Set $\boxed{\vec{E}' \equiv \vec{E} + \frac{i}{\omega\epsilon_0}\vec{J}}$,

which allows us to recast these equations for divergenceless "fields", and of course, outside the range of the sources,

$$\vec{E}' \to \vec{E} \quad \text{and} \quad \vec{H}' \to \vec{H}$$

whereas inside the source,

$$\vec{H}' = \frac{\vec{B}}{\mu_0} = \frac{\mu_0(\vec{H}+\vec{M})}{\mu_0} = \boxed{\vec{H} + \vec{M} = H'}$$

This gives new primed equations:

$$\boxed{\begin{aligned} \nabla\cdot\vec{H}' &= 0 \\ \nabla\cdot\vec{E}' &= 0 \end{aligned}} \qquad \boxed{\begin{aligned} \nabla\times\vec{E}' - ikZ_0\vec{H}' &= \frac{i}{\omega\epsilon_0}\nabla\times\vec{J} \\ \nabla\times\vec{H}' + \frac{ik}{Z_0}\vec{E}' &= \nabla\times\vec{M} \end{aligned}}$$

Taking the curl of these equations gives:

$$\boxed{\begin{aligned} (\nabla^2+k^2)\vec{H}' &= -\nabla\times(\vec{J}+\nabla\times\vec{M}) \\ (\nabla^2+k^2)\vec{E}' &= -iZ_0k\,\nabla\times\left(\vec{M}+\frac{1}{k^2}\nabla\times\vec{J}\right) \end{aligned}}$$

= inhomogeneous versions of Eqs. 9.108, 9.109, relevant when sources are present

Now, we saw from Eqs. 9.122, 9.123 that it is enough to know $\vec r\cdot\vec E'$ and $\vec r\cdot\vec H'$ to determine the multipole expansion coefficients.

$\Rightarrow$ Let's utilize this idea, through identities, namely

$$\nabla^2(\vec r\cdot\vec A) = \vec r\cdot\nabla^2\vec A + 2\cancel{\nabla\cdot\vec A}$$

0 if $\vec A$ = divergenceless

and

$$\vec r\cdot(\nabla\times\vec A) = (\vec r\times\nabla)\cdot\vec A = i\vec L\cdot\vec A \qquad \left(\text{for any vector field } \vec A\right)$$

whereby we have

$$(\nabla^2+k^2)(\vec r\cdot\vec H') = -i\vec L\cdot(\vec J+\nabla\times\vec M)$$

and the solution having outgoing waves at $r\to\infty$ is

$$\boxed{\vec r\cdot\vec H' = \frac{i}{4\pi}\int\frac{e^{ik|\vec x-\vec x'|}}{|\vec x-\vec x'|}\vec L'\cdot\left[\vec J(\vec x')+\nabla'\times\vec M(\vec x')\right]d^3x'}$$

and likewise

$$(\nabla^2+k^2)(\vec r\cdot\vec E') = Z_0 k\vec L\cdot\left(\vec M+\frac{1}{k^2}\nabla\times\vec J\right)$$

whose relevant solution is:

$$\boxed{\vec r\cdot\vec E' = -\frac{Z_0 k}{4\pi}\int\frac{e^{ik|\vec x-\vec x'|}}{|\vec x-\vec x'|}\vec L'\cdot\left[\vec M(\vec x')+\frac{1}{k^2}\nabla'\times\vec J(\vec x')\right]d^3x'}$$

Now, plugging in the spherical expansion of the Green's function, and setting $r_>=r$, $r_<=r'$,

i.e. $$\frac{e^{ik|\vec x-\vec x'|}}{|\vec x-\vec x'|} = ik\sum_{\ell m} h_\ell^{(1)}(kr)Y_{\ell m}(\theta,\phi)j_\ell(kr')Y^*_{\ell m}(\theta',\phi')$$

and invoking 9.123 again, gives

the multipole coefficients,

$$a_M^{(lm)} = \frac{k}{\sqrt{l(l+1)}} \int Y^*_{lm}\, \vec{r}\cdot\vec{H}\, d\Omega \Big/ h_l^{(1)}(kr)$$

$$a_E^{(lm)} = \frac{k}{Z_0\sqrt{l(l+1)}} \int Y^*_{lm}\, \vec{r}\cdot\vec{E}\, d\Omega \Big/ h_l^{(1)}(kr)$$

and since

$$\int d\Omega\, Y^*_{lm}\, \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = 4\pi i k\, h_l^{(1)}(kr)\, j_l(kr')\, Y_{lm}(\theta',\phi'),$$

these multipole coefficients reduce to:

$$\boxed{a_E^{(lm)} = \frac{ik^3}{\sqrt{l(l+1)}} \int j_l(kr)\, Y^*_{lm}\, \vec{L}\cdot\left(\vec{M} + \frac{1}{k^2}\nabla\times\vec{J}\right) d^3x}$$

and

$$\boxed{a_M^{(lm)} = \frac{-k^2}{\sqrt{l(l+1)}} \int j_l(kr)\, Y^*_{lm}\, \vec{L}\cdot\left(\vec{J} + \nabla\times\vec{M}\right) d^3x}$$

Which Jackson further simplifies via identities to

$$\boxed{a_E^{(lm)} = \frac{k^2}{i\sqrt{l(l+1)}} \int Y^*_{lm}\left\{c\rho\frac{\partial}{\partial r}\left(r j_l(kr)\right) + ik(\vec{r}\cdot\vec{J})\, j_l(kr) - ik\,\nabla\cdot(\vec{r}\times\vec{M})\, j_l(kr)\right\} d^3x}$$

↑ shows how "electric multipoles" are mostly determined by ρ, and "magnetic multipoles" determined mainly by $\vec{J}, \vec{M}$ ↓

$$\boxed{a_M^{(lm)} = \frac{k^2}{i\sqrt{l(l+1)}} \int Y^*_{lm}\left\{\nabla\cdot(\vec{r}\times\vec{J})\, j_l(kr) + \nabla\cdot\vec{M}\,\frac{\partial}{\partial r}\left(r j_l(kr)\right) - k^2(\vec{r}\cdot\vec{M})\, j_l(kr)\right\} d^3x}$$

These results are essentially EXACT! Read Jackson pp 441-2

Sec. 9.11 Multipole radiation in atoms and nuclei

Next explore radiative lifetimes of atoms and nuclei, using quantum ideas in the semiclassical approximation. For instance, we can estimate the lifetime τ through the formula

$$P\tau = \hbar\omega = \text{energy radiated}$$

or setting

$$\Gamma = \frac{1}{\tau} = \frac{\text{transition probability}}{\text{unit time}}$$

$$\Rightarrow \boxed{\Gamma = \frac{P}{\hbar\omega}}$$

Now, combining Jackson's Eq. 9.155, 9.169–9.172 we have for an electric multipole:

$$\Gamma_E^{(\ell m)} = \frac{\omega Z_0 k^{2\ell}}{2\hbar\left[(2\ell+1)!!\right]^2} \frac{\ell+1}{\ell} \left|Q_{\ell m} + Q'_{\ell m}\right|^2$$

and the same expression holds for a magnetic multipole decay rate $\Gamma_M^{(\ell m)}$ except for the replacement of

$$Q_{\ell m} + Q'_{\ell m} \longrightarrow \frac{1}{c}\left(M_{\ell m} + M'_{\ell m}\right)$$

order-of-magnitude estimates:

$$Q_{lm} \equiv \int \rho\, r^l Y_{lm} d^3x \simeq e R^l$$

where $R \simeq$ size of the system, whereas

$$Q'_{lm} \equiv \frac{-ik}{(l+1)c} \int r^l Y^*_{lm} \nabla \cdot (\vec{r} \times \vec{M}) d^3x$$

so assuming that the relevant particles have mass m ⇒ these magnetic moments are of order

$$\frac{e\hbar}{m} \Rightarrow |\vec{M}| \simeq \frac{e\hbar}{mR^3}$$

$$\Rightarrow Q'_{lm} \simeq \frac{\hbar\omega}{mc^2} e R^l$$

And since for both atoms + nuclei, $\hbar\omega \ll mc^2$,

$$\Rightarrow \quad Q_{lm} \gg Q'_{lm}$$

A similar estimate for magnetic multipoles gives $\frac{1}{c} | M_{lm} + M'_{lm} | \simeq \frac{e\hbar}{mc} R^{l-1}$

$$\Rightarrow \quad \frac{\Gamma_M^{(lm)}}{\Gamma_E^{(lm)}} \simeq \frac{\left(\frac{e\hbar}{mc} R^{l-1}\right)^2}{e^2 R^{2l}} \simeq \frac{\hbar^2}{m^2c^2} \frac{1}{R^2}$$

For an atomic electron in a shell that sees an "average effective charge" Z_{eff}, the radius is about $R \simeq a_{Bohr} / Z_{eff}$

$$\Rightarrow \quad \frac{\Gamma_M^{(\ell)}}{\Gamma_E^{(\ell)}} \simeq \left(\frac{Z_{eff}}{137} \right)^2$$, since $c \simeq 137$ atomic units

e.g. $Z_{eff} \simeq 1$ for valence shell transitions

$Z_{eff} \simeq Z$ for deep inner-shell X-ray transitions

$\Rightarrow$ For valence shell radiative transitions, electric multipoles dominate over magnetic multipoles of the same order, but for deep inner shell x-ray transitions, they can become comparable

Also note that lower-ℓ multipoles usually dominate, since

$$\frac{\Gamma_E^{(\ell+1)}}{\Gamma_E^{(\ell)}} \simeq k^2 R^2 \simeq \frac{\omega^2}{c^2} R^2 \sim (Z_{eff}\, \alpha)^2 = \left(\frac{Z_{eff}}{137} \right)^2$$

$\Rightarrow$ for atoms, $\Gamma_{M1} \simeq \Gamma_{E2}$, etc.

For NUCLEI, on the other hand, these estimates are rather inaccurate for $l=1$.

In particular, magnetic dipole transitions are far more common than for atoms, and usually just as intense as electric dipole transitions

See Jackson pp 443–444

Chapter 10 - Scattering + diffraction

When radiation strikes an object, its electric or magnetic field can induce a dipole or a higher multipole. The radiation by the induced moment is called scattering.

Sec. 10.1 Scattering at long wavelengths

example Consider light incident in vacuum that strikes a dielectric sphere, ϵ_r, of radius a. The incident light is monochromatic, wavelength $\lambda >> a$, so it makes sense to treat $\vec{E}_{inc}$ as uniform across the sphere,

$\Rightarrow$ assume that $e^{i\vec{k}\cdot\vec{x}} \simeq 1$ for $r \lesssim a$

Then an electric dipole $\vec{p}$ will be induced with the same harmonic time-dependence as $\vec{E}_{inc}$, i.e. $e^{-i\omega t}$, and we will work out the E' radiation emitted by this dipole as the scattered light.

DETAILS

(i) The incident radiation is

$$\vec{E}_{inc} = E_0 \hat{e}_0 e^{i(kz-\omega t)}$$

where $\hat{e}_0$ = polarization vector of the incident wave

(ii) Then $\vec{p}(t)$ can be obtained as a simple time-dependent generalization of Eq. 4.56, for a dielectric sphere $\mu_r \simeq 1$, $\epsilon_r \neq 1$, namely

$$\vec{p}(t) = 4\pi\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) a^3 E_0 \hat{e}_0 e^{-i\omega t}$$

(iii) The fields radiated by this oscillating electric dipole are then (9.19):

$$\left.\begin{aligned} \vec{H}_{scatt} &= \frac{ck^2}{4\pi} \hat{n} \times \vec{p} \, \frac{e^{ikr}}{r} \\ \vec{E}_{scatt} &= Z_0 \vec{H}_{scatt} \times \hat{n} \end{aligned}\right\} \text{ in radiation zone}$$

(iv) The angular distribution of the scattered radiation is:

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left| (\hat{n}\times\vec{p}) \times \hat{n} \right|^2 \qquad (9.22)$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left| 4\pi\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) a^3 E_0 \right|^2 \left|\hat{n}\times\hat{e}_0\right|^2$$

and recalling that $Z_0 = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}$, $Z_0\epsilon_0 = \frac{1}{c}$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{1}{Z_0} |E_0|^2 \left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 \frac{k^4 a^6}{2} \left|\hat{n}\times\hat{e}_0\right|^2}$$

Polarization properties We also care about the polarization state of the scattered radiation, not just the average radiated power.

Recall, polarization is linear, if $\hat{e}_0$ is ANY real linear combination of $\hat{x}, \hat{y}, \hat{z}$

Circular polarization, for light travelling along $+\hat{z}$, is conveniently represented by spherical unit vectors, $\hat{e}_{\mu=\pm1} = \mp\frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$

which along with $\hat{e}_{\mu=0} = \hat{z}$ are complete, since any vector $\vec{A}$ can be expanded as:

$$\vec{A} = \sum_{\mu} \hat{e}_{\mu} \left(\hat{e}^{*}_{\mu} \cdot \vec{A} \right) \quad , \quad \hat{e}_{\mu} \cdot \hat{e}^{*}_{\mu'} = \delta_{\mu\mu'}$$

and $\hat{n} \cdot \hat{e}_{\mu} = \left(\frac{4\pi}{3} \right)^{1/2} Y_{1\mu}(\hat{n})$

e.g. a circularly polarized light wave that is incident on a scatterer looks like

$$\vec{E}_{inc} = E_0 \hat{e}_{\lambda} e^{i(kz - \omega t)}$$

with $\hat{e}_{\lambda} = \hat{e}_{\pm 1}$

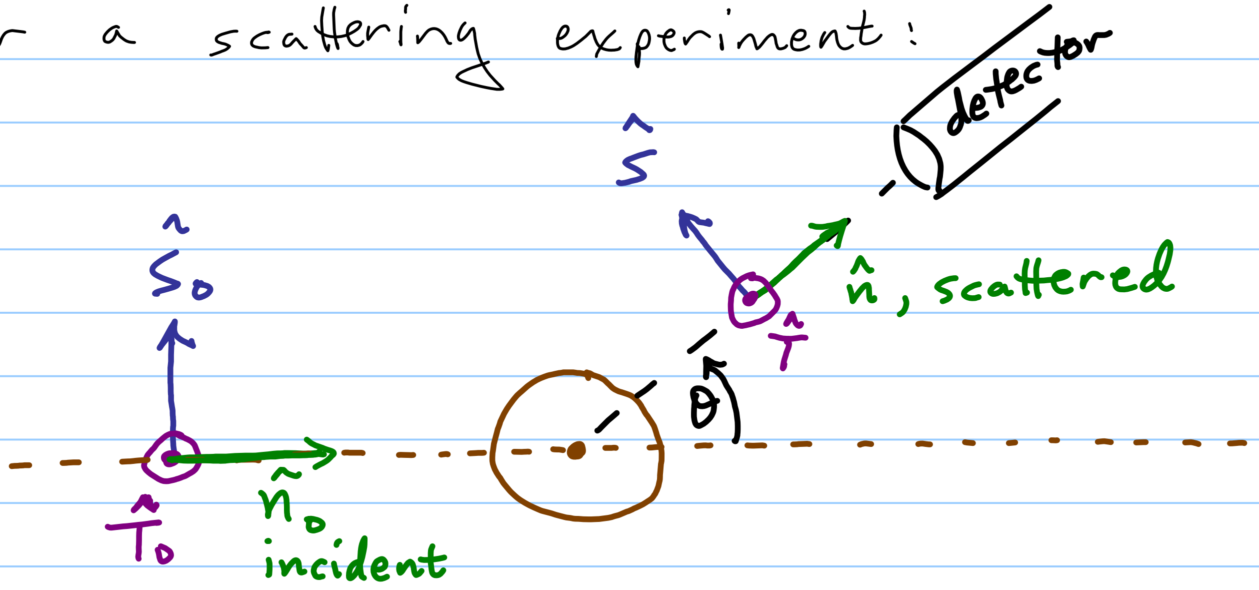
Suppose we now consider the angular distribution of radiated power that will pass through an ideal polarizer at the detector, which is 100% efficient in detecting only radiation of polarization $\hat{e}$. Then the detected radiation power per unit solid angle is

$$\frac{dP}{d\Omega}(\hat{n},\hat{e};\hat{n}_0,\hat{e}_0) = \frac{1}{Z_0} |E_0|^2 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \frac{k^4 a^6}{2} \times \left| \hat{e}^{*} \cdot \left\{ \hat{n} \times (\hat{n} \times \hat{e}_0) \right\} \right|^2$$

or in terms of the E-field vector $\vec{E}_{sc}$,

$$\frac{dP}{d\Omega}(\hat{n},\hat{e},\hat{n}_o,\hat{e}_o) = \lim_{r\to\infty} r^2 \frac{1}{2Z_o} \left| \hat{e}^* \cdot \vec{E}_{sc} \right|^2$$

Now consider the following geometry, for a scattering experiment:



The vectors $\hat{n}_o$ and $\hat{n}$ form a plane, which we call the <u>SCATTERING PLANE</u>.

And we define a unit vector normal to the scattering plane as

$$\hat{T} = \frac{\hat{n}_o \times \hat{n}}{|\hat{n}_o \times \hat{n}|} = \hat{T}_o$$

and a unit vector lying in the scattering plane,

$\hat{s}_o = \hat{T}_o \times \hat{n}_o$ (incident light), $\hat{s} = \hat{T} \times \hat{n}$ (scattered light)

$\Rightarrow \{\hat{s}_o, \hat{T}_o, \hat{n}_o\}$ = right-handed coord. system.

Next - write the incident and scattered E-fields in terms of these unit vectors:

e.g. $\vec{E}_{inc} = \hat{S}_0(\hat{S}_0 \cdot \vec{E}_{inc}) + \hat{T}_0(\hat{T}_0 \cdot \vec{E}_{inc})$

and then $$\vec{H}_{inc} = \frac{1}{Z_0}\hat{n}_0 \times \vec{E}_{inc}$$
$$= \frac{1}{Z_0}\hat{T}_0(\hat{S}_0 \cdot \vec{E}_{inc}) - \frac{1}{Z_0}\hat{S}_0(\hat{T}_0 \cdot \vec{E}_{inc})$$

and the time-averaged Poynting vector is

$$\vec{S}_{inc} = \frac{1}{2}\mathrm{Re}\left(\vec{E}_{inc} \times \vec{H}^*_{inc}\right)$$

$$\Rightarrow \boxed{\vec{S}_{inc} = \frac{1}{2Z_0}\left(|\hat{S}_0 \cdot \vec{E}_{inc}|^2 + |\hat{T}_0 \cdot \vec{E}_{inc}|^2\right)\hat{n}_0}$$

Note that there is NO INTERFERENCE between orthogonal polarization components in the power flux.

Next, consider the scattered radiation, decomposed into $\hat{S}$, $\hat{T}$ - polarization components, starting from:

$$\vec{E}_{sc} = Z_0 \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}(\hat{n} \times \vec{p}) \times \hat{n}$$

where $\vec{p} = 4\pi\epsilon_0\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)a^3 \vec{E}_{inc}$ $\left(\times e^{-i\omega t}\right)$

and $\vec{H}_{sc} = \frac{1}{Z_0}\hat{n} \times \vec{E}_{sc}$

The differential scattering cross section $\frac{d\sigma}{d\Omega}$ with polarization selectivity is defined as:

$$\frac{1}{|S_{inc}(\hat{n}_0,\hat{e}_0)|}\,\frac{dP}{d\Omega}(\hat{n},\hat{e},\hat{n}_0,\hat{e}_0) \equiv \frac{d\sigma}{d\Omega}(\hat{n},\hat{e},\hat{n}_0,\hat{e}_0)$$

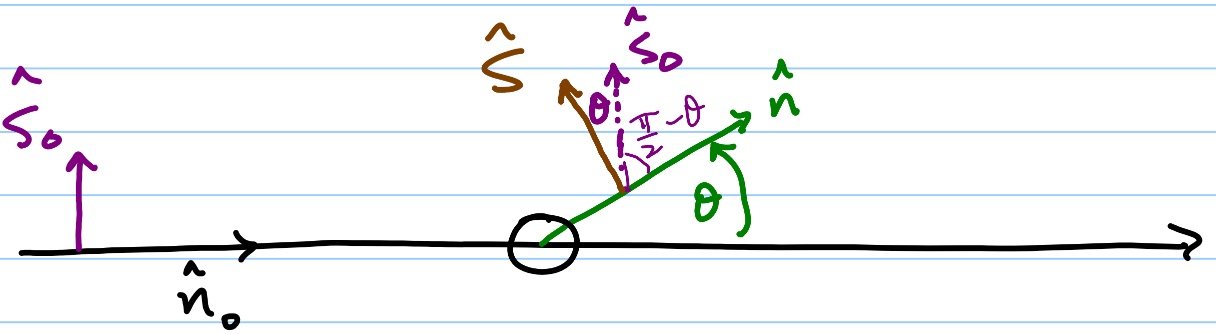
where

$$|S_{inc}(\hat{e}_0)| = \frac{1}{2Z_0}\left|\hat{e}_0^{*}\cdot\vec{E}_{inc}\right|^2$$

<u>Consider 4 separate cases:</u> Notation, set $\sigma_0 \equiv \left|\frac{\epsilon_r-1}{\epsilon_r+2}\right|^2 k^4 a^6$

(1) $$\frac{d\sigma}{d\Omega}(\hat{e}=\hat{s},\ \hat{e}_0=\hat{s}_0) = \sigma_0\left|\hat{s}^{*}\cdot[\hat{n}\times(\hat{n}\times\hat{s}_0)]\right|^2$$

(Recall, $\hat{s}_0$, $\hat{s}$ both lie IN the scattering plane)



Geometry: $\hat{n}\times\hat{s}_0 = \hat{T}_0\sin(\frac{\pi}{2}-\theta) = \cos\theta\,\hat{T}$

whereas $\hat{n}\times\hat{T} = -\hat{S} \Rightarrow \hat{n}\times(\hat{n}\times\hat{s}_0) = -\hat{S}\cos\theta$

and $\hat{s}^{*}\cdot[\hat{n}\times(\hat{n}\times\hat{s}_0)] = -\cos\theta$

Thus $$\boxed{(1)\quad \frac{d\sigma}{d\Omega}(\hat{e}=\hat{s},\ \hat{e}_0=\hat{s}_0) = \sigma_0\cos^2\theta}$$

and analogously,

(2) $\boxed{\dfrac{d\sigma}{d\Omega}(\hat{e}=\hat{T}, \hat{e}_0=\hat{S}_0)=0}$, since $\hat{T}\cdot\hat{S}=0$ and $\hat{T}=\hat{T}_0$

(3) $$\frac{d\sigma}{d\Omega}(\hat{e}=\hat{S}, \hat{e}_0=\hat{T}_0)=\sigma_0\left|\hat{S}^*\cdot[\hat{n}\times(\hat{n}\times\hat{T}_0)]\right|^2$$

with $\hat{n}\times\hat{T}=-\hat{S} \Rightarrow \hat{n}\times(\hat{n}\times\hat{T})=-\hat{T}_0=-\hat{T}$

and $\hat{S}^*\cdot\hat{T}_0=\hat{S}^*\cdot\hat{T}=0$

so this $\boxed{\dfrac{d\sigma}{d\Omega}(\hat{e}=\hat{S}, \hat{e}_0=\hat{S}_0)=0}$

(4) $$\frac{d\sigma}{d\Omega}(\hat{e}=\hat{T}, \hat{e}_0=\hat{T}_0)=\sigma_0\left|\hat{T}^*\cdot[\hat{n}\times(\hat{n}\times\hat{T}_0)]\right|^2$$
$$=\sigma_0\left|-\hat{T}\cdot\hat{T}_0\right|^2=\sigma_0$$
$$=\sigma_0$$

Very frequently, the incident light is UNPOLARIZED, in which case we need to AVERAGE $\frac{d\sigma}{d\Omega}$ over the 2 initial polarizations

Then

(i) Differential cross section for unpolarized incident light to be scattered through θ with the measured light polarization ∥ to the scattering plane. The formula for this case is:

$$\frac{d\sigma}{d\Omega}_{\parallel} = \left\langle \frac{d\sigma}{d\Omega}(\hat{e}=\hat{S}) \right\rangle$$

$$= \frac{1}{2}\left[\frac{d\sigma}{d\Omega}(\hat{e}=\hat{S}, \hat{e}_0=\hat{S}_0) + \frac{d\sigma}{d\Omega}(\hat{e}=\hat{S}, \hat{e}_0=\hat{T}_0) \right.$$

giving $\boxed{\frac{d\sigma}{d\Omega}_{\parallel}(\theta) = \frac{1}{2}\sigma_0 \cos^2\theta}$

and

(ii) For the measured light polarization $\perp$ to the scattering plane, unpolarized incident,

$$\frac{d\sigma}{d\Omega}_{\perp} = \left\langle \frac{d\sigma}{d\Omega}(\hat{e}=\hat{T}) \right\rangle = \frac{1}{2}\left[\frac{d\sigma}{d\Omega}(\hat{e}=\hat{T}, \hat{e}_0=\hat{S}_0) + \frac{d\sigma}{d\Omega}(\hat{e}=\hat{T}, \hat{e}_0=\hat{T}_0) \right]$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega}_{\perp}(\theta) = \frac{\sigma_0}{2}}$$

(iii) If there is no polarizer at the detector to block one polarization, we must sum over the final polarization states. This is the SAME RULE as in QM, namely to AVERAGE over initial states, but SUM over final states.

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\sigma_0}{2}(1+\cos^2\theta)$$ for unpolarized incident light and an unpolarized detector

It is conventional to define the POLARIZATION of the scattered radiation at some angle as:

$$\Pi(\theta) = \frac{\frac{d\sigma}{d\Omega}_{\perp} - \frac{d\sigma}{d\Omega}_{\parallel}}{\frac{d\sigma}{d\Omega}_{\perp} + \frac{d\sigma}{d\Omega}_{\parallel}} = \frac{\frac{\sigma_0}{2}(1-\cos^2\theta)}{\frac{\sigma_0}{2}(1+\cos^2\theta)}$$

or

$$\boxed{\Pi(\theta) = \frac{\sin^2\theta}{1+\cos^2\theta}}$$

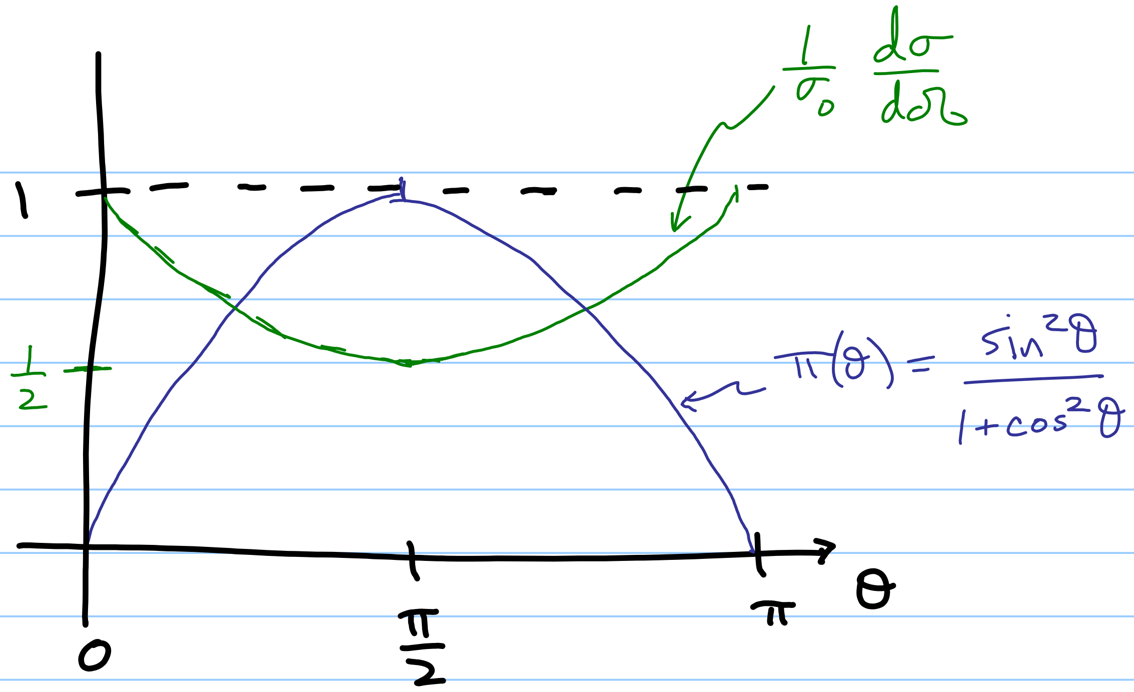
Finally, the total cross section for scattering of unpolarized radiation by a dielectric sphere in the electric dipole approximation ($ka \ll 1$) is:

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{\sigma_0}{2} 2\pi \underbrace{\int_{-1}^{1} dx\,(1+x^2)}_{=2+\frac{2}{3}}$$

or
$$\sigma = \frac{8}{3}\pi\sigma_0 = \frac{8}{3}\pi k^4 a^6 \left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2$$

or
$$\sigma = \frac{8}{3}\left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 (ka)^4 A$$

where $A = \pi a^2$ is the cross-sectional area of the sphere



Scattering from a small, perfectly-conducting sphere

In sec 2.5, we found that a perfectly-conducting sphere in a static, uniform E-field acquires an electric dipole moment

$$\vec{P} = 4\pi\epsilon_0 a^3 \vec{E}_{inc}$$

Alternatively, we saw that this result could be obtained by letting $\epsilon \to \infty$ in the result for a dielectric sphere.

In addition, a uniform, static applied B-field induces a magnetic dipole moment $\vec{m}$ in the conducting sphere, as can be seen by setting $\mu \to 0$ in Eq. 5.115. The magnetization is then

$$\vec{M} = \frac{3}{\mu_0}\left(\frac{\mu-\mu_0}{\mu+2\mu_0}\right)\vec{B}_{inc} \xrightarrow[\mu\to 0]{} -\frac{3}{2}\vec{H}_{inc}$$

so

$$\vec{m} = \frac{4}{3}\pi a^3 \vec{M} = -2\pi a^3 \vec{H}_{inc}$$

Another way to understand this is to recall Eq 5.112, for a uniformly magnetized sphere,

$$\vec{B}_{inside} = \vec{B}_{inc} + \frac{2\mu_0}{3}\vec{M}$$

$\Rightarrow$ Then in order to get $\vec{B}_{inside} = 0$, we must have $\vec{M} = -\frac{3}{2\mu_0}\vec{B}_{inc} \Rightarrow \underbrace{\vec{m} = \frac{4}{3}\pi a^3 \vec{M} = -2\pi a^3 \vec{H}_{inc}}$

and where $\vec{H}_{inc} = \frac{\hat{n}_0 \times \vec{E}_{inc}}{Z_0}$ if these are now radiation fields

The total fields of the scattered radiation are then obtained using

$$\vec{E}_{scatt} = \vec{E}_{E1} + \vec{E}_{M1}$$

where:

$$(1)\quad \vec{E}_{E1} = \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}(\hat{n}\times\vec{p})\times\hat{n}$$

$$= -k^2a^3 E_{inc}\frac{e^{ikr}}{r}\hat{n}\times(\hat{n}\times\hat{e}_0) \qquad \text{from } 10.2$$

$$(2)\quad \vec{E}_{M1} = -\frac{Z_0k^2}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{m} = \frac{-(\mu_0\epsilon_0)^{1/2}}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\hat{n}\times\vec{m}$$

$$= \frac{k^2a^3}{2}E_0\frac{e^{ikr}}{r}\hat{n}\times(\hat{n}_0\times\hat{e}_0) \qquad (\text{See } 10.13)$$

(Recall – can get $\vec{E}_{M1}$ and $\vec{H}_{M1}$ from $\vec{E}_{E1}$ and $\vec{H}_{E1}$ by making the replacements $\vec{E}\to Z_0\vec{H}$, $\vec{H}\to -\frac{\vec{E}}{Z_0}$, $\vec{p}\to\frac{\vec{m}}{c}$)

and thus

$$\frac{d\sigma}{d\Omega}(\hat{e},\hat{n};\hat{e}_0,\hat{n}_0) = k^4a^6\left|\hat{e}^*\cdot\left\{\overbrace{\hat{n}\times(\hat{n}\times\hat{e}_0)}^{\hat{n}(\hat{n}\cdot\hat{e}_0)-\hat{e}_0} - \frac{1}{2}\hat{n}\times(\hat{n}_0\times\hat{e}_0)\right\}\right|^2$$

$$= k^4a^6\left|-\hat{e}^*\cdot\hat{e}_0 - \frac{1}{2}\hat{e}^*\cdot\hat{n}\times(\hat{n}_0\times\hat{e}_0)\right|^2$$

or

$$\boxed{\frac{d\sigma}{d\Omega}(\hat{e},\hat{n};\hat{e}_0,\hat{n}_0) = k^4a^6\left|\hat{e}^*\cdot\hat{e}_0 - \frac{1}{2}(\hat{n}\times\hat{e}^*)\cdot(\hat{n}_0\times\hat{e}_0)\right|^2}$$

Recalling that $(\hat{S},\hat{T},\hat{n})$ and $(\hat{S}_0,\hat{T},\hat{n}_0)$ form a right-handed coordinate system with $\hat{s}$ and $\hat{s}_0$ in the scattering plane

Applying this formula, verify on your own the following specialized cases:

(1) $\frac{d\sigma}{d\Omega}(\hat{e}=\hat{S}, \hat{e}_o=\hat{S}_o)=k^4a^6\left|\cos\theta-\frac{1}{2}\right|^2$

(2) $\frac{d\sigma}{d\Omega}(\hat{e}=\hat{T}, \hat{e}_o=\hat{S}_o)=0$

(3) $\frac{d\sigma}{d\Omega}(\hat{e}=\hat{S}, \hat{e}_o=\hat{T}_o)=0$

(4) $\frac{d\sigma}{d\Omega}(\hat{e}=\hat{T}, \hat{e}_o=\hat{T}_o)=k^4a^6\left|1-\frac{1}{2}\cos\theta\right|^2$

and the results for unpolarized incident radiation are found by averaging are

$$\frac{d\sigma_{\parallel}}{d\Omega}=\frac{k^4a^6}{2}\left(\cos\theta-\frac{1}{2}\right)^2 \qquad (\text{average } (1) + (3))$$

$$\frac{d\sigma_{\perp}}{d\Omega}=\frac{k^4a^6}{2}\left(1-\frac{1}{2}\cos\theta\right)^2 \qquad (\text{average } (2) + (4))$$

And the differential scattering cross section without polarization selection is

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma_{\parallel}}{d\Omega}+\frac{d\sigma_{\perp}}{d\Omega}=k^4a^6\left[\frac{5}{8}(1+\cos^2\theta)-\cos\theta\right]$$

and total scattering cross section is

$$\sigma_{tot}=\frac{10\pi}{3}k^4a^6$$

Perturbation Theory of Scattering (sec. 10.2)

So far we have only treated the long-wavelength limit, namely E1, M1 only.

Now we develop a method that remains valid and useful even when $\lambda \ll d$, provided the amount of scattering is <u>WEAK</u>, i.e.

$$\frac{|\epsilon - \epsilon_0|}{\epsilon_0} \ll 1$$

Start from Maxwell's equations in a source-free region:

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i Z_0 k \vec{H}$$

$$\nabla \cdot \vec{D} = 0 \qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = -ikc\vec{D}$$

Consider

$$\text{(a)} \quad \nabla \times [\nabla \times (\vec{D} - \epsilon_0 \vec{E})] = \nabla(\cancelto{0}{\nabla \cdot \vec{D}}) - \nabla^2 \vec{D} + \epsilon_0 \nabla \times \frac{\partial \vec{B}}{\partial t}$$

$$\text{(b)} \quad \mu_0 \epsilon_0 \frac{\partial}{\partial t}\frac{\partial \vec{D}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \times \vec{H} = 0$$

see Jackson P. 463

which combine to give:

(*) = exact

$$\boxed{\nabla^2 \vec{D} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{D}}{\partial t^2} = -\nabla \times \left\{[\nabla \times (\vec{D} - \epsilon_0 \vec{E})] - \epsilon_0 \frac{\partial}{\partial t}(\vec{B} - \mu_0 \vec{H})\right\}}$$

Note that (*) has additional terms on the RHS if there are specified sources, namely add: $\left(-\nabla\rho - \mu_0\epsilon_0\frac{\partial\vec{J}}{\partial t}\right)$ ← derive on your own

Now, the medium will be assumed to have a localized region where ϵ, μ have small changes from ϵ_0, μ_0, namely

$$\epsilon = \epsilon_0 + \delta\epsilon(\vec{x}')$$
$$\mu = \mu_0 + \delta\mu(\vec{x}')$$

Then specializing to a harmonic t-dependence,

$$\Rightarrow (\nabla^2 + k^2)\vec{D} = -\vec{W}_{source}(\vec{x})$$

where

$$-\vec{W}_{source} = -\nabla\times[\nabla\times(\vec{D}-\epsilon_0\vec{E})] - i\epsilon_0\omega\nabla\times(\vec{B}-\mu_0\vec{H})$$

and if the fields are known $\overbrace{\text{on this RHS}}^{\text{(i.e. zeroth-order unperturbed)}}$, this is an inhomogeneous Helmholtz equation for $\vec{D}$, whose solution is simply written as

$$\vec{D}(\vec{x}) = \vec{D}_0(\vec{x}) + \int d^3x' \, G(\vec{x}, \vec{x}', \omega)\, \vec{W}_{source}(\vec{x}')$$

where G obeys $(\nabla^2 + k^2)G = -\delta(\vec{x} - \vec{x}')$

$$\Rightarrow G = \frac{e^{ikR}}{R}, \quad \vec{R} = \vec{x} - \vec{x}'$$

this is the free-space GF, assumes no boundaries and in the radiation zone,

$$G \xrightarrow[r>>r']{} \frac{1}{4\pi}\frac{e^{ikr}}{r}\, e^{-ik\hat{n}\cdot\vec{x}'}$$

The concept of this perturbative derivation is that the incident fields inserted into the RHS of (*) then allow (*) to be solved for the 1st-order corrected field, given by

$$\vec{D}(\vec{x}) = \vec{D}_0(\vec{x}) + \frac{e^{ikr}}{r}\left\{\int d^3x'\, \frac{e^{-ik\hat{n}\cdot\vec{x}'}}{4\pi}\, W_{source}(\vec{x}')\right\}$$

$$= \vec{D}_0(\vec{x}) + \vec{A}_{scatt}(\vec{k})\,\frac{e^{ikr}}{r}$$

or to be explicit, the scattering amplitude is

$$\vec{A}_{scatt}(\vec{k}) = \frac{1}{4\pi}\int d^3x'\, e^{-ik\hat{n}\cdot\vec{x}'}\left\{\nabla'\times[\nabla'\times(\vec{D}-\epsilon_0\vec{E}) + i\epsilon_0\omega\nabla'\times(\vec{B}-\mu_0\vec{H})\right\} \quad (**)$$

Time for some vector calculus identities:

$$\int \psi\, \nabla\times\vec{a}\, d^3x' = \int\left[\nabla'\times(\psi\vec{a}) - \nabla'\psi\times\vec{a}\right]d^3x'$$

and $\displaystyle\int_V \nabla'\times(\psi\vec{a})\, d^3x' = \oint_S \hat{n}'\times(\psi\vec{a})\, d^3x' = 0$ for localized sources

and since here $\psi = e^{-ik\hat{n}\cdot\vec{x}'} \Rightarrow \nabla\psi = -ik\hat{n}\, e^{-ik\hat{n}\cdot\vec{x}'}$

$$\Rightarrow \int e^{-ik\hat{n}\cdot\vec{x}'}(\nabla'\times\vec{a})\, d^3x' = ik\int d^3x'\, e^{-ik\hat{n}\cdot\vec{x}'}\, \hat{n}\times\vec{a}(\vec{x}')$$

Thus we can replace each curl in (**) by

$$\nabla'\times[\ \ ] \longrightarrow ik\hat{n}\times[\ \ ]$$

which gives:

$$\vec{A}_{sc} = \frac{k^2}{4\pi}\int d^3x' \, e^{-ik\hat{n}\cdot\vec{x}'}\Big\{\underbrace{[\hat{n}\times(\vec{D}-\epsilon_0\vec{E})]\times\hat{n}}_{\text{scattering by electric dipoles in scatterer}} - \underbrace{\frac{\epsilon_0\omega}{k}\hat{n}\times(\vec{B}-\mu_0\vec{H})}_{\text{by magnetic dipoles}}\Big\}$$

Compare this with Eq. 10.2!

And the differential cross section for the observation of polarization $\hat{e}$ at the detector is:

$$\frac{d\sigma}{d\Omega} = \frac{|\hat{e}^*\cdot\vec{A}_{sc}|^2}{|\vec{D}_0|^2}$$

This treatment is often referred to as the

FIRST-ORDER BORN APPROXIMATION

history: M. Born in QM ≃ 1930
Rayleigh in E+M ≃ 1881

⇒ in the inhomogeneous terms, make the approximations:

$$\vec{D}-\epsilon_0\vec{E} \simeq \frac{\delta\epsilon(\vec{x})}{\epsilon_0}\vec{D}_0(\vec{x})$$

$$\vec{B}-\mu_0\vec{H} \simeq \frac{\delta\mu(\vec{x})}{\mu_0}\vec{B}_0(\vec{x})$$

where $$\vec{D}_0(\vec{x}) = \hat{e}_0 D_0 e^{ik\hat{n}_0\cdot\vec{x}}$$

$$\vec{B}_0(\vec{x}) = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} \hat{n}_0 \times \vec{D}_0(\vec{x})$$

$$\Rightarrow \frac{\hat{e}^*\cdot A_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}\int d^3x\, e^{i\vec{q}\cdot\vec{x}}\left\{\hat{e}^*\cdot\hat{e}_0 \frac{\delta\epsilon(\vec{x})}{\epsilon_0} + (\hat{n}\times\hat{e}^*)\cdot(\hat{n}_0\times\hat{e}_0)\frac{\delta\mu(\vec{x})}{\mu_0}\right\}$$

and $\vec{q} \equiv k(\hat{n}_0 - \hat{n})$ = "momentum transfer" vector
(recall, $\vec{p} = \hbar\vec{k}$ in QM)

e.g. for a uniform dielectric sphere, the integral is simple,

$$\Rightarrow \frac{\hat{e}^*\cdot\vec{A}_{sc}}{D_0} = k^2\frac{\delta\epsilon}{\epsilon_0}(\hat{e}^*\cdot\hat{e}_0)\left(\frac{\sin qa - qa\cos qa}{q^3}\right)$$

approaches $\frac{a^3}{3}$ at $q \to 0$,
i.e. either at low frequencies
or else in the forward
direction at any frequency

And observe that

$$\lim_{q\to 0}\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\frac{\delta\epsilon}{\epsilon_0}\right|^2 |\hat{e}^*\cdot\hat{e}_0|^2$$

so as expected, this agrees with Eq 10.6 for a small dielectric sphere.

Sections 10.3, 10.4 - scattering theory for transverse vector fields

Recall that in scalar scattering theory, e.g. for Schrödinger waves or sound waves, the following identities are useful:

$$e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_{l=0}^{\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y^*_{lm}(\hat{k})\, Y_{lm}(\hat{x})$$

The analogous expansion for a transverse vector field is important in E+M, e.g. for an incident circularly-polarized plane wave,

$$\vec{E}^{(\pm)}(\vec{x}) = (\hat{x} \pm i\hat{y})\, e^{ikz} \qquad (a)$$

$$\hat{z}\times(\hat{x}\pm i\hat{y}) = \hat{y} \mp i\hat{x} = \mp i(\hat{x}\pm i\hat{y})$$

$$c\,\vec{B}^{(\pm)}(\vec{x}) = \hat{z}\times\vec{E}(\vec{x}) = \mp i\,\vec{E}^{(\pm)}(\hat{x}) \qquad (b)$$

then the spherical expansion of this transverse vector plane wave can generally be written as an expansion in terms of our general multipole solutions about an arbitrary origin (typically chosen at or near the scatterer "center"):

$$\vec{E}^{(\pm)}(\vec{x}) = \sum_{l,m}\left[a^{(\pm)}_{lm}\, j_l(kr)\, \vec{X}_{lm} + \frac{i}{k}\, b^{(\pm)}_{lm}\, \nabla\times j_l(kr)\,\vec{X}_{lm}\right] \qquad (a')$$

$$c\,\vec{B}^{(\pm)}(\vec{x}) = \sum_{l,m}\left[-\frac{i}{k}\, a^{(\pm)}_{lm}\, \nabla\times j_l(kr)\,\vec{X}_{lm} + b^{(\pm)}_{lm}\, j_l(kr)\,\vec{X}_{lm}\right] \qquad (b')$$

Now, equating these expansions to the plane-wave fields above, we use "Fourier's Trick" to find the coefficients $a_{\ell m}^{(\pm)}$, $b_{\ell m}^{(\pm)}$.

Three orthogonality relations are key!

(I) $\int [f_{\ell'}(r)\vec{X}_{\ell' m'}]^{*} \cdot [g_\ell(r)\vec{X}_{\ell m}]\, d\Omega = f_{\ell'} g_\ell\, \delta_{\ell\ell'}\delta_{mm'}$

(II) $\int [f_{\ell'}(r)\vec{X}_{\ell' m'}]^{*} \cdot [\nabla\times g_\ell(r)\vec{X}_{\ell m}]\, d\Omega = 0$

(III) $\frac{1}{k^2}\int [\nabla\times f_{\ell'}(r)\vec{X}_{\ell' m'}]^{*} \cdot [\nabla\times g_\ell(r)\vec{X}_{\ell m}]\, d\Omega$

$$= \delta_{\ell\ell'}\delta_{mm'}\left\{ f_\ell^{*} g_\ell + \frac{1}{k^2 r^2}\frac{\partial}{\partial r}\left[r f_\ell^{*} \frac{\partial}{\partial r}(r g_\ell)\right]\right\}$$

Now equate (a) to (a'), and apply $\int d\Omega\, \vec{X}_{\ell' m'}$ to both sides of the equation

$$\Rightarrow a_{\ell m}^{(\pm)} j_\ell(kr) = \int \vec{X}_{\ell m}^{*}\cdot \vec{E}^{(\pm)}(\vec{x})\, d\Omega$$

and doing the same to the equality between (b), (b'):

$$\Rightarrow b_{\ell m}^{(\pm)} j_\ell(kr) = c\int \vec{X}_{\ell m}^{*}\cdot \vec{B}^{(\pm)}(\vec{x})\, d\Omega$$

but observe that

$$\vec{X}_{\ell m}^{*}\cdot \vec{E}^{(\pm)}(\vec{x}) = \left[\frac{\overset{\swarrow L_\mp}{(L_x \mp i L_y)}\, Y_{\ell m}(\hat{r})}{\sqrt{\ell(\ell+1)}}\right]^{*} 4\pi \sum_{\ell m} i^{\ell} j_\ell(kr)\, Y_{\ell m}^{*}(\hat{z})\, Y_{\ell m}(\hat{r})$$

now use

$$L_\mp Y_{\ell m} = [(\ell \pm m)(\ell \mp m + 1)]^{1/2}\, Y_{\ell, m\mp 1} \qquad \text{Eq. 9.104}$$

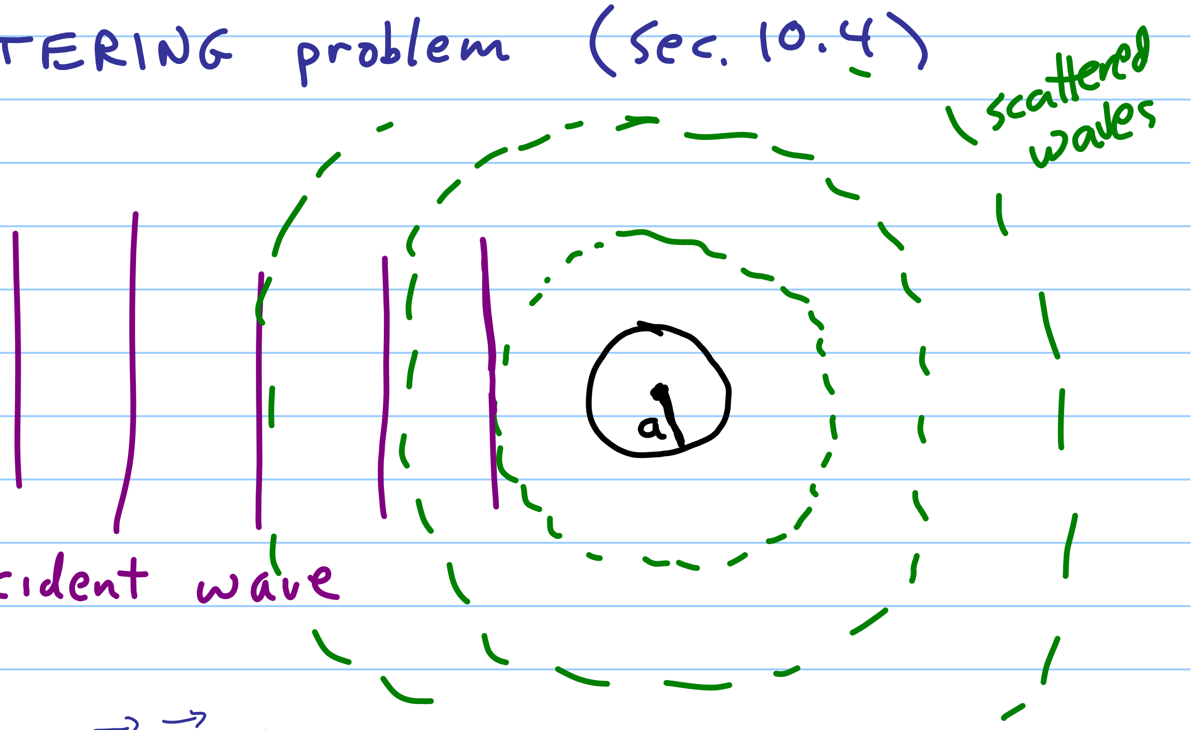
giving {From QM, recall that $J_\pm|jm\rangle = \hbar[j(j+1)-m(m\pm1)]^{1/2}|j,m\pm1\rangle$

$$\boxed{\begin{aligned} a^{(\pm)}_{lm} &= i^l[4\pi(2l+1)]^{1/2}\delta_{m,\pm1} \\ b^{(\pm)}_{lm} &= \mp i\, a^{(\pm)}_{lm} \end{aligned}}$$

In conclusion, we have the spherical expansion of our vector E + M plane wave:

$$\boxed{\begin{aligned} \vec{E}^{(\pm)}(\vec{x}) &= (\hat{x}\pm i\hat{y})e^{ikz} = \sum_{l=1}^{\infty} i^l[4\pi(2l+1)]^{1/2}\left[j_l\vec{X}_{l,\pm1} \pm \frac{1}{k}\nabla\times j_l\vec{X}_{l,\pm1}\right] \\ c\vec{B}^{(\pm)}(\vec{x}) &= \mp i\vec{E}^{(\pm)} = \sum_{l=1}^{\infty} i^l[4\pi(2l+1)]^{1/2}\left[-\frac{i}{k}\nabla\times j_l\vec{X}_{l,\pm1} \mp i\, j_l\vec{X}_{l,\pm1}\right] \end{aligned}}$$

Now apply this to treat a SPHERICAL SCATTERING problem (Sec. 10.4)



We expect the $\vec{E}, \vec{B}$ fields should have the structure

$$\vec{E}(\vec{x}) = \vec{E}_{inc} + \vec{E}_{sc}$$
$$\vec{B}(\vec{x}) = \vec{B}_{inc} + \vec{B}_{sc}$$

And we expect that, outside the sphere, the scattered wave radial solutions must be <u>outgoing-wave</u> spherical Hankel solutions $h_\ell^{(1)}(kr)$, i.e.

and
$$\vec{E}_{sc} = \frac{1}{2}\sum_{\ell=1}^{\infty} i^\ell \left[4\pi(2\ell+1)\right]^{1/2} \left\{ \alpha_\ell^{(\pm)} h_\ell^{(1)} \vec{X}_{\ell,\pm1} \pm \frac{\beta_\ell^{(\pm)}}{k} \nabla\times h_\ell^{(1)} \vec{X}_{\ell,\pm1} \right\}$$

$$c\vec{B}_{sc} = \frac{1}{2}\sum_{\ell=1}^{\infty} i^\ell \left[4\pi(2\ell+1)\right]^{1/2} \left\{ -\frac{i\alpha_\ell^{(\pm)}}{k} \nabla\times h_\ell^{(1)} \vec{X}_{\ell,\pm1} \mp i\beta_\ell^{(\pm)} h_\ell^{(1)} \vec{X}_{\ell,\pm1} \right\}$$

And the coefficients $\alpha_\ell^{(\pm)}$, $\beta_\ell^{(\pm)}$ are found by either:

(i) matching these formulas to appropriate boundary conditions at the sphere surface, e.g. if the sphere is a <u>perfect conductor</u>

or

(ii) matching these to the actual short-range solution <u>inside</u> the sphere

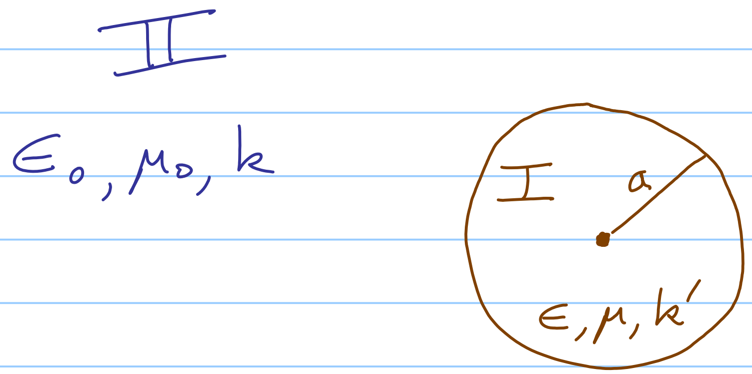
Our two-phase strategy:

Ⓐ First solve for the complete set of TM, TE modes for the sphere by itself, imposing boundary conditions at the origin but NOT at ∞, initially.

Ⓑ Next, superpose those solutions to describe physical SCATTERING BCs, i.e. demanding that

$$\vec{E} \xrightarrow[r\to\infty]{} \hat{e}_o e^{ikz} + \vec{A}_{sc}(\hat{k},\hat{e}_o)\frac{e^{ikr}}{r}$$

Application of this strategy to scattering by a sphere having ϵ, μ, radius a:



Recall

$$i\epsilon\omega\vec{E} + \nabla\times\vec{H} = 0$$
$$-i\mu\omega\vec{H} + \nabla\times\vec{E} = 0$$
$$k' = \omega\sqrt{\mu\epsilon}$$
$$k = \omega\sqrt{\mu_o\epsilon_o}$$
$$\frac{k'}{k} = \left(\frac{\mu\epsilon}{\mu_o\epsilon_o}\right)^{1/2}$$

Region I, r<a

$$\vec{E}_{lm}^{(I,TE)} = j_l(k'r)\vec{X}_{lm}$$
$$\vec{H}_{lm}^{(I,TE)} = \frac{1}{i\mu\omega}\nabla\times j_l(k'r)\vec{X}_{lm}$$

$$\vec{H}_{lm}^{(I,TM)} = j_l(k'r)\vec{X}_{lm}$$
$$\vec{E}_{lm}^{(I,TM)} = \frac{-1}{i\epsilon\omega}\nabla\times j_l(k'r)\vec{X}_{lm}$$

Region II, r>a We have the identical forms just written (I), except now $\epsilon\to\epsilon_o$, $\mu\to\mu_o$, $k'\to k$ AND we will now need to include terms with BOTH $j_l(kr)$ AND $n_l(kr)$, in order to match the BCs, which are:

$$\boxed{\begin{array}{l} (i)\ \vec{D}\cdot\hat{r} = \text{continuous} \\ (ii)\ \vec{B}\cdot\hat{r} = \text{continuous} \\ (iii)\ \hat{r}\times\vec{E} = \text{"} \\ (iv)\ \hat{r}\times\vec{H} = \text{"} \end{array}}$$

i.e.

$$\Rightarrow \vec{E}_{\ell m}^{(II, TE)} = \left(a_{\ell m}^{TE} j_\ell(kr) + b_{\ell m}^{TE} n_\ell(kr)\right)\vec{X}_{\ell m}$$

and continuity of $E_{\parallel}$ demands the B.C.

$$j_\ell(k'a) = a_{\ell m}^{TE} j_\ell(ka) + b_{\ell m}^{TE} n_\ell(ka)$$

and since the continuity of $D_{\perp}$ is irrelevant for any TE mode, consider $H_{\parallel}$, (iv):

$$\vec{H}_{\ell m}^{(II, TE)} = \frac{1}{i\mu_0\omega}\nabla\times\left(a_{\ell m}^{TE} j_\ell(kr) + b_{\ell m}^{TE} n_\ell(kr)\right)\vec{X}_{\ell m}$$

and note the identity:

$$\hat{n}\times\left(\nabla\times Z_\ell\vec{X}_{\ell m}\right) = \frac{k}{2\ell+1}\left(\ell Z_{\ell+1} - (\ell+1)Z_{\ell-1}\right)\vec{X}_{\ell m}$$

provided $Z_\ell(kr)$ = any spherical Bessel function of order ℓ

and to simplify notation, introduce

$$u_\ell(x) \equiv (\ell+1) j_{\ell-1}(x) - \ell j_{\ell+1}(x)$$

$$v_\ell(x) \equiv (\ell+1) n_{\ell-1}(x) - \ell n_{\ell+1}(x)$$

So the $H_{\parallel}$ continuity B.C. reads:

$$\frac{k'}{i\mu\omega} u_\ell(k'a) = \frac{k}{i\mu_0\omega}\left(a_{\ell m}^{TE} u_\ell(ka) + b_{\ell m}^{TE} v_\ell(ka)\right)$$

For notational brevity, define also $\boxed{\gamma \equiv \frac{k' \mu_0}{k \mu}}$ and our 2 equations, 2 unknowns can be solved:

$$\boxed{\begin{aligned} a_\ell^{TE} &= \frac{(ka)^2}{2\ell+1}\left[j_\ell(k'a)\, v_\ell(ka) - \gamma\, u_\ell(k'a)\, n_\ell(ka) \right] \\ b_\ell^{TE} &= \frac{(ka)^2}{2\ell+1}\left[-j_\ell(k'a)\, u_\ell(ka) + \gamma\, u_\ell(k'a)\, j_\ell(ka) \right\} \end{aligned}}$$

note identity: $n_\ell(z)\, u_\ell(z) - j_\ell(z)\, v_\ell(z) = -(2\ell+1)/z^2$

Phaseshifts: It is convenient & physically useful to replace a_ℓ^{TE} and b_ℓ^{TE} by an overall amplitude A_ℓ^{TE} and a phaseshift δ_ℓ^{TE}, i.e. set

$$\boxed{\begin{aligned} a_\ell^{TE} &= A_\ell^{TE} \cos \delta_\ell^{TE} \\ b_\ell^{TE} &= -A_\ell^{TE} \sin \delta_\ell^{TE} \end{aligned}} \quad \text{or} \quad \boxed{\tan \delta_\ell^{TE} = \frac{-b_\ell^{TE}}{a_\ell^{TE}}}$$

We do this because it simplifies the asymptotic form of these waves, namely:

$$a_\ell^{TE} j_\ell(kr) + b_\ell^{TE} n_\ell(kr) \xrightarrow[r\to\infty]{} \frac{A_\ell^{TE}}{kr} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell^{TE}\right)$$

where we have used

$$j_\ell \longrightarrow \frac{\sin(x - \frac{\ell\pi}{2})}{x}$$

$$n_\ell \longrightarrow \frac{-\cos(x - \frac{\ell\pi}{2})}{x}$$

and observe that

δ_ℓ^{TE} = TE-mode scattering phaseshift for partial wave ℓ is IMPORTANT!

whereas the amplitude $\cancel{A_\ell^{TE}}{}^{!}$ is largely unimportant, and can be divided out, or set to unity

Of course one can similarly introduce phaseshifts δ_ℓ^{TM} for the TM-modes, whereby the most general fields outside the sphere have the form

$$\vec{E} = \sum_{\ell m}\left(d_{\ell m}^{TE}\,\vec{E}_{\ell m}^{TE} + d_{\ell m}^{TM}\,\vec{E}_{\ell m}^{TM}\right)$$

$$\vec{H} = \sum_{\ell m}\left(d_{\ell m}^{TE}\,\vec{H}_{\ell m}^{TE} + d_{\ell m}^{TM}\,\vec{H}_{\ell m}^{TM}\right)$$

which have the following asymptotic forms at $r \to \infty$:

$$\vec{E} \xrightarrow[r\to\infty]{} \sum_{\ell m} d_{\ell m}^{TE}\,\frac{\sin(kr - \frac{\ell\pi}{2} + \delta_\ell^{TE})}{kr}\,\vec{X}_{\ell m}$$

$$+ \sum_{\ell m}\left(-\frac{d_{\ell m}^{TM}}{i\epsilon_0\omega}\right)\nabla\times \frac{\sin(kr - \frac{\ell\pi}{2} + \delta_\ell^{TM})}{kr}\,\vec{X}_{\ell m}$$

and

$$\vec{H} \xrightarrow[r\to\infty]{} \sum_{\ell m} d_{\ell m}^{TM}\,\frac{\sin(kr - \frac{\ell\pi}{2} + \delta_\ell^{TM})}{kr}\,\vec{X}_{\ell m}$$

$$+ \sum_{\ell m}\frac{d_{\ell m}^{TE}}{i\mu_0\omega}\,\nabla\times \frac{\sin(kr - \frac{\ell\pi}{2} + \delta_\ell^{TE})}{kr}\,\vec{X}_{\ell m}$$

and if we now demand that, for $\vec{E}_{inc} = \hat{e}_\pm e^{ikz}$,

$$\vec{E}-\vec{E}_{inc}\xrightarrow[r\to\infty]{}\vec{A}_{sc}(\hat{k},\hat{e}_0)\frac{e^{ikr}}{r}$$, i.e. with no incoming waves $\propto e^{-ikr}/r$

and then plug in the spherical expansion for $\vec{E}_{inc}$, we can immediately apply "Fourier's Trick" to determine the $d_{\ell m}$ above, and find the scattering amplitude

i.e. $$\vec{E}_{inc}^{(\pm)}\xrightarrow[r\to\infty]{}\sum_\ell i^\ell[4\pi(2\ell+1)]^{1/2}\left\{\frac{\sin(kr-\frac{\ell\pi}{2})}{kr}\vec{X}_{\ell,\pm1}\pm\frac{1}{k}\nabla\times\frac{\sin(kr-\frac{\ell\pi}{2})}{kr}\vec{X}_{\ell,\pm1}\right\}$$

So clearly, the application of Fourier's trick tells us that only $m=\pm1$ contribute, giving:

$$\vec{E}-\vec{E}_{inc}^{(\pm)}\xrightarrow[r\to\infty]{}\sum_\ell\left\{d^{TE}_{\ell,\pm1}\frac{\sin(kr-\frac{\ell\pi}{2}+\delta_\ell^{TE})}{kr}-\frac{i^\ell[4\pi(2\ell+1)]^{1/2}}{kr}\sin(kr-\frac{\ell\pi}{2})\right\}\vec{X}_{\ell,\pm1}$$
$$+\sum_\ell\left\{d^{TM}_{\ell,\pm1}\left(\text{---- TM terms in } \vec{E}-\vec{E}_{inc}\right)\right.$$

So after writing out as exponentials, using

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i},$$

one readily sees that, to kill the incoming TE waves proportional to e^{-ikr}/r, we must have

$$d^{TE}_{\ell,\pm1}=i^\ell[4\pi(2\ell+1)]^{1/2}e^{i\delta_\ell^{TE}}$$

and plugging this in & simplifying gives

$$\vec{E}-\vec{E}^{(\pm)}_{inc} \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{r}\left(-\frac{i}{k}\right)\sum_l [\pi(2l+1)]^{1/2}\left(e^{2i\delta_l^{TE}}-1\right)$$

$$+ (\text{TM terms in } \vec{E}-\vec{E}_{inc})$$

and similarly

to kill the incoming-waves in the TM terms of $\vec{H}$, one must have $d^{TM}_{l,\pm 1} = \pm[4\pi(2l+1)]^{1/2} i^l e^{i\delta_l^{TM}}$,

giving after some algebra:

$$\vec{H}-\vec{H}_{inc} \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{r}\left\{\mp\sum_l \left(\frac{\epsilon_0}{\mu_0}\right)^{1/2}[\pi(2l+1)]^{1/2}\frac{e^{2i\delta_l^{TM}}-1}{k}\vec{X}_{l,\pm 1}\right\}$$

$$+(\text{TE terms in } \vec{H}-\vec{H}_{inc})$$

Then the analysis in Jackson, p.474 gives the scattering amplitude, and the integrated scattering cross section, e.g. Eq. 10.61:

$$\sigma_{sc} = \frac{\pi}{2k^2}\sum_{l=1}^{\infty}(2l+1)\left\{\left|e^{2i\delta_l^{TE}}-1\right|^2+\left|e^{2i\delta_l^{TM}}-1\right|^2\right\}$$

so if the δ_l are all real (nondissipative media) we get the simple result

$$\boxed{\sigma_{sc} = \frac{2\pi}{k^2}\sum_{l=1}^{\infty}(2l+1)\left(\sin^2\delta_l^{TE}+\sin^2\delta_l^{TM}\right)}$$

Scalar diffraction theory (10.5)

Recall that the Helmholtz equation GF is

$$G = \frac{e^{ikR}}{4\pi R}, \quad \text{for which } \nabla' G = \left(\frac{ik\vec{R}}{R} - \frac{\vec{R}}{R^2}\right)\frac{e^{ikR}}{4\pi R}$$

and Green's theorem applied to a volume V bounded by a closed surface S reads for a solution of $(\nabla^2 + k^2)\Psi = 0$, $\boxed{\text{with } \hat{n}' = \underline{\text{inward}} \text{ normal}}$

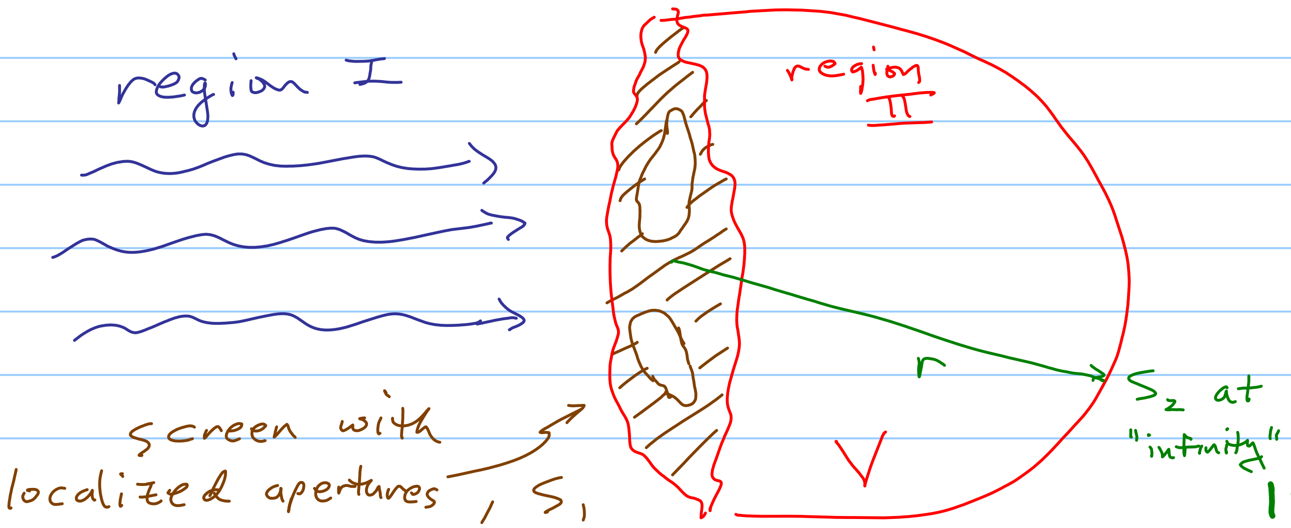
$$\Psi(\vec{x}) = \oint_S \left[\Psi(\vec{x}')\hat{n}'\cdot\nabla' G(\vec{x},\vec{x}') - G(\vec{x},\vec{x}')\hat{n}'\cdot\nabla'\Psi(\vec{x}')\right] da'$$

and plugging in our above G,

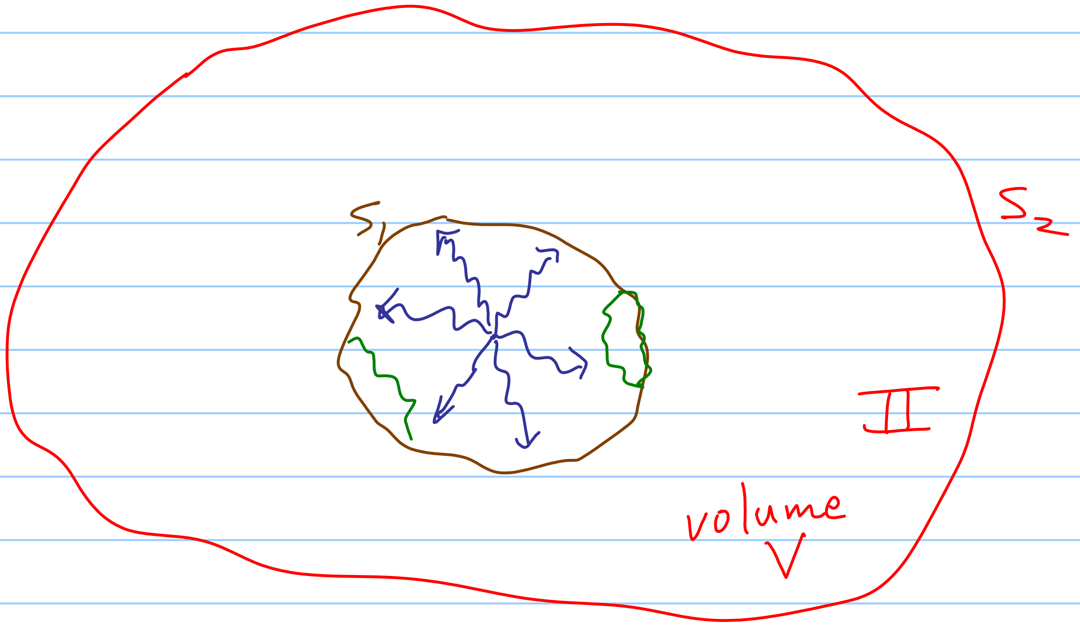
$$\Rightarrow \boxed{\Psi(\vec{x}) = -\frac{1}{4\pi}\oint_S \frac{e^{ikR}}{R}\hat{n}'\cdot\left[\nabla'\Psi + ik\Psi(\vec{x}')\left(1+\frac{i}{kR}\right)\frac{\vec{R}}{R}\right] da'} \quad (*)$$

This formula can be applied to diffraction problems such as the following two types:

EITHER radiation comes from sources, e.g. at $-\infty$, and encounters a screen with apertures:



OR radiation comes from an inside volume, passes through a surface S_1 with apertures, and produces a pattern on surface S_2 at "∞":



In either of these cases, light diffracts through apertures in S_1 and is detected at S_2, and far from the apertures, in both cases, the solution looks like:

$$\psi \xrightarrow[r\to\infty]{} f(\theta,\phi)\,\frac{e^{ikr}}{r} \Rightarrow \frac{1}{\psi}\frac{\partial\psi}{\partial r} \to ik - \frac{1}{r}$$

So neglecting the contribution of S_2 to the integral (*), we can apply it with an integration over S_1 only. This is the well-known Kirchoff approximation of optics, obtained if we now make two assumptions:

(i) $\psi = 0$ and $\frac{\partial \psi}{\partial n} = 0$ on S_1, except in the apertures

(ii) Use unperturbed values of ψ and $\frac{\partial \psi}{\partial n}$ in the apertures, taken from the incident wave.

Jackson comments on the resulting inconsistency that results from this overspecification of boundary conditions, but points out that fixing this inconsistency gives little improvement in practice.

But if we care about this point we should follow our logic in Chaps 2,3 last semester, and develop a consistent treatment using either Dirichlet or Neumann BC's

e.g. if ψ is known exactly on S_1, or at least approximately, we should use a Dirichlet GF obeying

$$G_D(\vec{x}, \vec{x}\,') = 0 \quad \text{for all } \vec{x}\,' \text{ on } S_1$$

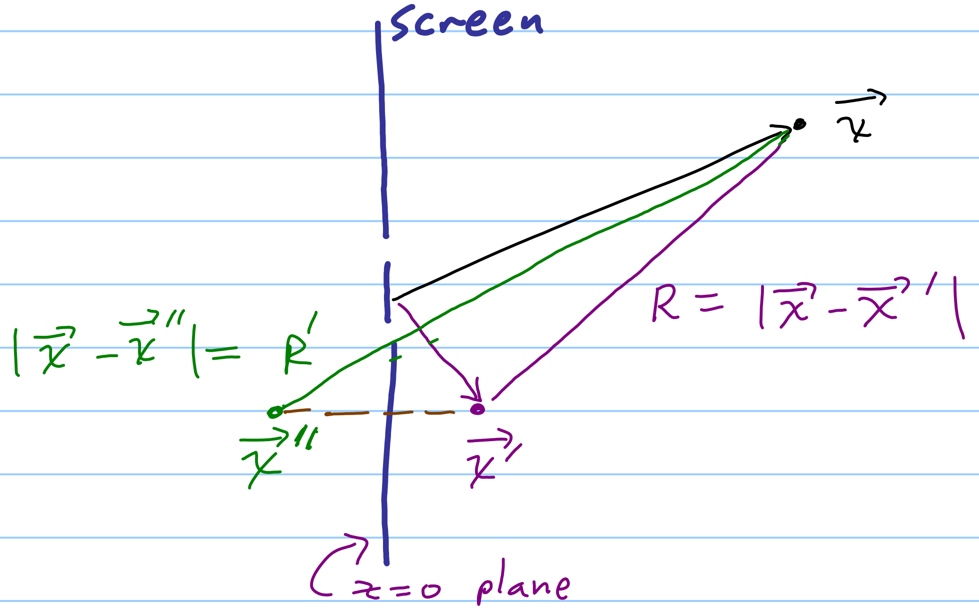
and then the solution desired will be

$$\boxed{\psi(\vec{x}) = \int_{S_1} \psi(\vec{x}\,') \frac{\partial G_D(\vec{x}, \vec{x}\,')}{\partial n'} \, da'}$$

Special case, S_1 = planar screen

$\Rightarrow$ the Dirichlet GF can be obtained with the method of images, namely (10.84)

$$G_D(\vec{x},\vec{x}\,') = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}\right)$$



here $R = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}$

while
$$R' = [(x-x')^2 + (y-y')^2 + (z+z')^2]^{1/2}$$

and the normal surface derivative needed is seen to be (algebra spared here)

$$\frac{\partial G_D}{\partial n'} = \left.\frac{\partial G_D}{\partial z'}\right|_{z'=0} = \frac{k}{2i\pi}\,\frac{e^{ikR}}{R}\left(1+\frac{i}{kR}\right)\frac{\hat{n}'\cdot\vec{R}}{R}$$

resulting in another, more consistent, approximation,

$$\boxed{\Psi(\vec{x}) = \frac{k}{2i\pi}\int_{S_1}\frac{e^{ikR}}{R}\left(1+\frac{i}{kR}\right)\frac{\hat{n}'\cdot\vec{R}}{R}\,\Psi(\vec{x}\,')\,da'} \quad {}^{\ast\ast}$$

More important is the <u>vector</u> nature of the fields, so we now turn to

Vector diffraction theory (10.6)

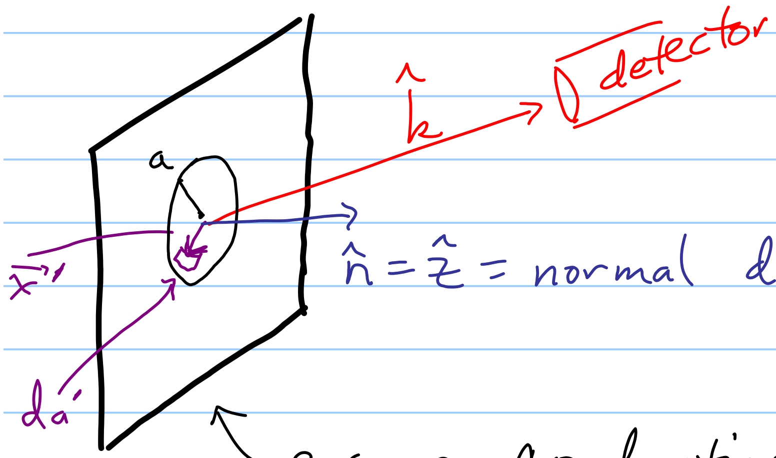
If we apply our Green's theorem formula (above) to each Cartesian component of $\vec{E}$, we obtain

$$\vec{E}(\vec{x}) = \oint_S \left[\vec{E}(\vec{x}\,')(\hat{n}'\cdot\nabla' G) - G(\vec{x},\vec{x}\,')(\hat{n}'\cdot\nabla')\vec{E}(\vec{x}\,') \right] da'$$

Because the general theory is quite complicated, I will specialize to a planar S_1 diffraction screen, and jump to an approximation for the diffraction E-field, Eq. 10.101, which in the large-r limit is

$$\vec{E}_{diff}(\vec{x}) = \frac{i}{2\pi r} e^{ikr}\, \vec{k} \times \int_{S_1} \hat{n} \times \underbrace{\vec{E}(\vec{x}\,')}\, e^{-i\vec{k}\cdot\vec{x}\,'}\, da'$$

↑ approximate $\vec{E}$ here by $\vec{E}_{inc}$



$\hat{n} = \hat{z}$ = normal directed into observation side

← e.g. a conducting planar screen with a circular hole.

To apply this, we choose coordinates as in Jackson, e.g. the wavevector incident, $\vec{k}$, lies in the xz-plane:

$$\vec{k} = k(\hat{z}\cos\alpha + \hat{x}\sin\alpha)$$

and $$\hat{e}_0 = -\hat{z}\sin\alpha + \hat{x}\cos\alpha$$

=> the polarization vector $\hat{e}_0$ is also in the xz-plane, i.e.

$$\vec{E}_{inc} = E_0\,(\hat{x}\cos\alpha - \hat{z}\sin\alpha)\,e^{ik(z\cos\alpha + x\sin\alpha)}$$

which equals, in the aperture where $z'=0$,

$$\hat{n}' \times \vec{E}_{inc}\Big|_{z'=0} = E_0\,\hat{y}\cos\alpha\, e^{ikx'\sin\alpha}$$

and $$\vec{E}(\vec{x}) = \frac{i e^{ikr}}{2\pi r}\,(\hat{k}\times\hat{y})\int_0^a \rho' d\rho' \int_0^{2\pi} d\phi'$$

$$\times\, e^{ik\rho'\sin\alpha\cos\phi' - ik\rho'\sin\theta\cos(\phi-\phi')}$$

↳ the exponent simplifies to

$$ik\rho'\{(\sin\alpha - \sin\theta\cos\phi)\cos\phi' + \sin\theta\sin\phi\sin\phi'\}$$

$$\equiv A\cos\phi' + B\sin\phi'$$

$$= \sqrt{A^2+B^2}\,(\cos\Phi\cos\phi' + \sin\Phi\sin\phi')$$

$$= \xi\cos(\phi' - \Phi)$$

where $\tan\Phi \equiv \dfrac{\sin\theta\sin\phi}{\sin\alpha - \sin\theta\cos\phi}$

and $\xi = \left[(\sin\alpha - \sin\theta\cos\phi)^2 + \sin^2\theta\sin^2\phi\right]^{1/2}$

so our integral becomes

$$\int_0^a \rho'\,d\rho' \int_0^{2\pi} d\phi'\, e^{ik\xi\rho'\cos(\phi'-\Phi)} = 2\pi\int_0^a \rho' J_0(k\rho'\xi)\,d\rho'$$

$$= \frac{2\pi a}{k\xi} J_1(ka\xi)$$

And finally we have

$$\vec{E}(\vec{x}) = \frac{i e^{ikr}}{r} a^2 E_0 \cos\alpha\, (\hat{k}\times\hat{y}) \frac{J_1(ka\xi)}{ka\xi}$$

and $\vec{H} = \dfrac{\hat{k}\times\vec{E}}{Z_0}$, whereby

$$r^2\vec{S} = \frac{1}{2}\mathrm{Re}\left\{ \frac{k^2|E_0|^2 a^4\cos^2\alpha}{Z_0} \left| \frac{J_1(ka\xi)}{ka\xi}\right|^2 \right\}$$

$$\times \underbrace{(\hat{k}\times\hat{y})\times[\hat{k}\times(\hat{k}\times\hat{y})]\cdot\hat{k}}$$

$$\parallel$$

$$|\hat{k}\times\hat{y}|^2 = |(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)\times\hat{y}|^2$$

$$= |\hat{z}\sin\theta\cos\phi - \hat{x}\cos\theta|^2$$

$$= \sin^2\theta\cos^2\phi + \cos^2\theta$$

and the time-averaged diffracted power, per unit solid angle, is

$$\frac{dP}{d\Omega} = \frac{k^2}{2} \frac{|E_0|^2 a^4}{Z_0} \cos^2\alpha \left(\cos^2\theta + \sin^2\theta\cos^2\phi\right) \left|\frac{J_1(ka\xi)}{ka\xi}\right|^2$$

Jackson recasts this in terms of the component of the incident power that is NORMAL to the aperture plane, i.e. $P_i = \frac{|E_0|^2}{2Z_0} \pi a^2 \cos\alpha$

$$\boxed{\frac{dP}{d\Omega} = P_i \cos\alpha \frac{(ka)^2}{4\pi} \left(\cos^2\theta + \sin^2\theta\cos^2\phi\right) \left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2}$$

Observe that for short wavelengths, $\lambda << a$, this is sharply peaked around the incidence angle, $\theta = \alpha$, $\phi = 0$, as one expects in the limit of geometrical optics.

For comparison, a scalar diffraction calculation (10.119) gives

$$\frac{dP}{d\Omega} = P_i \frac{(ka)^2}{4\pi} \cos\alpha \left(\frac{\cos\alpha + \cos\theta}{2\cos\alpha}\right)^2 \left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

Chapter 11 – Special Relativity

(Jackson changes here to Gaussian units.
See the Appendix for SI ⟷ Gaussian conversions)

Albert Einstein postulated (1905):

(1) The laws of physics are the same in all inertial (nonaccelerated) reference frames

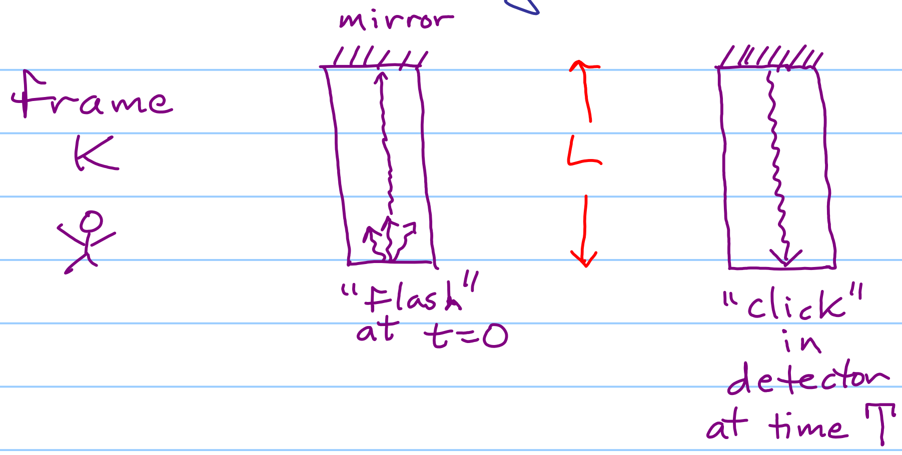
(2) The speed of light in vacuum, c, is the same in every inertial frame. Experiments have confirmed this to many decimal places over the years. For this reason, c is now taken as a DEFINED fundamental constant:

$$\boxed{c = 299792458 \ \frac{m}{s}}$$

Let's briefly review the elementary derivations of special relativity.

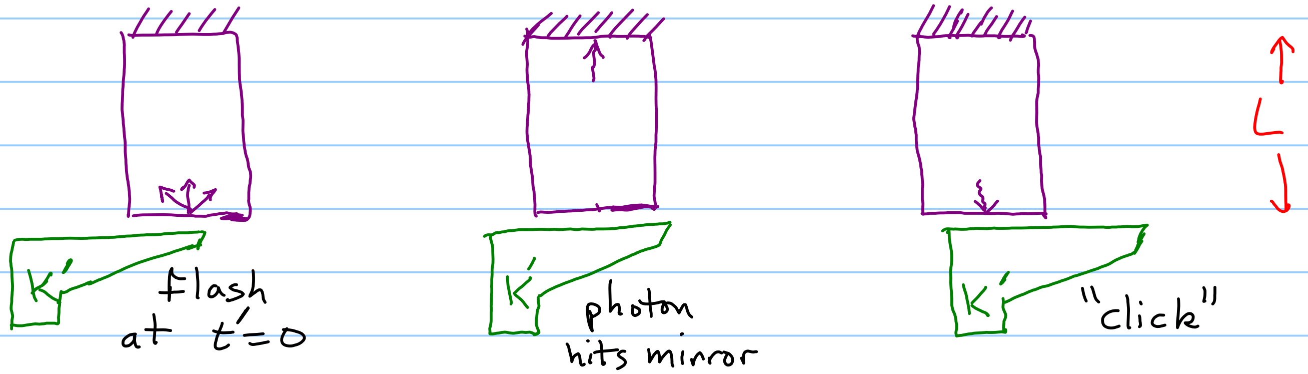
<u>TIME DILATION</u> – To measure the duration of a time interval between 2 events in some inertial reference frame K, we use a set of synchronized clocks.

For definiteness, consider a light box that is stationary in K

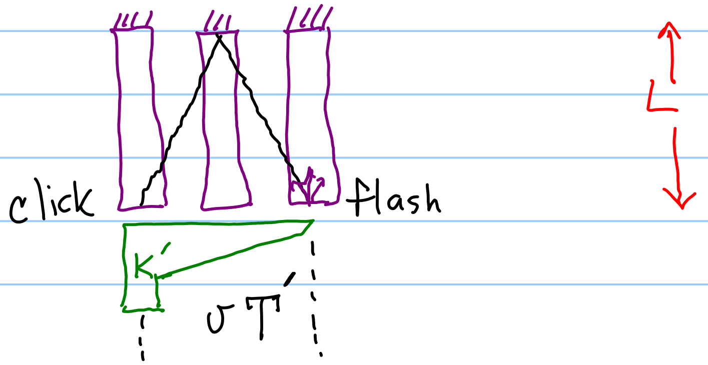


simple kinematics: The time of the click for observer K is $T = \frac{2L}{c}$

Next consider these same events seen by observer K′ moving at velocity v relative to K, e.g. imagine that K′ is attached to a long coasting spaceship



From the viewpoint of observers on ship K′, this looks as follows:



Thus the K′ observers say that the time between the two events is

$$T' = \frac{2}{c}\left[L^2 + \left(\frac{vT'}{2}\right)^2\right]^{1/2}$$

or

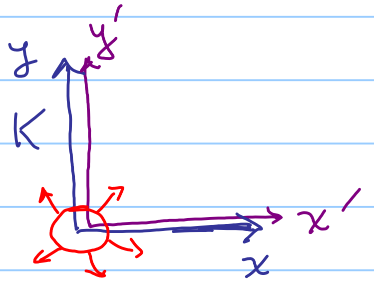
$$\boxed{T' = \frac{T}{\sqrt{1-\frac{v^2}{c^2}}}} \equiv \gamma T, \quad \boxed{\text{where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \geq 1}$$

And we note that the SHORTEST time interval between 2 events will always be observed in that inertial frame where BOTH events occur at the SAME LOCATION.

This shortest possible time is also called the PROPER TIME τ between those events.

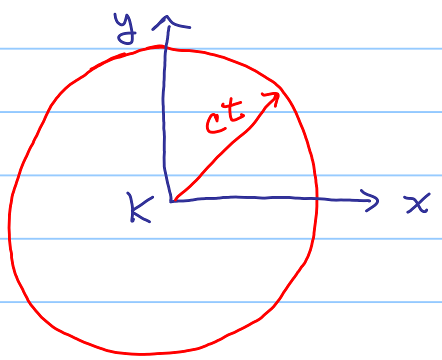
Other inertial frames measure the time interval to be DILATED, i.e. $\Delta t' = \gamma\tau$

What about a spherical light pulse in K?

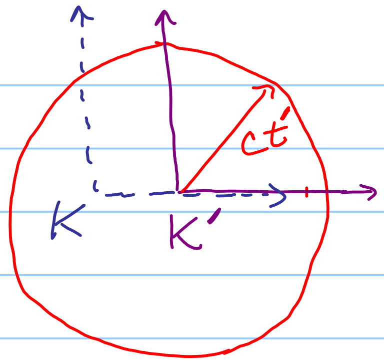


light flashes on and off at $t = t' = 0$
$x = x' = 0$, $y = y' = 0 = z = z'$

What does K see?



What does K' see?



Remarkably, counterintuitively, BOTH observers see a spherical shell expanding outwards in all directions at speed c.

i.e. the shell location obeys

$$c^2t^2 - x^2 - y^2 - z^2 = 0 \text{ to } K$$

while to K':

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = 0 \text{ to } K'$$

Let's set up the relationship between the coordinates of K and K', setting this up with time as a "0th coordinate", as ct, i.e.

<u>K coordinates</u>	<u>K' coordinates</u>
$x_0 = ct$	$x_0' = ct'$
$x_1 = x$	$x_1' = x'$
$x_2 = y$	$x_2' = y'$
$x_3 = z$	$x_3' = z'$

Let's concentrate on the spatial coordinate parallel to $\vec{v} = v\hat{x}$, i.e. x and x', since we know/expect $y = y'$ and $z = z'$.

$\Rightarrow$ from the above example, we know that

$$c^2t^2 - x^2 = c^2t'^2 - x'^2$$

or $$x_0^2 - x_1^2 = x_0'^2 - x_1'^2$$

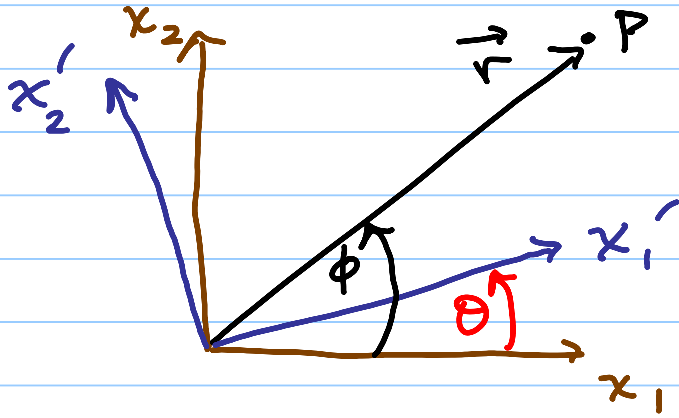
Goal: Replace the Galilean transformation between (x_0, x_1) and (x_0', x_1') with a new one, the Lorentz transformation

- Want the transformation to be LINEAR, and replacing $v \to -v$ should give the inverse transformation.
- It must preserve an invariant, i.e.

$$x_0^2 - x_1^2 = x_0'^2 - x_1'^2$$

How to find such a general transformation?

For comparison, consider an ordinary 2D rotation (passive):



$$\Rightarrow \vec{r} = x_1 \hat{x} + x_2 \hat{y} = r\cos\phi\, \hat{x} + r\sin\phi\, \hat{y}$$

or $$\vec{r} = r\cos(\phi - \theta)\, \hat{x}' + r\sin(\phi-\theta)\, \hat{y}' = x_1' \hat{x}' + x_2' \hat{y}'$$

$$= (r\cos\phi\cos\theta + r\sin\phi\sin\theta)\hat{x}'$$
$$+ (r\sin\phi\cos\theta - r\cos\phi\sin\theta)\hat{y}'$$
$$= (x_1\cos\theta + x_2\sin\theta)\hat{x}' + (-x_1\sin\theta + x_2\cos\theta)\hat{y}'$$

or $$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \mathbb{R}(\theta)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and note that this transformation preserves an invariant, the length of $\vec{r}$, i.e.

$$x_1^2 + x_2^2 = x_1'^2 + x_2'^2$$

But, for a Lorentz transformation, our invariant is a DIFFERENCE of squares, which we can write as

$$(i x_0')^2 + x_1'^2 = (i x_0)^2 + x_1^2$$

Mathematically, this connects with a rotation by an imaginary angle, $\theta \to -i\xi$, i.e. rotate the vector $\begin{pmatrix} i x_0 \\ x_1 \end{pmatrix}$ as above:

Recall $\cos i\xi = \cosh\xi$
$\sin i\xi = i\sinh\xi$

$$\Rightarrow \begin{pmatrix} i x_0' \\ x_1' \end{pmatrix} = \begin{pmatrix} \cosh\xi & -i\sinh\xi \\ i\sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} i x_0 \\ x_1 \end{pmatrix}$$

or writing it out,

$$i x_0' = i\cosh\xi\, x_0 - i\sinh\xi\, x_1$$
$$x_1' = -\sinh\xi\, x_0 + \cosh\xi\, x_1$$

or dropping the i: $\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$

which indeed obeys $x_0'^2 - x_1'^2 = x_0^2 - x_1^2$

since $\cosh^2\xi - \sinh^2\xi = 1$

Now, to relate ξ to the physics:

Let's consider the coordinate origin of K' to be $x_1' = 0 = -x_0 \sinh\xi + x_1 \cosh\xi$

$$\Rightarrow \quad \frac{x_1}{x_0} = \tanh\xi = \frac{x_1/t}{c}$$

But $\frac{x_1}{t} = v =$ velocity of K' with respect to K.

$$\Rightarrow \quad \boxed{\xi = \tanh^{-1}\left(\frac{v}{c}\right)} = \tanh^{-1}\beta, \quad \beta \equiv \frac{v}{c}$$

and then $\tanh^2\xi = \beta^2$,

and $\sinh^2\xi = \beta^2\cosh^2\xi = \cosh^2\xi - 1$

$$\Rightarrow \boxed{\cosh\xi = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma}, \quad \text{so} \quad \boxed{\sinh\xi = \beta\gamma}$$

Hence our Lorentz transformation reads

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

or $x_0' = \gamma(x_0 - \beta x_1)$

$x_1' = \gamma(-\beta x_0 + x_1)$

$$\Rightarrow \boxed{\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}}$$

and ξ is called "the boost parameter"

or "the rapidity"

Four-vectors Where we write a 3-vector as $\vec{x} = (x_1, x_2, x_3)$, it is useful to introduce notation for a 4-vector,

i.e. $x = (ct, x^1, x^2, x^3) = x^\alpha$

(we will be more systematic and careful about defining & using this notation later, e.g. in Secs. 11.6, 11.7)

Any collection of 4 objects obeying this same Lorentz transformation law is called a

4-vector

e.g. for a quantity $(A^0, \overbrace{A^1, A^2, A^3}^{``\vec{A}"})$, it is a 4-vector if it becomes $(A^{0\prime}, A^{1\prime}, A^{2\prime}, A^{3\prime})$ after Lorentz transformation, where

$$\boxed{\begin{aligned} A^{0\prime} &= \gamma(A_0 - \vec{\beta}\cdot\vec{A}) \qquad \vec{A}^{\perp\prime} = \vec{A}^{\perp} \\ A^{\parallel\prime} &= \gamma(A^{\parallel} - \beta A^0) \end{aligned}}$$

(Here $\parallel$, $\perp$ signify components parallel, perp. to $\vec{\beta}$)

Mathematically, the Lorentz $\overbrace{\text{transformation}}^{\text{between 2 inertial frames}}$ is written as a matrix-vector multiplication, as

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, x^{\beta},$$

where Einstein's convention for summation on all repeated indices is employed.

Then any 4-vector quantity A^{α} obeys:

$$A'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, A^{\beta}$$

Aside (Eq. 11.19): if $\vec{\beta}$ is not along one of the Cartesian unit vectors, the Lorentz transf. is slightly more complicated, namely

$$\boxed{\begin{aligned} x_0' &= \gamma(x_0 - \vec{\beta}\cdot\vec{x}) \\ \vec{x}\,' &= \vec{x} + \frac{(\gamma-1)}{\beta^2}(\vec{\beta}\cdot\vec{x})\vec{\beta} - \gamma x_0 \vec{\beta} \end{aligned}}$$

The generalization of the invariant scalar product, which in ordinary 3D is $\vec{x}\cdot\vec{y} = \sum_i x_i y_i$ is written for two 4-vectors as

$$x\cdot y = x_0 y_0 - \vec{x}\cdot\vec{y} = x'\cdot y' = x_0' y_0' - \vec{x}\,'\cdot\vec{y}\,'$$

and $x\cdot y$ is a scalar w.r.t. GENERALIZED Lorentz transformations, a subset of which are the ordinary 3D rotations.

A more careful notation

Henceforth we will write the invariant (or scalar product)

$$x\cdot y = \sum_{\alpha=0}^{3} x^\alpha y_\alpha = x^\alpha y_\alpha = x_\alpha y^\alpha$$

where $x^\alpha = (x_0, x_1, x_2, x_3) = (x_0, \vec{x})$

are called the CONTRAVARIANT components of the 4-vector x, and

$y_\alpha = (y_0, -y_1, -y_2, -y_3) = (y_0, -\vec{y})$ are the COVARIANT components of a 4-vector y

Note that invariant or scalar contractions always involve one contravariant index and one covariant index.

e.g. the invariant interval between two 4-displacements is written in this notation as

$$ds^2 = dx^\alpha dx_\alpha = dx_0^2 - |d\vec{x}|^2$$

And we can convert between contravariant and covariant components using the METRIC TENSOR

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\alpha\beta}$$

i.e. $x^\alpha = g^{\alpha\beta} x_\beta$ and $x_\alpha = g_{\alpha\beta} x^\beta$

and we write $x \cdot y = x^\alpha y_\alpha = x^\alpha g_{\alpha\beta} x^\beta$

and hence $ds^2 = g_{\alpha\beta} x^\alpha x^\beta$

This notation and terminology is adopted from standard differential geometry or tensor analysis.

General transformation theory

Suppose the transformation from one inertial frame to another is written as

$$x'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} x^{\beta}$$

and we can visualize $\frac{\partial x'^{\alpha}}{\partial x^{\beta}} \equiv \Lambda^{\alpha}{}_{\beta}$ as a transformation matrix

and ANY vector quantity A^{α}, whose transformation law is

$$A'^{\alpha} = \Lambda^{\alpha}{}_{\beta} A^{\beta},$$

is a contravariant vector.

Similarly, a quantity whose transformation law is $B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta}$ is a covariant vector (or tensor of rank 1)

Relativistic Doppler Shift

– a derivation by invariance arguments

Observe that in a plane wave like

$$e^{i(\vec{k}\cdot\vec{r}-\omega t)} \equiv e^{i\phi},$$

ϕ must be an INVARIANT, independent of reference frame, since it simply counts the number of wave crests

$\Rightarrow$ if we write $-\phi = \frac{\omega}{c} ct - \vec{k}\cdot\vec{x} = k^\alpha x_\alpha$
where $k^\alpha = \left(\frac{\omega}{c}, \vec{k}\right)$,

then k^α must be a 4-vector since its contraction with a known 4-vector x^α is an invariant (scalar)

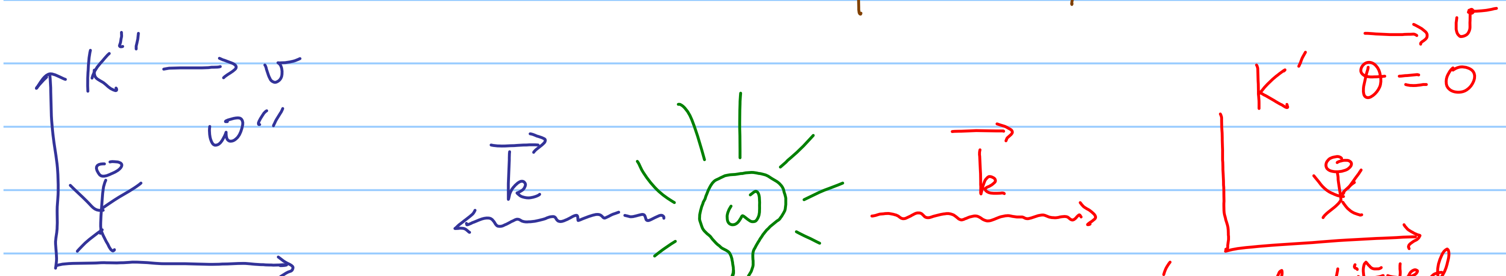
Accordingly, this is all we need to know in order to write down the transformation law for k^α. First break its spatial part $\vec{k}$ into parallel + (to $\vec{v} = c\vec{\beta}$) perpendicular components, $\vec{k} = \vec{k}_\parallel + \vec{k}_\perp$

i.e. $\vec{\beta}\cdot\vec{k}_\perp = 0$, then

$$\boxed{\begin{array}{l} (1)\ \vec{k}_\perp' = \vec{k}_\perp \\ (2)\ k_0' = \frac{\omega'}{c} = \gamma\left(k_0 - \beta k_\parallel\right) \\ (3)\ k_\parallel' = \gamma\left(k_\parallel - \beta k_0\right) \end{array}}$$

Now, recall that for EM radiation in free space, $|\vec{k}| = \frac{\omega}{c} \Rightarrow v k_\parallel = v|\vec{k}|\cos\theta = \frac{v}{c}\omega\cos\theta$
where θ = angle between $\vec{v}$ and $\vec{k}$ (note: $\theta \neq \theta'$)

Thus $\omega' = \gamma\omega(1-\beta\cos\theta)$



ω'' = blue-shifted from ω

ω' = red-shifted from ω

$\underline{\theta = 0}$ $\quad \vec{k} \parallel \vec{v} \Rightarrow \omega' = \gamma(1-\beta)\omega = \dfrac{1-\beta}{\sqrt{1-\beta^2}}\,\omega$

or $\boxed{\omega' = \left(\dfrac{1-\beta}{1+\beta}\right)^{1/2} \omega < \omega}$

$\underline{\theta = \pi}$ $\quad \vec{k} \parallel -\vec{v} \quad \omega'' = \gamma(1+\beta)\omega = \dfrac{\gamma(1+\beta)}{\sqrt{1-\beta^2}}\,\omega$

or $\boxed{\omega'' = \left(\dfrac{1+\beta}{1-\beta}\right)^{1/2} \omega > \omega}$

$\Rightarrow$ This is an example of the longitudinal Doppler shift. Note that the nonrelativistic result would be $\omega' = (1-\beta\cos\theta)\omega$, which would have $\omega' = \omega$ at $\theta = 90°$

But in relativity, even the TRANSVERSE Doppler shift is nonzero, i.e, at $\theta = 90°$, $\omega' = \gamma\omega = \dfrac{\omega}{\sqrt{1-\beta^2}}$

Experiment – see Saathoff et al, Phys. Rev. Lett. 91, 190403 (2003)

– has confirmed this to a limit of $\pm 2.2 \times 10^{-7}$

Velocity Addition Theorem

Suppose a particle in frame K is observed to move from (x_1, x_2, x_3) to $(x_1+dx_1, x_2+dx_2, x_3+dx_3)$ in the time interval from t to $t+dt$.

Observer K' sees these 2 events occurring at (x_1', x_2', x_3') and $(x_1'+dx_1', x_2'+dx_2', x_3'+dx_3')$ during the interval t' to $t'+dt'$

$\Rightarrow$ Then if frame K′ moves at velocity $v\hat{x}_1$, relative to K, these observations are related by a Lorentz transformation, namely

$$dx_0 = \gamma_v (dx_0' + \beta\, dx_1') \qquad dx_2 = dx_2'$$

$$dx_1 = \gamma_v (dx_1' + \beta\, dx_0') \qquad dx_3 = dx_3'$$

Now, the ordinary velocity 3-vector components of the particle in each frame are:

$$u_i = c\,\frac{dx_i}{dx_0}\,, \quad u_i' = c\,\frac{dx_i'}{dx_0'}$$

whereby $u_\parallel = c\,\dfrac{dx_1' + \beta\, dx_0'}{dx_0' + \beta\, dx_1'} = c\left(\dfrac{\frac{dx_1'}{dx_0'} + \beta}{1 + \beta\,\frac{dx_1'}{dx_0'}}\right)$

or

$$\boxed{u_\parallel = \frac{u_\parallel' + v}{1 + \frac{u_\parallel' v}{c^2}}}$$

And $u_\perp = c\,\dfrac{dx_\perp}{dx_0} = c\,\dfrac{dx_\perp'}{\gamma_v (dx_0' + \beta\, dx_\parallel')}$

or

$$\boxed{u_\perp = \frac{u_\perp'}{\gamma_v \left(1 + \frac{u_\parallel' v}{c^2}\right)}}$$

These are the Einstein velocity addition formulas.

Limiting cases

(i) $u', v \ll c \Rightarrow \vec{u} = \vec{u}' + \vec{v}$, coincides with the Galilean result

(ii) either $u' \to c$, or $v \to c$

$\Rightarrow u \to c$

Important: velocities do NOT transform according to a Lorentz transformation!

Terminology: SPACELIKE versus TIMELIKE separations between events

Consider 2 events, $P(t_a, \vec{x}_a)$ and $P(t_b, \vec{x}_b)$

The squared invariant interval between them is

$$S_{ab}^2 = c^2(t_a - t_b)^2 - |\vec{x}_a - \vec{x}_b|^2$$

$$\equiv (x_a - x_b)^2 = c^2 \Delta t^2 - R^2$$

Case 1 $S_{ab}^2 > 0 \Rightarrow |c\Delta t| > R$

Claim: For this case we can always find an inertial frame K' for which

$$\vec{x}_a' - \vec{x}_b' = (x_0', 0),$$

in which case we say that $x_a' - x_b'$ is *purely timelike*.

Proof Align the x_1-axis along the vector
$$\vec{x}_a - \vec{x}_b \equiv \vec{x}_{ab}$$

then $$\vec{x}_{ab}' = \gamma(\vec{x}_{ab} - \vec{v}t)$$

so to find the frame where $\vec{x}_{ab}' = 0$, require $R = |\vec{x}_{ab}| = vt$, which is possible since $|ct| < R$,

i.e. $v < c$ which is physically acceptable

Case 2 $S_{ab}^2 < 0 \Rightarrow |c\Delta t| < R$

or $|c\Delta t| < |\vec{x}_{ab}|$

Claim: we can find an inertial frame K' in which $t_a' - t_b' = 0$,

i.e. for which $x_a' - x_b' = (0, \vec{x}_{ab}')$ is PURELY SPACELIKE.

Proof $x_0' = \gamma(x_0 - \beta R)$

So to have $x_0' = 0$, we require $x_0 = \beta R$

or $c\Delta t = \frac{v}{c} R$, and since $c\Delta t < R$

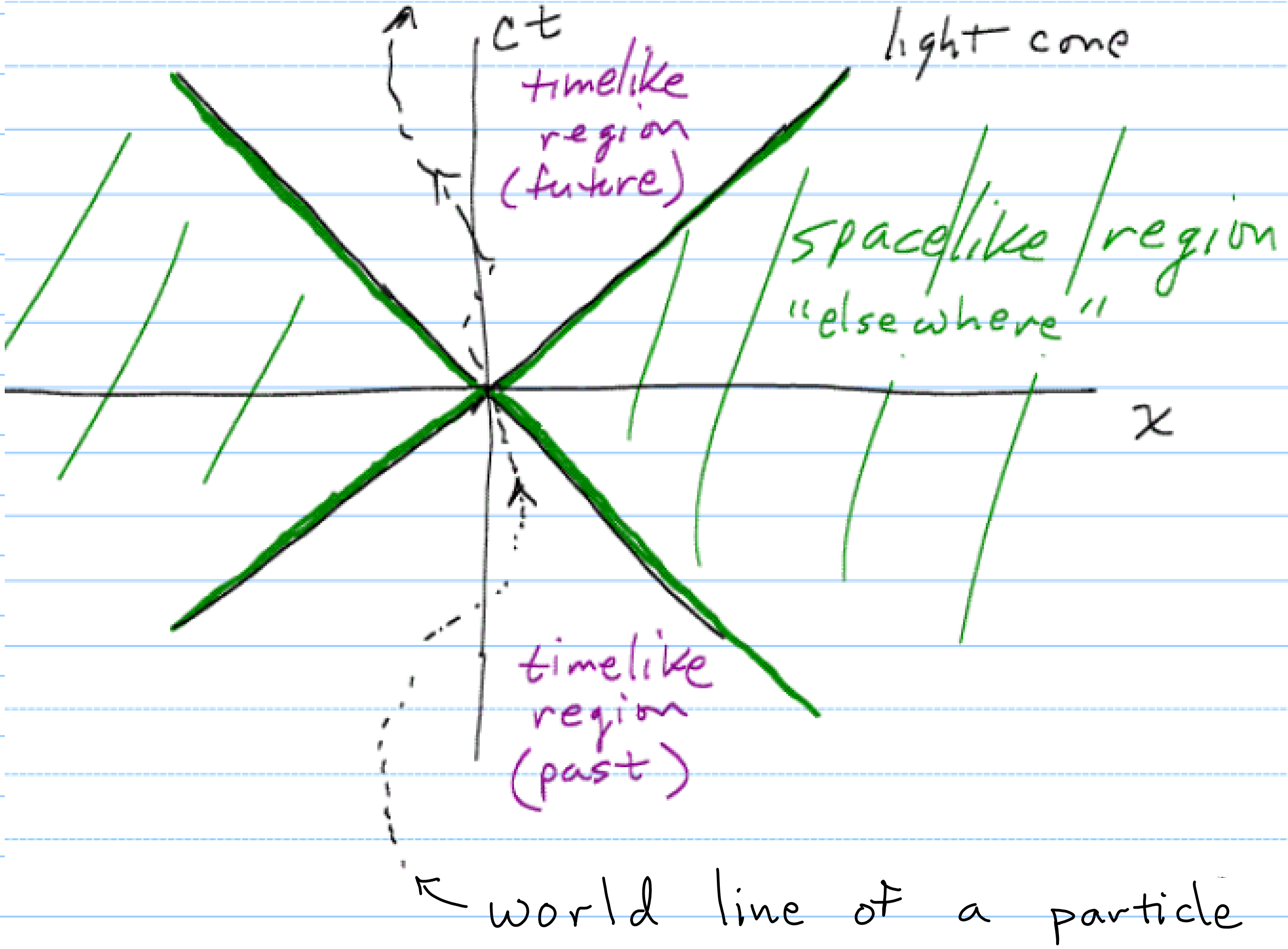
$\Rightarrow \beta = \frac{v}{c} < 1$, this is physically acceptable

Case 3 If $S_{ab}^2 = 0 \Rightarrow$ A LIGHTLIKE separation

$\Rightarrow$ These 2 events can be connected only by light pulses.

CAUSALITY

2 events with a spacelike separation cannot be causally related, because they would require faster-than-c communication, which is not possible.



Causality - consider a particle at the origin

⇒ 1. It can only be affected then by events in the timelike past.

2. It can only affect events in the future timelike region.

Proper time and 4-velocity

The Einstein velocity addition formula differs from the Lorentz transformation because:

velocity = ratio between the spatial components of a 4-vector x and the timelike component of a 4-vector

Observe, however, that the ratio of a 4-vector to ANY invariant (scalar) would transform like a 4-vector.

In particular, an appropriate invariant to consider is the PROPER TIME.

e.g. recall that $ds^2 = dx^\alpha dx_\alpha$ = invariant

or $ds^2 = c^2 dt^2 - d\vec{x}^2$, which equals, in the particle's instantaneous rest frame, $ds^2 \to c^2 dt'^2$

But this dt' is precisely the proper time between the "two events", when the particle is at $\vec{x}', t'$ and when it is at $\vec{x}' + d\vec{x}'$, at time $t' + dt'$.

Hence we can write this as $ds^2 = c^2 d\tau^2$

Now, to express $d\tau$ in terms of dt in any other frame, we apply our time dilation result, i.e.

$$dt = \gamma\, d\tau \;=\; \text{time interval in } K$$

Alternative argument: We can instead consider ds^2, which must be the same in the two frames, i.e.

$$ds^2 = \underbrace{c^2 d\tau^2}_{\text{in particle's rest frame}} = \underbrace{c^2 dt^2 - \left(\frac{d\vec{x}}{dt}\right)^2 dt^2}_{\text{in reference frame } K} = c^2(1-\beta^2)\,dt^2$$

$$= \frac{c^2 dt^2}{\gamma^2}$$

$$\Rightarrow \boxed{dt = \gamma\, d\tau} \text{ by this argument too}$$

Thus we define the 4-velocity as $\boxed{U^\alpha = \dfrac{dx^\alpha}{d\tau}}$

which now transforms as a 4-vector!

Interpretation

Since $x^\alpha = (ct, \vec{x})$, the space part of U^α is

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \gamma_u \frac{d\vec{x}}{dt} \Rightarrow \boxed{\vec{U} = \gamma_u \vec{u}}$$

while the time part is

$$U^0 = \frac{dx^0}{d\tau} = c\gamma_u \frac{dt}{dt} = c\gamma_u$$

so in other words, $U^\alpha = \gamma_u(c, \vec{u})$

with $\gamma_u = \dfrac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

Notation: capital U^α is the 4-velocity, l.c. $\vec{u}$ = ordinary velocity

Sec. 11.5 Relativistic energy + momentum of a particle

Our goal: Find the relativistic generalizations of

$$\vec{p} = m\vec{u} \quad \text{and} \quad T = \frac{1}{2}mu^2$$

We would hope that the new generalizations will give the old (nonrelativistic) conservation laws in the limit $u \ll c$, and which hopefully identify quantities conserved even in relativity.

e.g. consider the nonrelativistic momentum conservation in a 1D collision in frame K'

$$① + ② \longrightarrow ③ + ④$$

$$u_1' \qquad u_2' \qquad u_3' \qquad u_4'$$

Then the nonrelativistic equality in K' we know is

$$m_1 u_1' + m_2 u_2' = m_3 u_3' + m_4 u_4'$$

Now perform a Lorentz transf. to frame K moving at velocity $-v$ w.r.t. K':

$$\Rightarrow m_1 \frac{v + u_1'}{1 + \frac{v u_1'}{c^2}} + m_2 \frac{v + u_2'}{1 + \frac{v u_2'}{c^2}}$$

$$\overset{?}{=} m_3 \frac{v + u_3'}{1 + \frac{v u_3'}{c^2}} + m_4 \frac{v + u_4'}{1 + \frac{v u_4'}{c^2}}$$

It is hard to imagine that this equality could be true!

In fact it is not true!

whereas the nonrelativistic momentum conservation equation in K would read simply

$$m_1(v+u_1')+m_2(v+u_2')=m_3(v+u_3')+m_4(v+u_4')$$

which holds nonrelativistically since

$$m_1+m_2=m_3+m_4,$$

and mass is conserved in Galilean kinematics

Idea of the relativistic approach:

Attempt to define the relativistic momentum in terms of the 4-velocity, i.e. try

$$\vec{P}=m\vec{U}=\gamma_u m\vec{u}\ , \text{ where } \gamma_u=\left(1-\frac{u^2}{c^2}\right)^{-1/2}$$

This is because, if this momentum is conserved in a collision in one frame K', i.e. if the components along $\hat{x}_1$ obey:

$$\boxed{m_1U_1'(1)+m_2U_1'(2)=m_3U_1'(3)+U_1'(4)} \qquad \boxed{\text{I}}$$

Now carry out a Lorentz transformation to K, assuming each mass m_i = Lorentz scalar

$\Rightarrow$ Let K' move at $v=\beta c$ w.r.t. K and set $\gamma_v=(1-\beta^2)^{-1/2}$, and then for each particle,

$$U_0=\gamma_v(U_0'+\beta U_1')$$

$$U_1=\gamma_v(U_1'+\beta U_0')$$

Now check whether momentum conservation still holds in K, see if this equality holds,

$$m_1 U_1(1) + m_2 U_1(2) \overset{?}{=} m_3 U_1(3) + m_4 U_1(4)$$

This holds iff

II

$$\boxed{\begin{aligned} & m_1 \gamma_v \left[U_1'(1) + \beta U_0'(1)\right] + m_2 \gamma_v \left[U_1'(2) + \beta U_0'(2)\right] \\ & \quad = m_3 \gamma_v \left[U_1'(3) + \beta U_0'(3)\right] + m_4 \gamma_v \left[U_1'(4) + \beta U_0'(4)\right] \end{aligned}}$$

and in fact this <u>does</u> hold provided I holds <u>AND</u> provided the following holds as an additional conservation law,

$$m_1 U_0'(1) + m_2 U_0'(2) = m_3 U_0'(3) + m_4 U_0'(4)$$

$\Rightarrow$ mass is clearly not conserved, but something else is. What?

Recall that $m_1 U_0'(1) = \gamma_1 m_1 c$ with $\gamma_1 = \left[1 - \frac{u_1'(1)^2}{c^2}\right]^{-\frac{1}{2}}$ and in the low velocity limit, $|u_1'(1)| \ll c$,

$$m_1 U_0'(1) \longrightarrow m_1 c$$

Thus this new conserved quantity is consistent with mass conservation in the Galilean limit

⇒ Look at the next order corrections:

$$m\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} m = m\left[1 + \frac{u^2}{2c^2} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\frac{u^4}{c^4} + \dots\right]$$

$$= \frac{1}{c^2}\left[mc^2 + \underbrace{\frac{1}{2}mu^2} + \frac{3}{8}\frac{mu^4}{c^4} + \dots\right]$$

this next correction to the mass is the ordinary nonrelativistic kinetic energy

$$\Rightarrow \gamma mc^2 = mc^2 + T + O\left(\frac{u^4}{c^4}\right)$$

Based on this, we interpret

$mc^2 \equiv$ REST ENERGY of a particle

$\gamma mc^2 \equiv$ TOTAL ENERGY of a particle

$(\gamma - 1)mc^2 \equiv$ KINETIC ENERGY in relativity

And thus our newly-defined 3-momentum

$$\vec{p} = m\vec{U}$$

will be conserved IF our newly-defined total energy is <u>also</u> <u>conserved</u>.

⇒ These combine to define the 4-momentum, or the energy-momentum 4-vector:

$$p^\alpha = mU^\alpha = (mU_0, \vec{p}) = \gamma_u m(c, \vec{u}) = \left(\frac{E}{c}, \vec{p}\right)$$

where $E = \gamma_u mc^2$ = total energy in this frame

Note also that $\vec{u} = \frac{\vec{p}\,c}{E}$

And as for any 4-vector, the "length" is INVARIANT, i.e.

$$p^\alpha p_\alpha = \frac{E^2}{c^2} - \vec{p}^{\,2} = \text{Lorentz invariant}$$

$\Rightarrow$ In practice, we can calculate it in ANY convenient reference frame, and this is a ubiquitous strategy in special relativity.

e.g. in the particle's rest frame, $\vec{p}=0$, $\gamma = 1$, and $E = mc^2$

$$\Rightarrow p^\alpha p_\alpha = m^2c^2 = \frac{E^2}{c^2} - \vec{p}^{\,2}$$

or $E = \sqrt{\vec{p}^{\,2}c^2 + m^2c^4}$

and $T = E - mc^2 = (\gamma - 1)mc^2$

Secs. 11.6, 11.9 On the idea of COVARIANCE

Recall: so far we have discussed

contravariant 4-vectors, $a^\alpha = (a^0, \vec{a})$

covariant " $a_\alpha = (a_0, -\vec{a})$

$\Big\}\; a^0 = a_0$

connected by the metric tensor,

$$g_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix}$$

with the properties $g_{\alpha\gamma}\, g^{\gamma\beta} = \delta_\alpha{}^\beta$

and $\delta_\alpha{}^\alpha = 4$

And the Lorentz transformation is $x'^\alpha = \Lambda^\alpha{}_\beta\, x^\beta$

where
$$\Lambda^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\beta} = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or
$$\Lambda^\alpha{}_\beta = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
for a boost along the x-axis

Other relations: $\Lambda^\alpha{}_\beta = g^{\alpha\eta}\,\Lambda_{\eta\beta} = \Lambda^{\alpha\eta}\, g_{\eta\beta}$

Try to stay consistent and, when possible, write these formulas with adjacent indices matching

Generalizing further, a rank 2 contravariant tensor transforms as:

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\sigma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\sigma\delta}$$

or $$F'^{\alpha\beta} = \Lambda^{\alpha}{}_{\sigma} \Lambda^{\beta}{}_{\delta} F^{\sigma\delta}$$

which can be rewritten in matrix notation as

$$F' = \Lambda F \tilde{\Lambda} \leftarrow \text{means matrix transpose}$$

see Eq. 11.147.

One can similarly show that if b_β is a covariant 4-vector, then

$$Y^{\alpha} = F^{\alpha\beta} b_{\beta}$$ transforms as a covariant 4-vector

which is analogous to what we saw in Chap. 6, e.g. $\overleftrightarrow{T} \cdot \vec{b}$ transforms like an ordinary vector under 3D rotations

more generally transformations look like

$$F'^{\alpha'\beta'\ldots\gamma'\delta'} = \Lambda^{\alpha'}{}_{\alpha} \Lambda^{\beta'}{}_{\beta} \cdots \Lambda^{\gamma'}{}_{\gamma} \Lambda^{\delta'}{}_{\delta} F^{\alpha\beta\ldots\gamma\delta}$$

etc...

There is also a 4-vector operator that generalizes the 3D gradient operator, motivated by the 3D case. Let $\vec{a}$ = constant, 3D vector. Then we know that $\nabla(\vec{a} \cdot \vec{x}) = \vec{a}$

Similarly, consider a constant 4-vector a^α which we can use to make the invariant

$$a \cdot x = a^\alpha x_\alpha$$

$$\Rightarrow \quad \frac{\partial}{\partial x_\beta}(a \cdot x) = a^\beta$$

From this we conclude that $\frac{\partial}{\partial x_\alpha}$ transforms like a contravariant 4-vector.

$\Rightarrow$ With this motivation, let's write

$$\boxed{\frac{\partial}{\partial x_\alpha} \equiv \partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\nabla\right)}$$

and similarly $\boxed{\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla\right)}$

is a <u>covariant</u> 4-vector

These results further imply that the contracted operator

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x_0^2} - \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

is a Lorentz scalar / invariant

Now let's re-express some old results, to gain familiarity with this notation:

(1) The inhomogeneous wave eqns for $\Phi, \vec{A}$ in the Lorentz gauge, in Gaussian units:

Eqs (6.14 - 6.16)

$$\Box \begin{Bmatrix} \Phi \\ \vec{A} \end{Bmatrix} = -\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix} \Phi \\ \vec{A} \end{Bmatrix} = \frac{4\pi}{c}\begin{Bmatrix} c\rho \\ \vec{J} \end{Bmatrix}$$

plus the Lorentz gauge condition,

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \vec{A} = 0$$

(2) The charge continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

which suggests that we define a 4-current,

$$J^\alpha = (c\rho, \vec{J})$$

because now it is clear that the continuity equation holds in every inertial frame, i.e.

$$\boxed{\partial_\alpha J^\alpha = \frac{\partial}{\partial x^0}(c\rho) + \nabla \cdot \vec{J} = 0 = \partial^\alpha J_\alpha}$$

and from (1), $\Box(\Phi, \vec{A}) = \frac{4\pi}{c}(c\rho, \vec{J}) = \frac{4\pi}{c} J^\alpha$

Since $\Box$ = Lorentz scalar, this equation requires that

$\boxed{(\Phi, \vec{A}) \equiv A^\alpha}$ is a contravariant 4-vector,

and the Lorentz gauge condition is simply

$$\boxed{\partial_\alpha A^\alpha = 0}$$

Now the potential equations of motion and the gauge condition are expressed in covariant form.

Expressing key equations, like these, in manifestly covariant form means that their behavior under Lorentz transformation is trivially obvious.

$\Rightarrow$ We also see that under the transf. from $A^{\alpha} \xrightarrow{LT} A'^{\alpha}$

$\Rightarrow$ we find a mixing of $\Phi, \vec{A}$

and from $J^{\alpha} \rightarrow J'^{\alpha}$

we see a mixing of $\rho, \vec{J}$

Assertion without proof (read Sec. 11.7 on your own):

The most general proper Lorentz transformation can be written as $\Lambda = e^{L}$

where $L = 4\times 4$ matrix. For the simplest case where the transformation is only a boost ξ along the x_1-axis,

$$\Lambda = \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

while a rotation about the z-axis by ω looks like:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

But a more general boost transformation by a vector $\vec{\beta}$ looks like

$$\Lambda = e^{-\vec{\zeta}\cdot\vec{K}}$$

where $\vec{\zeta} \equiv \hat{\beta}\tanh^{-1}\beta$, and K_1, K_2, K_3 are given in 11.91, i.e. $K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & & & \\ 0 & & \bigcirc & \\ 0 & & & \end{pmatrix}$ $K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ 1 & & \bigcirc & \\ 0 & & & \end{pmatrix}$

and $K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & & & \\ 0 & & \bigcirc & \\ 1 & & & \end{pmatrix}$ = 3 generators of infinitesimal boosts

whereas a pure rotation is governed by the 3 infinitesimal rotation generators,

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix};\ S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix};\ S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the most general proper Lorentz transf depends on 6 parameters, $\zeta_1, \zeta_2, \zeta_3, \omega_1, \omega_2, \omega_3$ and the transf. matrix is

$$\boxed{\Lambda = e^{-\vec{\omega}\cdot\vec{S} - \vec{\zeta}\cdot\vec{K}}}$$

see Jackson pp. 546-7

These 6 matrices obey commutation relations that define the LORENTZ GROUP, namely

$$\boxed{\begin{aligned} [S_i, S_j] &= \epsilon_{ijk} S_k \\ [K_i, K_j] &= -\epsilon_{ijk} S_k \\ [S_i, K_j] &= \epsilon_{ijk} K_k \end{aligned}}$$

Note: boosts do not in general commute

One intriguing implication of this noncommutation is the remarkable story of the

THOMAS PRECESSION EFFECT

Recall - the spin-orbit interaction in hydrogen can be viewed as the interaction energy between the e^- spin magnetic moment and the B-field created at the e^- position by the "current loop" of the "proton orbital motion" as viewed by the e^-.

This is usually derived in elementary courses by finding the $\vec{B}$-field at the e^- using the BIOT-Savart law, and then use the e^- spin magnetic moment, $\vec{\mu}_{e,spin} = -\frac{g_e e}{2 m_e c}\vec{S}$ (Gaussian units)

and then

$$H^{\text{elementary}}_{spin-orbit} = -\vec{\mu}_{e,spin}\cdot\vec{B}$$

giving

$$W^{\text{elem.}}_{spin-orbit} = \frac{g_e}{2m^2c^2}\vec{S}\cdot\vec{L}\ \frac{1}{r}\frac{dU}{dr}$$

$U =$ pot. energy of e-p

turns out to be approximately 2 times too large!

(e.g. in Gaussian units, for a 1-electron atom,

$$U = -\frac{Ze^2}{r}$$
)

Alternatively, this can be derived by making a Lorentz transformation of the $\vec{E}, \vec{B}$-fields from the NUCLEUS (lab) rest frame, into the e^- rest frame, using

$$\vec{B}_{e-\text{frame}} = \gamma\left(\cancel{\vec{B}} - \frac{\vec{v}}{c}\times\vec{E}\right)$$

0 here

The classical equation for spin precession is

used $\gamma \simeq 1$

$$\frac{d\vec{S}}{dt} = \vec{\mu}_s \times \vec{B}_{e-\text{frame}} = -\vec{\mu}_s \times \left(\frac{\vec{v}_e}{c}\times\vec{E}\right)$$

where $\vec{E} = -\nabla\Phi = -\frac{\vec{r}}{r}\frac{d\Phi}{dr} = +\frac{1}{e}\frac{\vec{r}}{r}\frac{dU}{dr}$

The energy associated with this precession is

$$W' = -\vec{\mu}_s \cdot \vec{B}_{e-\text{frame}} = -\vec{\mu}_s \cdot \left(-\frac{\vec{v}_e}{c}\times\vec{E}\right)$$

$$= \vec{\mu}_s \cdot \left(\frac{\vec{v}_e}{c}\times\frac{1}{e}\frac{\vec{r}}{r}\frac{dU}{dr}\right)$$, but $m_e\vec{v}_e\times\vec{r} = -\vec{L}$

so $\boxed{W' = \frac{g_e}{2m^2c^2}\,\vec{S}\cdot\vec{L}\,\frac{1}{r}\frac{dU}{dr}}$ as before (and still wrong!)

To correct the error in this derivation, recall from mechanics that the rate of change of any vector in a rotating frame relates to the rate of change in a nonrotating frame:

$$\left(\frac{d\vec{S}}{dt}\right)_{\text{nonrot}} = \left(\frac{d\vec{S}}{dt}\right)_{\text{rot}} + \vec{\omega}_T \times \vec{S}$$

where $\vec{\omega}_T$ is the angular velocity of rotation found by Thomas (1927), whereby

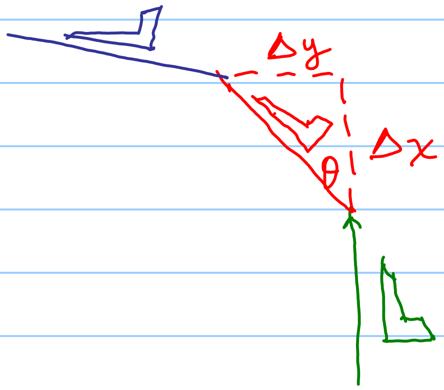
$$\left(\frac{d\vec{S}}{dt}\right)_{\text{nonrot}} = \vec{\mu}_s \times \vec{B}_{\text{e-frame}} - \vec{S} \times \omega_T$$

and this precession rate corresponds to an interaction energy in the lab frame of

$$W = W' + \vec{S} \cdot \vec{\omega}_T$$

Next – here is a simple derivation of the "Thomas precession frequency" $\vec{\omega}_T$

⇒ Approximate a circular orbit of an electron (= space shuttle) by an N-sided polygon with N large. When the craft traverses ONE of the N sides, it must alter its angle of flight by $\theta = \frac{2\pi}{N}$ radians:



$$\Rightarrow \theta \simeq \frac{\Delta y}{\Delta x} \ll 1$$

⇒ After N segments, the shuttle is back at its original point, having rotated by 2π radians in the LAB frame.

But in the shuttle rest frame, the rotation angle is LARGER, namely

$$\theta' = \gamma\theta = \frac{\Delta y}{(\Delta x/\gamma)}$$

because the length Δx is Lorentz-contracted but Δy is not

⇒ After a complete revolution, the e (shuttle) frame experiences a rotation $2\pi\gamma$

⇒ Relativity causes an EXTRA amount of rotation in the rotating frame, equal to

$$\Delta\theta' = 2\pi(\gamma - 1)$$, or in the LAB frame,

this corresponds to a frequency ratio

$$\frac{\omega_T}{\omega} = -\frac{\Delta\theta/T}{2\pi/T} = -(\gamma-1)$$

i.e. $$\omega_T = -(\gamma-1)\frac{v}{r} \simeq -\frac{1}{2}\frac{v^2}{c^2}\frac{v}{r}$$

and now $$\frac{mv^2}{r} = eE = e\frac{1}{e}\frac{dU}{dr}$$

$$\Rightarrow \quad \omega_T = -\frac{1}{2}\frac{1}{mc^2}\frac{vr}{r}\frac{dU}{dr}$$

or $$\vec{\omega}_T = -\frac{1}{2m^2c^2}\vec{L}\,\frac{1}{r}\frac{dU}{dr}$$

Hence the total energy in the lab frame is

$$W = W' + \vec{S}\cdot\vec{\omega}_T$$

or

$$W = \frac{g}{2}\frac{1}{m^2c^2}\vec{S}\cdot\vec{L}\,\frac{1}{r}\frac{dU}{dr} - \frac{1}{2}\frac{1}{m^2c^2}\vec{S}\cdot\vec{L}\,\frac{1}{r}\frac{dU}{dr}$$

so the correct spin-orbit term in the lab frame is (to order $\frac{v^2}{c^2}$):

$$\boxed{W = \frac{g-1}{2}\frac{1}{m^2c^2}\vec{S}\cdot\vec{L}\,\frac{1}{r}\frac{dU}{dr}}$$

The electron g - factor is of course

$$g \simeq 2 + \frac{\alpha}{\pi} - 0.657\left(\frac{\alpha}{\pi}\right)^2 + \ldots$$

$\Rightarrow \frac{g-1}{2} = \frac{1}{2}$ instead of the value 1, before making the Thomas precession correction!

and this modified form agrees with expt.

Note: Dirac's relativistic QM theory includes this effect "automatically", though it predicts $g=2$ exactly

$\Rightarrow$ Need QED + renormalization to get a more accurate value of g. (see QM 3)

Transformation of the fields

Recall that
$$\boxed{\begin{aligned} \partial^\alpha &= (\partial_0, -\nabla) \\ \partial_0 &= \frac{1}{c}\frac{\partial}{\partial t} \\ A^\alpha &= (\Phi, \vec{A}) \end{aligned}}$$

We start by writing out the fields in this notation, component-by-component:

(1) $\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \Rightarrow E^1 = \partial^1 A^0 - \partial^0 A^1$
$$= -(\partial^0 A^1 - \partial^1 A^0)$$

and likewise, $E^2 = -(\partial^0 A^2 - \partial^2 A^0)$, ...etc

(2) $\vec{B} = \nabla \times \vec{A} \Rightarrow B^1 = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$

and $B^2 = -(\partial^3 A^1 - \partial^1 A^3)$, etc.

and this suggests that a key entity in electromagnetism should be the following:

$$\boxed{F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha}$$

which we call the antisymmetric, rank-2,

FIELD-STRENGTH TENSOR

Writing out its contravariant form explicitly, this is:

$$F^{\alpha\beta} = \begin{array}{c|cccc} \alpha \backslash \beta & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & -E_x & -E_y & -E_z \\ 1 & E_x & 0 & -B_z & B_y \\ 2 & E_y & B_z & 0 & -B_x \\ 3 & E_z & -B_y & B_x & 0 \end{array}$$

and its covariant form is $F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta}$
which is identical, except
that everywhere $\vec{E} \to -\vec{E}$, $\vec{B} \longrightarrow \vec{B}$

Another quantity of interest is the
"DUAL FIELD-STRENGTH TENSOR":

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}$$

(simply get this by replacing $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ in $F^{\alpha\beta}$)

where $\epsilon^{\alpha\beta\gamma\delta}$ = totally antisymmetric rank-4 tensor, i.e.

$\epsilon^{0123} = 1$ and all other even permutations

$\epsilon^{1023} = -1$ " " " odd permutations

$\epsilon^{1123} = 0$ and all other elements having 2 or more equal indices

In Mathematica, e.g.:

```
epsilon = LeviCivitaTensor[4]
epsilon[[1,2,3,4]]   (returns 1)
epsilon[[2,1,3,4]]   (returns -1)
epsilon[[2,2,3,4]]   (returns 0)
```

Covariant form of Maxwell's equations (microscopic case, no media)

(a) inhomogeneous equations (remember – Gaussian units now)

1) $\nabla \cdot \vec{E} = 4\pi\rho \Rightarrow \sum_{i=1}^{3} \partial_i E^i = \frac{4\pi}{c} J^0$

and recall that $E^i = F^{i,0}$, $F^{0,0} = 0$

Hence we can write this as

$$\partial_\alpha F^{\alpha,0} = \frac{4\pi}{c} J^0$$

2) $\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$

(*)

e.g. $\boxed{\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{\partial E_x}{\partial x^0} = \frac{4\pi}{c} J_x}$

and now $B_z = F^{2,1}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial x^2} = \partial_2$

$B_y = -F^{3,1}$, $\frac{\partial}{\partial z} = \frac{\partial}{\partial x^3} = \partial_3$

$E_x = -F^{0,1}$, $\frac{\partial}{\partial x^0} = \partial_0$

Thus (*) reads:

$$\partial_0 F^{0,1} + \partial_2 F^{2,1} + \partial_3 F^{3,1} = \frac{4\pi}{c} J^1$$

for convenience, insert here $0 = \partial_1 F^{1,1}$ since $F^{1,1} = 0$

$\Rightarrow \partial_\alpha F^{\alpha,1} = \frac{4\pi}{c} J^1$, and more generally we get the simple covariant form of the <u>inhomogeneous</u> Maxwell's equations,

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$$

(b) homogeneous equations

1) $\nabla \cdot \vec{B} = 0$ 2) $-\nabla \times \vec{E} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$

and observe that these can be obtained from the INHOMOGENEOUS Maxwell equations by making these substitutions:

$$\boxed{\begin{aligned} \vec{E} &\longrightarrow \vec{B} \\ \vec{B} &\longrightarrow -\vec{E} \\ \rho &\longrightarrow 0 \\ \vec{J} &\longrightarrow 0 \end{aligned}}$$

same replacement used to get $\mathcal{F}^{\alpha\beta}$ from $F^{\alpha\beta}$ above

Therefore we have at once that the covariant form of Maxwell's <u>homogeneous</u> equations are

$$\boxed{\partial_\alpha \mathcal{F}^{\alpha\beta} = 0}$$

<u>Aside</u> Jackson asserts that this can also be recast as Eq. 11.143, $\boxed{\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0}$ with $\alpha, \beta, \gamma =$ any 3 <u>nonequal</u> indices of $(0,1,2,3)$

e.g. $\alpha, \beta, \gamma = 0, 1, 2$ would give

$$\partial_0 F_{1,2} + \partial_1 F_{2,0} + \partial_2 F_{0,1} \overset{?}{=} 0$$

i.e. $\frac{\partial}{\partial x^0} F_{12} + \frac{\partial}{\partial x^1} F_{20} + \frac{\partial}{\partial x^2} F_{01} \stackrel{?}{=} 0$

and $$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad (11.138)$$

$$= \frac{1}{c}\frac{\partial}{\partial t}(-B_z) + \frac{\partial}{\partial x}(-E_y) + \frac{\partial}{\partial y}E_x$$

$$= -\frac{1}{c}\frac{\partial B_z}{\partial t} - (\nabla \times \vec{E})_z$$, which checks!

Summary Covariant electromagnetism can be expressed EITHER using potentials:

$$\boxed{\begin{aligned} \Box A^\alpha &= \frac{4\pi}{c} J^\alpha \\ \partial_\alpha J^\alpha &= 0 \\ \partial_\alpha A^\alpha &= 0 \end{aligned}}$$

OR using fields, in the form

$$\boxed{\begin{aligned} \partial_\alpha F^{\alpha\beta} &= \frac{4\pi}{c} J^\beta \\ \partial_\alpha \mathcal{F}^{\alpha\beta} &= 0 \end{aligned}}$$

Relativistic dynamics

Next let's put the equations of motion for a changed particle into covariant form. In any chosen inertial reference frame the appropriate form of Newton's 2nd law is

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{u}}{c}\times\vec{B}\right)$$

note: the Lorentz force has the same form in the relativistic limit!

and recall that the relativistic momentum 4-vector is

$$p^\alpha = (p_0, \vec{p}) = m(U^0, \vec{U})$$

with $p^0 = E/c$, $U^\alpha = \gamma(c, \vec{u})$

To get a version of this $\frac{d\vec{p}}{dt} = \vec{F}$ equation having simpler transformations, it would be better to work instead with $\frac{d\vec{p}}{d\tau}$.

$\Rightarrow$ Consider $\gamma \frac{d\vec{p}}{dt} = q\left(\gamma\vec{E} + \frac{\vec{U}}{c}\times\vec{B}\right)$

and, using $d\tau = \frac{dt}{\gamma}$, we obtain

$$\frac{d\vec{p}}{d\tau} = \frac{q}{c}\left(\gamma c\vec{E} + \vec{U}\times\vec{B}\right)$$

or $$\frac{d\vec{p}}{d\tau} = \frac{q}{c}\left(U_0 \vec{E} + \vec{U} \times \vec{B}\right)$$

where, for instance, the x - component looks like

$$\frac{dp^1}{d\tau} = \frac{q}{c}\left(\underline{F^{1,0}}\, U_0 - \underline{B_y}\overset{= -U_3}{(\vec{U})_z} + \underline{B_z}\overset{= -U_2}{(\vec{U})_y}\right)$$

now recall that
$$F^{\alpha\beta} = \begin{array}{c|cccc} \alpha\backslash\beta & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & -E_x & -E_y & -E_z \\ 1 & E_x & 0 & -B_z & B_y \\ 2 & E_y & B_z & 0 & -B_x \\ 3 & E_z & -B_y & B_x & 0 \end{array}$$

or with these identifications, we see that

$$\frac{dp^1}{d\tau} = \frac{q}{c}\left(F^{10}U_0 + F^{11}U_1 + F^{12}U_2 + F^{13}U_3\right)$$

or $$\frac{dp^1}{d\tau} = \frac{q}{c} F^{1\beta} U_\beta$$, which clearly generalizes for 2, 3 spatial components too

What about the TIME component (0) of this equation?

Eq. 6,110 implies that for a single charge q moving at velocity $\vec{u}$, and if $\mathcal{E}$ = mechanical energy, then

$$\frac{d\mathcal{E}}{dt} = q\,\vec{E} \cdot \vec{u} \leftarrow$$ describes the rate of change of mechanical energy

and multiplying through by γ_u/c gives

$$\frac{dp^0}{d\tau} = \frac{q}{c}\vec{E}\cdot\vec{U} = \frac{q}{c}\sum_{i=1}^{3} F^{0i}U_i \leftarrow$$

since $F^{0i}=(-\vec{E})_i$ and $U_i=(-\vec{u})_i$

$$= \frac{q}{c}F^{0\alpha}U_\alpha \leftarrow \text{since } F^{00}=0$$

And we see that the equations of motion are the following, now in manifestly covariant form, for any charged particle in an electromagnetic field:

$$\boxed{\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta}$$

plus, of course $\longrightarrow$ $\boxed{U^\alpha = \frac{d}{d\tau}x^\alpha}$

$\Rightarrow$ These are coupled differential equations whose solution gives the position 4-vector $x^\alpha(\tau)$ as a function of proper time, assuming that the fields are known and specified versus position & time. We also would have in classical Newtonian mechanics that $F^{\alpha\beta}=F^{\alpha\beta}(x(\tau))$ should already be known, to find the trajectory of charge q.

Sec 11.10 Lorentz transformation of electromagnetic fields

The fact that these fields are characterized by an antisymmetric rank 2 4-tensor,

$$F^{\alpha\beta} = \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}$$

tells us immediately that the Lorentz transf. from a frame K to another K′ moving at velocity $\vec{v}$ w.r.t. K MUST BE:

$$F'^{\alpha\beta} = \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} F^{\gamma\delta}$$

or in matrix notation,

$$F' = \Lambda F \tilde{\Lambda}$$

where $\Lambda = \exp(-\vec{\omega}\cdot\vec{S} - \vec{\zeta}\cdot\vec{K})$

Specializing to $\vec{v} = v\hat{x}$, where

$$\Lambda^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$F = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

See, e.g. the Mathematica notebook
LorentzTransformationMatrix.nb
in the lecture notes directory for the course, to do these matrix multiplications. They yield:

$$\boxed{\begin{array}{ll} E_x' = E_x & B_x' = B_x \\ E_y' = \gamma(E_y - \beta B_z) & B_y' = \gamma(B_y + \beta E_z) \\ E_z' = \gamma(E_z + \beta B_y) & B_z' = \gamma(B_z - \beta E_y) \end{array}}$$

Note also that the inverse transformation is obtained by:

(i) interchanging primed and unprimed quantities

and (ii) setting $\beta \to -\beta$

You might verify yourself that the general fields in K', moving at an arbitrary velocity $\vec{v}$ w.r.t. K are given by

$$\boxed{\begin{array}{l} \vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \dfrac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{E}) \\ \vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \dfrac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{B}) \end{array}}$$

Comments

- $\vec{E}$ and $\vec{B}$ get "mixed-up" when you change from one Lorentz frame to another

- What appears to be an exclusively electrical phenomenon to an observer K could be viewed as being partly or even mostly magnetic to observers in K'!

- In this sense, one might argue that $F^{\alpha\beta}$ is more "fundamental" than $\vec{E}$ and $\vec{B}$ separately

A simple but important example

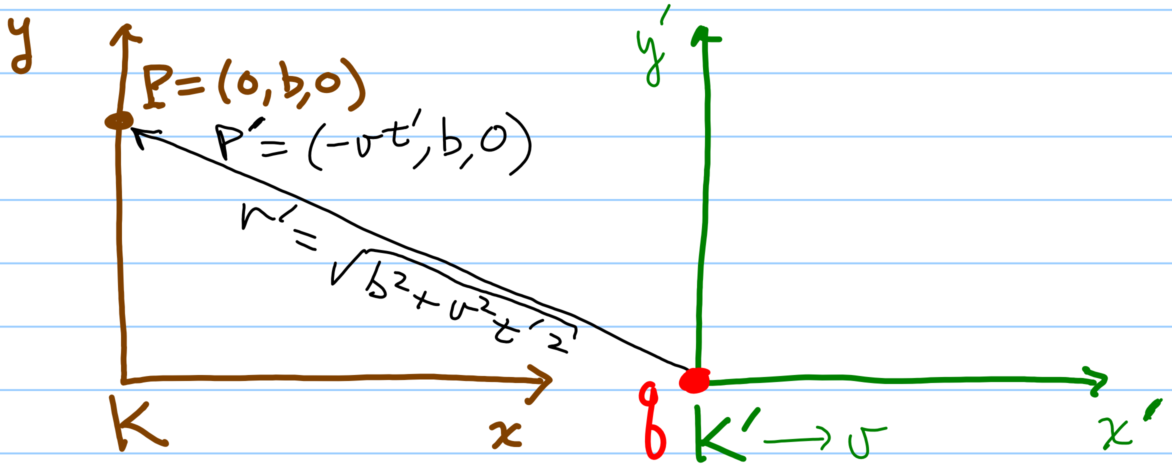
- Consider a point charge q moving at constant velocity $\vec{v} = v\,\hat{x}$ relative to an observer's frame K.

$\Rightarrow$ Find $\vec{E}$ and $\vec{B}$ seen by the observer K.

Solution call K' the rest frame of q, and let the origins of K, K' coincide at $t = t' = 0$

Let's further specify that the observer in K who measures the fields will be located at point $P = (0, b, 0)$ in K, i.e. on the y-axis a distance b from the origin.

Then in K' the observer's position at time t' is $(-vt', b, 0)$



and $t' = \gamma\left(t - \frac{vx}{c^2}\right) \to \gamma t$ here, since $x=0$ at P

Of course, in K' there is only an electric field at P', namely

$$E'_x = \frac{q}{r'^2}\left(\frac{-vt'}{r'}\right) = -\frac{qvt'}{r'^3} = \frac{-qv\gamma t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

and

$$E'_y = \frac{qb}{r'^3} = \frac{qb}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \quad , \quad E'_z = 0$$

$$(\text{and } \vec{B}' = 0)$$

To transform these into frame K, we need the <u>inverse</u> of the above transformation, namely

$$\boxed{\begin{array}{ll} E_x = E'_x & B_x = B'_x \\ E_y = \gamma(E'_y + \beta B'_z) & B_y = \gamma(B'_y - \beta E'_z) \\ E_z = \gamma(E'_z - \beta B'_y) & B_z = \gamma(B'_z + \beta E'_y) \end{array}}$$

Applying these formulas to the present example:

$$E_x = E'_x = \frac{-q\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$E_y = \gamma E'_y = \frac{\gamma q b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$B_z = \gamma \beta E'_y = \frac{\gamma \beta q b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$E_z = 0$

$B_x = 0$

$B_y = 0$

Observations

(1) To observer K at P, $\frac{E_x}{E_y} = -\frac{vt}{b}$

meaning that $\vec{E}$ points radially AWAY from the "present" position of the particle; i.e. NOT from its position at the retarded time.

(2) $B_z \neq 0$, as expected since this moving charge represents a CURRENT element to K.

i.e. recall the Biot-Savart law (Eq. 5.5 converted to Gaussian units) which says $\vec{B} = \frac{q}{c}\frac{\vec{v}\times\vec{r}}{r^3}$, and for $\vec{v} = v\hat{x}$ here,

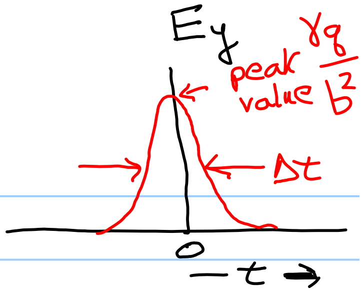
and $\vec{r} = -vt\hat{x} + b\hat{y}$ $\Rightarrow$ $\vec{B}(\text{Biot-Savart}) = \frac{\beta q b}{r^3}\hat{z}$

Equivalent to the above result at low velocities, when $\beta \ll 1$, $\gamma \simeq 1$

(3) At $t=0$, $E_y^{max} = \frac{\gamma q}{b^2} \xrightarrow[\beta\to1]{} \infty$

$\Rightarrow$ Observer K sees a pulse of E_y $\Rightarrow$ and a pulse of $B_z = \beta E_y$

or $B_z \simeq E_y$ for $\beta \simeq 1$



The duration of this pulse is of order

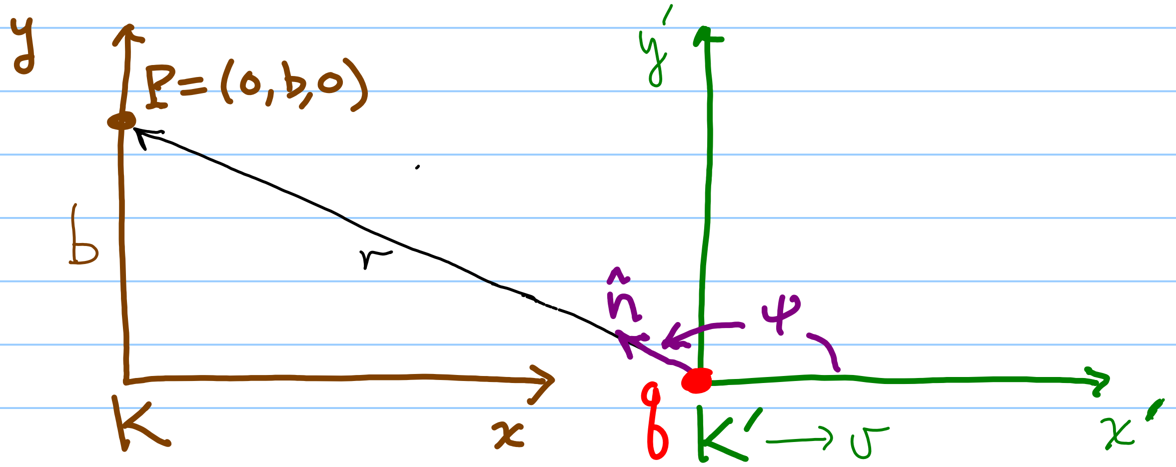
$$\Delta t \simeq \frac{b}{\gamma v} \xrightarrow[\beta\to1]{} 0$$

and the integrated pulse strength is

$$\int_{q\text{'s path}} E_y\,dx = \int E_y\,v\,dt = \int \frac{\gamma q v b\,dt}{(b^2+\gamma^2 v^2 t^2)^{3/2}} = \frac{2q}{b}$$

and this, interestingly, is independent of v!

(4) Consider Fig 11.8 of Jackson



and observe that $b = r\sin(\pi-\psi) = r\sin\psi$

$vt = r\cos(\pi-\psi) = -r\cos\psi$

and thus we can rewrite the $\vec{E}$-Field as

$$E_x = \frac{q r \cos\psi}{\gamma^2 r^3 (1-\beta^2 \sin^2\psi)^{3/2}}$$

$$E_y = \frac{q r \sin\psi}{\gamma^2 r^3 (1-\beta^2 \sin^2\psi)^{3/2}}$$

Why? $b^2+\gamma^2 v^2 t^2 = r^2\left[\sin^2\psi + \gamma^2(1-\sin^2\psi)\right]$

$$= \gamma^2 r^2 \left\{ 1 + \sin^2\psi \, (\underbrace{\gamma^{-2}-1}_{= -\beta^2}) \right\}$$

$$= \gamma^2 r^2 (1-\beta^2\sin^2\psi)$$

So we deduce a simple, compact expression for the full $\vec{E}$-vector in K (lab frame):

$$\boxed{\vec{E} = \frac{q\vec{r}}{\gamma^2 r^3 (1-\beta^2\sin^2\psi)^{3/2}}}$$

Thus: (a) $\vec{E} \xrightarrow[\beta \to 0]{} \frac{q\hat{r}}{r^2}$ as expected

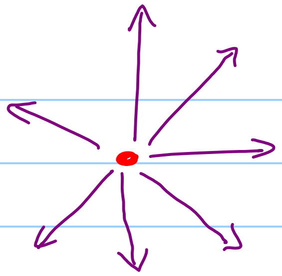
(b) $\vec{E} \propto \hat{r}$, i.e. radially outward from the particle's present position, as noted above.

(c) $\vec{E}$ is not isotropic in frame K

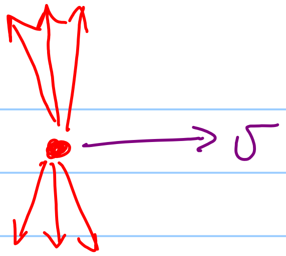
$\Rightarrow$ at any fixed r-value, $|\vec{E}|$ is maximum at $\sin^2\psi = 1 \Rightarrow \psi = \pm\frac{\pi}{2}$

while $|\vec{E}|$ is minimum at $\sin^2\psi = 0 \Rightarrow \psi = 0, \pi$

$\beta \ll 1$, isotropic



$\beta \simeq 1$, anisotropic



$\Rightarrow$ This compression of field lines in the direction of motion is somewhat analogous to a "length contraction"

Chap. 12 Relativistic dynamics

By now we know the equations of motion of a charged particle e through $\vec{E}, \vec{B}$ - fields, i.e.

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}\right), \quad \frac{d\mathcal{E}}{dt} = e\vec{u} \cdot \vec{E}$$

or their equivalent manifestly covariant form,

$$\frac{dU^{\alpha}}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_{\beta}$$

While these equations are classically "complete", it is desirable to recast them into the language of Lagrangian or Hamiltonian dynamics, anticipating eventual quantum treatments.

$\Rightarrow$ Start with the Principle of Least Action:

If a particle moves from one phase space point $a = (q_i^a, \dot{q}_i^a)$ at time t_a to another point $b = (q_i^b, \dot{q}_i^b)$ at time t_b, it will follow the trajectory that minimizes the ACTION:

$$\boxed{A = \int_{t_a}^{t_b} L[q_i(t), \dot{q}_i(t)]\, dt}$$

From this, one derives the equations of motion (the Euler-Lagrange eqns) by demanding the stationary minimum equation:

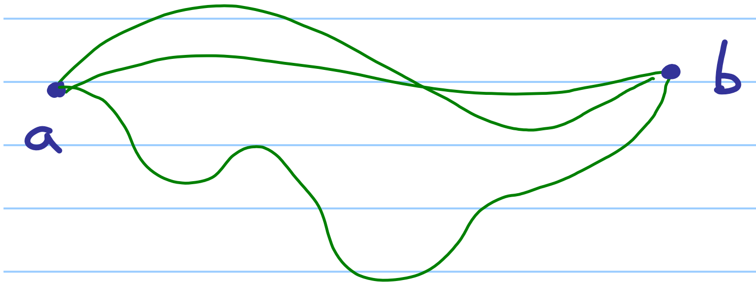
$$\delta\left(\frac{dq_i}{dt}\right) = \frac{d}{dt}\delta q_i$$

$$0 = \delta A = \sum_i \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot q_i}\delta \dot q_i \right) dt$$

integrate this term by parts

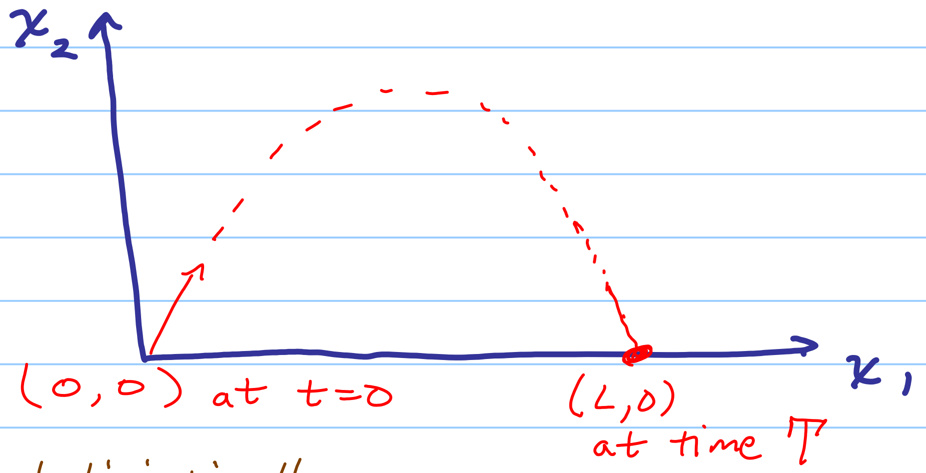
$$0 = \delta A = \sum_i \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} \right) \delta q_i \, dt + \cancel{\frac{\partial L}{\partial \dot q_i}\delta q_i}\Big|_{t_a}^{t_b}$$

vanishes if $\delta q_i = 0$ at the endpoints, i.e. if we only consider those paths that start at q_i^a at time t_a and end at q_i^b at time t_b



<u>Nonrelativistic example</u> — how Lagrangian dynamics works

⇒ consider a projectile near the Earth's surface



Recall that nonrelativistically,

$$L = T - V = \frac{m}{2}(\dot x_1^2 + \dot x_2^2) - mgx_2$$

For these initial and final points, the action is

$$A = \int_0^T \left\{ \tfrac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - m g x_2(t) \right\} dt$$

where $x_1(t)$, $x_2(t)$ are to be determined

$$\Rightarrow \delta A = 0 = \sum_{i=1}^{2} \int_0^T \left\{ \frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right\} \delta x_i \, dt$$

+ surface terms that vanish since $\delta x_i = 0$ there

$$\Rightarrow \text{(i)} \quad \cancelto{0}{\frac{\partial L}{\partial x_1}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_1} = 0 \Rightarrow \frac{d}{dt}(m \dot{x}_1) = 0$$

or $\dot{x}_1 =$ constant

$$\Rightarrow x_1(t) = \frac{L}{T} t$$

and

$$\text{(ii)} \quad \frac{\partial L}{\partial x_2} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_2} = 0 \Rightarrow -mg - \frac{d}{dt}(m \dot{x}_2) = 0$$

integration const.

$$\text{or} \quad \ddot{x}_2 = -g \Rightarrow x_2(t) = V_2 t - \tfrac{1}{2} g t^2$$

and V_2 is fixed by demanding that $x_2(T) = 0$

$$\Rightarrow V_2 T - \tfrac{1}{2} g T^2 = 0 \Rightarrow V_2 = \tfrac{1}{2} g T$$

and finally the full solution is

$$\boxed{\begin{aligned} x_1(t) &= \frac{L}{T} t \\ x_2(t) &= \tfrac{1}{2} g (tT - t^2) \end{aligned}}$$

Next, let's find the Lagrangian that gives the correct relativistic equations of motion, i.e. find L appropriate to

$$A = \int_{t_1}^{t_2} L\,dt = \int_{\tau_1}^{\tau_2} \gamma L\,d\tau$$

using $d\tau = \frac{dt}{\gamma}$ and $\gamma = \left(1 - \frac{u(t)^2}{c^2}\right)^{-1/2}$

Now, we seek extrema of A, and they must be extrema in ALL inertial frames, Therefore, we anticipate that A must be a Lorentz invariant (scalar)

$\Rightarrow$ inspecting the above A, this means that

$$\gamma L = \text{invariant}$$

Further considerations

(a) For a free particle, we expect that L should depend on speed only, not on position nor the direction of velocity.

$\Rightarrow$ The only plausible invariant constructed from velocity, recalling that $U^\alpha = \gamma(c, \vec{u})$, is

$$U_\alpha U^\alpha = \gamma^2 c^2 - \gamma^2 u^2 = c^2$$

This invariant has dimensions of energy if we multiply by mass m of the particle.

This suggests that L for a free particle might be chosen to be

$$L = -\frac{mc^2}{\gamma_u} = -mc^2\sqrt{1-\frac{u^2}{c^2}}$$

We must check to see whether the Euler-Lagrange equations make sense:

$$\cancelto{0}{\frac{\partial L}{\partial x_i}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial u_i} = 0$$

and $\frac{\partial L}{\partial u_i} = -mc^2 \frac{\left(\frac{1}{2}\right)\left(-\frac{2u_i}{c^2}\right)}{\left(1-\frac{u^2}{c^2}\right)^{1/2}} = \underbrace{\gamma_u m u_i}$

ith component of the relativistic momentum

$\Rightarrow \frac{d}{dt}\vec{p} = 0$, or $\vec{p}$ = constant is the solution, as expected

So we conclude that an appropriate relativistic free-particle Lagrangian is

$$\boxed{L_{free} = -mc^2\sqrt{1-\frac{u^2}{c^2}}}$$

which does satisfy γL_{free} = Lorentz invariant.

(b) Now consider a particle interacting with external $\vec{E}, \vec{B}$-fields

We know that in the low velocity limit the interaction energy is mainly electrostatic, i.e.

$$V_{int} = q\Phi \longrightarrow \frac{1}{c}\int c\rho\,\Phi\,d^3x$$

Then, since the nonrelativistic Lagrangian is $L = T - V$, we expect $L_{int} \xrightarrow[u\to 0]{} -q\Phi$

Thus the natural generalization to finite velocity is

$$\gamma V_{int} = \frac{q}{c}U^\alpha A_\alpha$$

$$\Rightarrow \quad \gamma L \longrightarrow \gamma L_{free} - \frac{q}{c}U^\alpha A_\alpha$$

or finally

$$\boxed{L = -mc^2\sqrt{1-\frac{u^2}{c^2}} - \frac{q}{c}\left(c\Phi - \vec{u}\cdot\vec{A}\right)}$$

This is completely correct, and γL is a Lorentz invariant, even though it is not manifestly covariant

And the Euler-Lagrange equations do give the correct equations of motion, as is readily verified: $\frac{d}{dt}\frac{\partial L}{\partial u_i} = \frac{\partial L}{\partial x_i}$

$$\Rightarrow \frac{d}{dt}\left(\gamma m u_i + \frac{q}{c} A_i\right) = -q\frac{\partial \Phi}{\partial x_i} + \frac{q}{c}\sum_j u_j \frac{\partial A_j}{\partial x_i}$$

Now recall that $\vec{A} = \vec{A}(\vec{x}(t), t)$

$$\Rightarrow \quad \frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + (\vec{u}\cdot\nabla)A_i$$

$$\Rightarrow \frac{d}{dt}(\gamma m u_i) = q\left[(-\nabla\Phi)_i - \frac{1}{c}\frac{\partial A_i}{\partial t}\right]$$

$$+ \frac{q}{c}\sum_j\left(u_j\frac{\partial A_j}{\partial x_i} - u_j\frac{\partial A_i}{\partial x_j}\right)$$

and $\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

Moreover $\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = \sum_k \epsilon_{ijk} B_k$

whereby $\sum_j u_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) = \sum_{jk}\epsilon_{ijk} u_j B_k = (\vec{u}\times\vec{B})_i$

and this gives finally the expected result,

$$\boxed{\frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{u}) = q\left(\vec{E} + \frac{\vec{u}}{c}\times\vec{B}\right)}$$

Next, with an eye toward quantum physics, the Hamiltonian is constructed

First of all, the momentum p_i conjugate to any generalized coordinate q_i is defined to be

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial u_i} = \gamma m u_i + \frac{q}{c} A_i$$

Then

$$H = \sum_i p_i \dot{q}_i - L = \vec{p}\cdot\vec{u} - L$$

and to use this H in either classical or QM, we should express H only in terms of p_i, q_i and eliminate $u_i = \dot{q}_i$, i.e. using $\boxed{\vec{p} = \gamma m \vec{u} + \frac{q}{c}\vec{A}}$

e.g. $\vec{p}^{\,2} = \gamma^2 u^2 m^2 + \frac{q^2}{c^2} A^2 + 2\frac{q}{c}\gamma\,\vec{u}\cdot\vec{A}$

or $\gamma^2 u^2 m^2 = \left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 = \dfrac{m^2 u^2}{1 - \frac{u^2}{c^2}}$

$\Rightarrow \vec{u} = \dfrac{c\left(\vec{p} - \frac{q}{c}\vec{A}\right)}{\left[\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + m^2c^2\right]^{1/2}}$, so plugging this in and simplifying gives ↙

$\Rightarrow H = \vec{p}\cdot\vec{u} - L$ equals

$$\boxed{H = \sqrt{\left(c\vec{p} - q\vec{A}\right)^2 + m^2c^4} + q\,\Phi(\vec{x})}$$

Aside: The so-called "first quantization" involves the replacement $\vec{p} \longrightarrow -i\hbar\nabla$, in QM

Sec 12.7 Lagrangian for the $\vec{E}, \vec{B}$-fields

Now let's find a Lagrangian for the fields, acknowledging that the dynamical variables $\vec{E}(\vec{x}, t)$, $\vec{B}(\vec{x}, t)$ are continuously infinite. Here $\vec{x}, t$ are the independent variables which the dependent variables $\vec{E}, \vec{B}$ depend on.

The following correspondences can be established with the case of particle motion:

Particles, $q_i(t)$		Fields or Potentials, $\phi_k(x)$
index i	→	index k
independent variable, t	→	variables, x^α
$q_i(t)$	→	$\phi_k(x)$
$\dot{q}_i(t)$	→	$\partial^\alpha \phi_k(x)$
$L = \sum_i L(q_i, \dot{q}_i)$	→	$\mathcal{L}$ = Lagrangian density $\int \mathcal{L}(\phi_k, \partial^\alpha \phi_k) d^3x$
action $A = \int L\, dt$	→	$A = \iint \mathcal{L}\, d^3x\, dt = \int \mathcal{L}\, d^4x$

i.e. For the EM field, the "positions" are the fields or potentials A^α, while the role of "velocities" are $\partial^\beta A^\alpha$

The Euler - Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \longrightarrow \partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta \phi_k)} = \frac{\partial \mathcal{L}}{\partial \phi_k}$$

What are the desired properties of the action, A?

$\Rightarrow$ As before, we desire that A should be a Lorentz invariant

$\Rightarrow$ $\mathcal{L}$ = Lorentz scalar, because d^4x = invariant

How to construct $\mathcal{L}_{EM}$?

$\Rightarrow$ By analogy with the case for nonrelativistic particles, we expect that $\mathcal{L}_{EM}$ might be $\underset{\sim\sim\sim}{\text{quadratic}}$ in the "velocities", i.e. in $\partial^\beta A^\alpha$ or $F^{\alpha\beta}$

Recalling that $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

and $\mathcal{F} = \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$

Therefore, two plausible candidates are

$$\underbrace{F_{\alpha\beta}F^{\alpha\beta}}_{\text{invariant under inversion}} \quad \text{or} \quad \underbrace{F_{\alpha\beta}\mathcal{F}^{\alpha\beta}}_{\substack{\text{changes sign under inversion} \\ \Rightarrow \text{pseudoscalar}}}$$

This suggests that we should try

$$\mathcal{L}^{\text{Free}}_{EM} \propto F_{\alpha\beta} F^{\alpha\beta}$$

For interactions with charges and currents, recall that we found earlier

$$\gamma L_{int} = -\frac{q}{c} U_\alpha A^\alpha$$

and this suggests that for continuous changes + currents,

$$\mathcal{L}_{int} \propto -\frac{J_\alpha A^\alpha}{c}$$

In fact, when this is "tried out", the Lagrangian density that gives the correct "equations of motion" = Maxwell's equations are found to be

$$\boxed{\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha}$$

Next let's verify that this works:

(1) $\dfrac{\partial \mathcal{L}}{\partial A^\beta} = -\dfrac{J_\beta}{c}$ is one partial derivative needed

(2) To simplify the algebra, call $K \equiv -\dfrac{1}{16\pi}$.

The next derivative we need is

$$\partial^\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial^\alpha A^\beta)}\right] = K\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial(\partial^\alpha A^\beta)}\left(F_{\gamma\delta}F^{\gamma\delta}\right)$$

$$= K\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial(\partial^\alpha A^\beta)}\left\{(\partial^\gamma A^\delta - \partial^\delta A^\gamma)\, g_{\gamma\eta}\, g_{\delta\mu}\,(\partial^\eta A^\mu - \partial^\mu A^\eta)\right\}$$

... some details follow... Observe that

$$\frac{\partial}{\partial(\partial^\alpha A^\beta)}\left(\partial^\gamma A^\delta - \partial^\delta A^\gamma\right) = \delta^\gamma{}_\alpha\,\delta^\delta{}_\beta - \delta^\delta{}_\alpha\,\delta^\gamma{}_\beta$$

Kronecker δ-functions

$$\Rightarrow \frac{\partial\mathcal{L}}{\partial(\partial^\alpha A^\beta)} = K\Big[(\delta^\gamma{}_\alpha\,\delta^\delta{}_\beta - \delta^\delta{}_\alpha\,\delta^\gamma{}_\beta)\, g_{\gamma\eta}\, g_{\delta\mu}\,(\partial^\eta A^\mu - \partial^\mu A^\eta)$$
$$+ (\partial^\gamma A^\delta - \partial^\delta A^\gamma)\, g_{\gamma\eta}\, g_{\delta\mu}\,(\delta^\eta{}_\alpha\,\delta^\mu{}_\beta - \delta^\mu{}_\alpha\,\delta^\eta{}_\beta)\Big]$$

$$= K\Big[(g_{\alpha\eta}\,g_{\beta\mu} - g_{\alpha\mu}\,g_{\beta\eta})(\partial^\eta A^\mu - \partial^\mu A^\eta)$$
$$+ (\partial^\gamma A^\delta - \partial^\delta A^\gamma)(g_{\gamma\alpha}\,g_{\delta\beta} - g_{\delta\alpha}\,g_{\gamma\beta})\Big]$$

and since $g_{\alpha\eta} = g_{\alpha\alpha}\,\delta_{\alpha\eta}$ (no sum on α here)

we have $g_{\alpha\eta}\partial^\eta = g_{\alpha\alpha}\partial^\alpha$ "

$$\text{So } \frac{\partial\mathcal{L}}{\partial(\partial^\alpha A^\beta)} = K\, g_{\alpha\alpha}\, g_{\beta\beta}\Big[(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - (\partial^\beta A^\alpha - \partial^\alpha A^\beta)$$
$$+ (\partial^\alpha A^\beta - \partial^\beta A^\alpha) - (\partial^\beta A^\alpha - \partial^\alpha A^\beta)\Big]$$

no sum on α,β

$$= 4K\, g_{\alpha\alpha}\, g_{\beta\beta}\, F^{\alpha\beta} \quad (\text{no sum on } \alpha, \beta)$$

$$= 4K\, g_{\alpha\gamma}\, g_{\beta\delta}\, F^{\gamma\delta} = 4K\, F_{\alpha\beta}$$

Accordingly, the Euler-Lagrange equations are

$$4K\,\partial^\alpha F_{\alpha\beta} = -J_\beta/c\,, \quad \text{with } K = \frac{-1}{16\pi}$$

or finally,

$$\boxed{\partial^\alpha F_{\alpha\beta} = \frac{4\pi}{c} J_\beta}$$ ← agrees with Eq. 11.141

These give only the INHOMOGENEOUS Maxwell's equations. However, the 2 HOMOGENEOUS equations have been automatically satisfied, by construction, i.e. one can verify that

$$\boxed{\partial^\alpha \mathcal{F}_{\alpha\beta} = 0}$$

This is enforced, in effect, because the field tensor was originally defined using potentials, and those were initially created to satisfy

$$\vec{B} = \nabla\times\vec{A} \Rightarrow \nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0 \checkmark$$

and since $\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

$$\Rightarrow \nabla\times\vec{E} = \cancelto{0}{\nabla\times(-\nabla\Phi)} - \frac{1}{c}\frac{\partial}{\partial t}\nabla\times\vec{A}$$

$$= -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\,, \text{ automatically } \checkmark$$

Note also that change conservation or continuity holds here, since $\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c} J_\alpha$

used antisymmetry of $F_{\alpha\beta}$

$$\Rightarrow \partial^\alpha\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c}\partial^\alpha J_\alpha = -\partial^\alpha\partial^\beta F_{\alpha\beta} = -\partial^\beta\partial^\alpha F_{\beta\alpha}$$

or $\boxed{\partial^\alpha J_\alpha = 0}$ ✓

Sec. 12.11 Invariant Green Functions

Recall that (the inhomogeneous) Maxwell's equations are

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

and in terms of the 4-potential A^α,

$$\partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = \Box A^\beta - \partial^\beta(\partial_\alpha A^\alpha)$$

so if we adopt the Lorentz gauge, $\partial_\alpha A^\alpha = 0$,

$$\Rightarrow \boxed{\Box A^\beta = \frac{4\pi}{c} J^\beta}$$

This inhomogeneous wave equation can be solved formally using a Green's function satisfying

$$\Box_x D(x,x') = \delta^{(4)}(x-x') = \delta(x_0 - x_0')\delta^{(3)}(\vec{x}-\vec{x}')$$

and if there are no boundary surfaces, D can only depend on $z^\alpha = x^\alpha - x'^\alpha$

$$\Rightarrow \Box_z D(z) = \delta^{(4)}(z)$$

In fact we already solved this for the retarded (+) and advanced (−) Green's function last semester (refer to pp 190 – 198, Fall 2011 notes). The solution obtained there, in the present notation, and with $R \equiv |\vec{x}-\vec{x}'|$, is:

$$D^{(\pm)}(z) = \frac{1}{4\pi R}\,\delta(x_0 - x_0' \mp R) \qquad \left(\begin{matrix}\text{Jackson}\\ \text{Eq. } 12.132\end{matrix}\right)$$

and if we like, we can add a factor $\theta(x_0 - x_0')$ to select the forward light cone with observation times AFTER$^{(+)}$ the source time or $\theta(x_0' - x_0)$ to select the backward light cone for the advanced GF which has observation times BEFORE$^{(-)}$ the source time

The text shows on p. 614 how to put these GFs into covariant form, e.g. using the identity $\delta[(x-x')^2] = \delta[(x_0-x_0')^2 - |\vec{x}-\vec{x}'|^2]$

$$= \delta[(x_0 - x_0' - R)(x_0 - x_0' + R)]$$

$$= \frac{1}{2R}\delta(x_0 - x_0' - R) + \frac{1}{2R}\delta(x_0 - x_0' + R)$$

and the covariant forms of these two GFs can then be expressed as

$$D_r(x-x') = \frac{1}{2\pi}\theta(x_0 - x_0')\,\delta[(x-x')^2]$$

and

$$D_a(x-x') = \frac{1}{2\pi}\theta(x_0' - x_0)\,\delta[(x-x')^2]$$

This θ-function <u>appears</u> to be noninvariant, but it is in fact when constrained by the δ-function i.e. θ here just selects the forward (backward) light cone for retarded (advanced) waves.

Then, e.g. as usual, the solution is

$$\boxed{A^\alpha(x) = A^\alpha_{inc}(x) + \frac{4\pi}{c}\int d^4x'\, D_r(x-x')\, J_\alpha(x')}$$

Aside: If it is desirable to make D_r manifestly invariant, we can rewrite $\theta(z_0)$, e.g. by introducing a 4-vector

$$\eta^\alpha = (1, 0, 0, 0) = (1, \vec{0}) \text{ in our frame}$$

$$\Rightarrow z^0 = z^\alpha \eta_\alpha \text{ is an invariant in any frame}$$

$$\Rightarrow \boxed{D_r(z) = \frac{\theta(z^\alpha \eta_\alpha)\,\delta(z^\beta z_\beta)}{2\pi}}$$

Later we will need the **covariant current** of a moving point charge. In an inertial frame K, call $\vec{r}(t)$ = position of the point charge q

$$\Rightarrow \rho(\vec{x}, t) = q\,\delta[\vec{x} - \vec{r}(t)]$$

and $\vec{J}(\vec{x}, t) = q\,\vec{v}(t)\,\delta[\vec{x} - \vec{r}(t)]$

where $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$ = charge's velocity at time t as measured in K

The appropriate contravariant 4-vector is

$$\boxed{\begin{aligned} &J^\alpha = qc \int d\tau\, U^\alpha(\tau)\,\delta^{(4)}[x - r(\tau)] \\ &\text{where } r(\tau) \equiv (ct, \vec{r}(t)) \text{ in frame K} \\ &\text{and } U^\alpha(\tau) = \gamma(c, \vec{v}(t)); \end{aligned}}$$

The source proper time τ is connected to frame K's clock through $d\tau = \gamma^{-1}_{v(t)}\,dt$, i.e. $\tau = \int dt\, \gamma^{-1}_{v(t)}$

and $J^\alpha(x) = qc \int dt'\, \gamma_v^{-1}\, \gamma_v (c, \vec{v}(t'))\,\delta(ct - ct')\,\delta(\vec{x} - \vec{r}(t'))$

which checks!

Chapter 14 Radiation by moving charges

Now let $r^\alpha(\tau)$ = spacetime position 4-vector of charge q, as a function of its proper time τ. In a specific inertial frame K it looks like

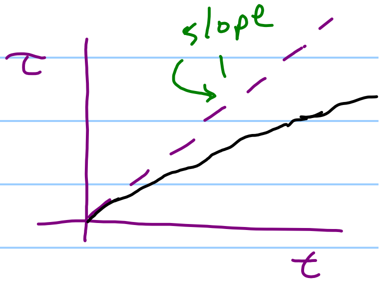
$$r^\alpha(\tau) = (ct(\tau), \vec{r}(\tau))$$

and $\tau(t) = \int_0^t \gamma^{-1}_{v(t)}\, dt$, i.e. $d\tau = \gamma^{-1}_{v(t)}\, dt$

$$\gamma^{-1}_{v(t)} = \sqrt{1 - \frac{v(t)^2}{c^2}}$$

So we can also write

$$\vec{V}(\tau) = \frac{1}{\gamma(\tau)} \frac{d\vec{r}(\tau)}{d\tau}$$



Then the 4-velocity of q is

$$U^\alpha(\tau) = \frac{d}{d\tau} r^\alpha(\tau) = \gamma_{\vec{v}(\tau)}(c, \vec{V}(\tau))$$

and the EM potentials generated by this charged particle motion are found using

$$\boxed{A^\alpha(x) = \frac{4\pi}{c} \int d^4x'\, D_r(x-x')\, J^\alpha(x')}$$

and this assumes: NO incident fields

NO boundary surfaces.

Therefore, plugging in $D_r(x-x') = \frac{\theta(x_0 - x_0')}{2\pi} \delta[(x-x')^2]$

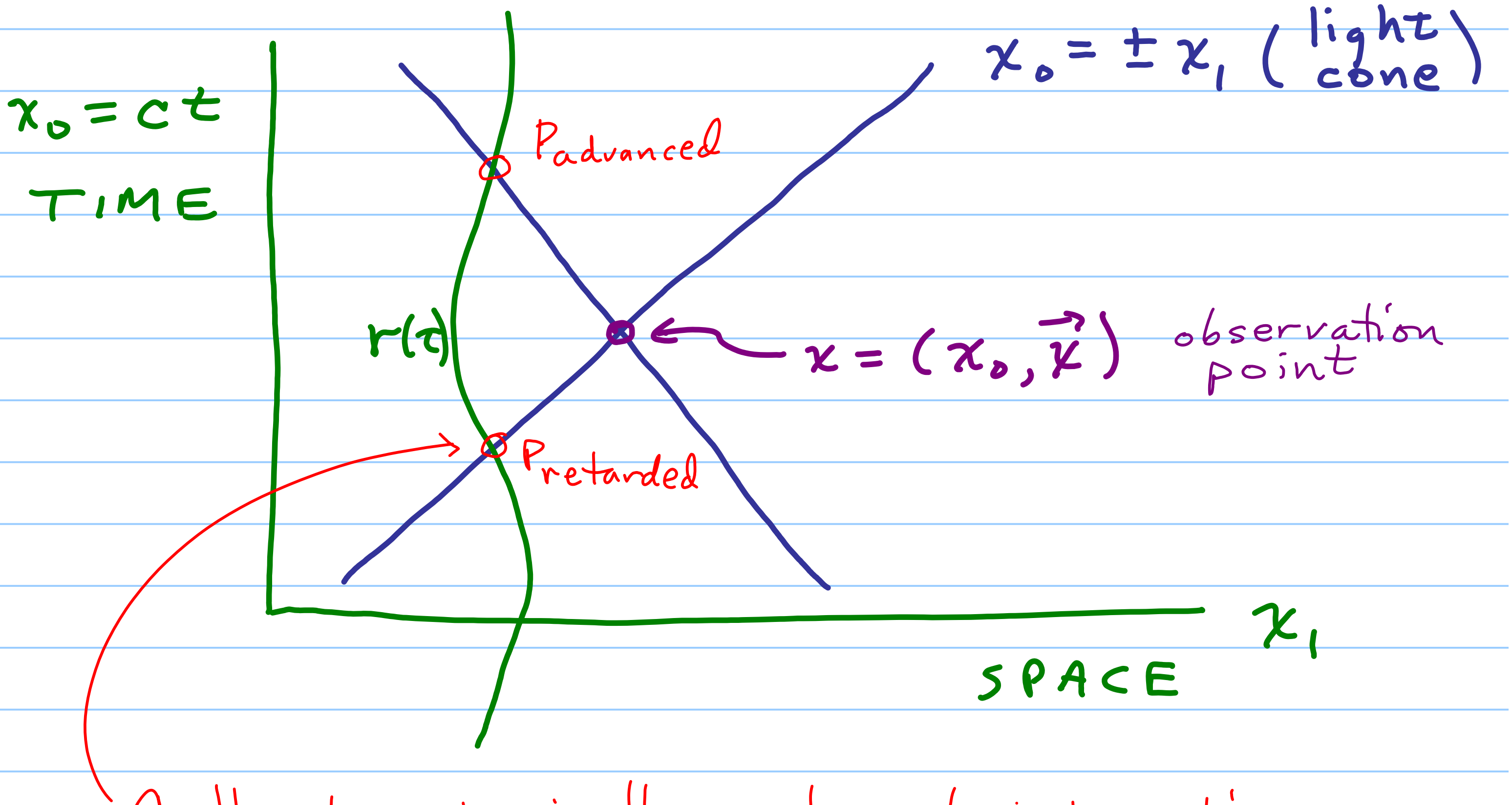
and

$$J^\alpha(x') = qc \int d\tau\, U^\alpha(\tau)\, \delta^{(4)}(x - r(\tau))$$

$$\Rightarrow A^\alpha(x) = 2q \int d\tau\, U^\alpha(\tau) \underbrace{\int d^4x'\, \theta(x_0 - x_0')\, \delta[(x-x')^2]\, \delta^{(4)}[x' - r(\tau)]}_{\text{Let's evaluate this integral first}}$$

$$\Rightarrow A^\alpha(x) = 2q \int d\tau\, U^\alpha(\tau)\, \theta(x_0 - r_0(\tau))\, \delta[(x-r(\tau))^2]$$

To see the big picture, consider the particle trajectory as viewed with respect to the observation point:



Call the physically-relevant intersection with the light cone, $\underline{r(\tau_0)}$, i.e. where the light cone intersects the particle's trajectory in the <u>PAST</u>.

Mathematical solution for the root:

$$[x - r(\tau)]^2 = (x_0 - r_0(\tau))^2 - |\vec{x} - \vec{r}(\tau)|^2$$

$$= 0 \quad \text{when} \quad x_0 - r_0(\tau_0) = \pm |\vec{x} - \vec{r}(\tau_0)|$$

only the '+' sign contributes because of the causality θ-function

Next use $$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df}{dx}\right|_{x=x_i}}$$ ← summed over all points where $f(x_i)=0$

Hence $$\delta[(x-r(\tau))^2] = \frac{\delta(\tau-\tau_0)}{\left|\frac{d}{d\tau}[x-r(\tau)]^2\right|_{\tau=\tau_0}}$$

where

$$\frac{d}{d\tau}[x-r(\tau)]^2 = \frac{d}{d\tau}\left\{(x^\alpha - r^\alpha(\tau))\, g_{\alpha\beta}(x^\beta - r^\beta(\tau))\right\}_{\tau_0}$$

$$= -2\frac{dr^\alpha(\tau)}{d\tau}(x_\alpha - r_\alpha(\tau))\Big|_{\tau_0}$$

So putting this all together into

$$A^\alpha(x) = 2q\int d\tau\, U^\alpha(\tau)\,\theta(x_0-r_0(\tau))\,\delta[(x-r(\tau))^2]$$

gives

$$\boxed{A^\alpha(x) = \frac{q\,U^\alpha(\tau)}{\left|U(\tau)\cdot(x-r(\tau))\right|_{\tau=\tau_0}}}$$

where τ_0 is the causality-appropriate root of $[x-r(\tau_0)]^2=0$, i.e. obeying $x_0 > r_0(\tau_0)$

Note that this denominator will be positive definite even without absolute value signs. (Convince yourself why!)

This formula for $A^\alpha(x)$ is called the Lienard-Wiechert potential.

Next we investigate these potentials and fields

First of all, notice that

$$U(\tau_0)\cdot\left(x-r(\tau_0)\right) = \overset{\gamma c}{U_0}(\tau_0)\overbrace{\left(x_0-r_0(\tau_0)\right)}^{c(t_{obs}-t_{ret})=R} - \vec{U}(\tau_0)\cdot\left(\vec{x}-\vec{r}(\tau_0)\right)$$

and the light cone constraint reads:

$$x_0 - r_0(\tau_0) = |\vec{x}-\vec{r}(\tau_0)| \equiv R$$

source - observer distance at the retarded time

The following notation is helpful:

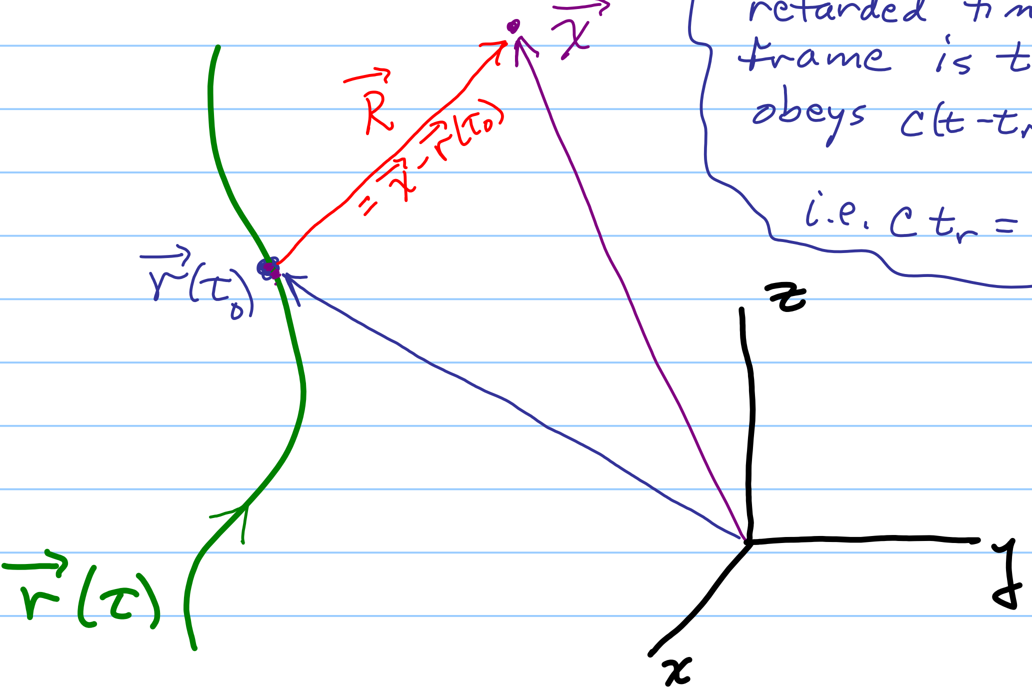
$$\hat{n} = \frac{\vec{R}}{R} = \frac{\vec{x}-\vec{r}(\tau_0)}{|\vec{x}-\vec{r}(\tau_0)|}$$

$$\Rightarrow U\cdot\left(x-r(\tau)\right) = \gamma c R - \gamma\vec{u}(\tau_0)\cdot\hat{n}R$$

$$= \gamma c R\left(1-\vec{\beta}\cdot\hat{n}\right)$$

where $\vec{\beta} \equiv \frac{\vec{u}(\tau_0)}{c}$

To reiterate, the retarded time in OUR frame is t_r, which obeys $c(t-t_r) = |\vec{x}-\vec{r}(\tau_0)|$
$= R$
i.e. $ct_r = r_0(\tau_0)$



And from the above derivation we have

$$A^\alpha(x) = q\,\frac{\gamma(c,\vec{u})}{\gamma c R(1-\vec{\beta}\cdot\hat{n})}$$

RHS evaluated at the retarded time

or

$$\boxed{\Phi(\vec{x},t) = \left[\frac{q}{(1-\vec{\beta}\cdot\hat{n})R}\right]_{ret}, \qquad \vec{A}(\vec{x},t) = \left[\frac{q\vec{\beta}}{(1-\vec{\beta}\cdot\hat{n})R}\right]_{ret}}$$

Now, the electromagnetic fields can be directly calculated from these $\Phi, \vec{A}$, but it turns out to be simpler to use the integral expression (14.3).

$$\Rightarrow A^\beta(x) = 2q\int d\tau\, U^\beta(\tau)\,\theta(x_0 - r_0(\tau))\,\delta[(x-r(\tau))^2]$$

and of course we need $\partial^\alpha A^\beta$ to determine $F^{\alpha\beta}$

$$\Rightarrow \partial^\alpha A^\beta = 2q\int d\tau\, U^\beta(\tau)\,\partial^\alpha\delta[(x-r(\tau))^2]\,\theta(x_0-r_0(\tau))$$

+ (terms involving $\partial^\alpha\theta$, involving $\delta(R)$ which do not survive at $R\neq 0$)

Recall also that ∂^α acts only on x, and abbreviate $\partial^\alpha\delta[(x-r(\tau))^2] \equiv \partial^\alpha\delta(f)$

where $f = (x-r(\tau))^2$

$$\Rightarrow \partial^\alpha\delta(f) = (\partial^\alpha f)\,\frac{d}{df}\delta(f) = (\partial^\alpha f)\,\frac{d\tau}{df}\,\frac{d}{d\tau}\delta(f)$$

And, moreover,

$$(i)\ \partial^\alpha f = \partial^\alpha (x-r(\tau))^2 = 2\,(x^\alpha - r^\alpha(\tau))$$

$$(ii)\ \frac{df}{d\tau} = -2\,(x-r(\tau))\cdot U(\tau)$$

So

$$\partial^\alpha \delta[(x-r(\tau))^2] = \frac{-(x^\alpha - r^\alpha(\tau))}{U(\tau)\cdot(x-r(\tau))}\,\frac{d}{d\tau}\,\delta[(x-r(\tau))^2]$$

So if we plug this into the above integral for $\partial^\alpha A^\beta$ and integrate by parts, this gives

$$\partial^\alpha A^\beta = 2q\int d\tau\ \theta(x_0 - r_0(\tau))\,\delta[(x-r(\tau))^2] \times \frac{d}{d\tau}\left[\frac{(x^\alpha - r^\alpha(\tau))\,U^\beta(\tau)}{(x-r(\tau))\cdot U(\tau)}\right]$$

+ surface terms that vanish because they are not on the light cone at $\tau \to \pm\infty$

Now, let's again use $\delta[(x-r(\tau))^2] = \dfrac{\delta(\tau-\tau_0)}{2\,U(\tau)\cdot(x-r(\tau))}$

Then,

$$F^{\alpha\beta}(x) = \frac{q}{(x-r(\tau_0))\cdot U(\tau_0)}\,\frac{d}{d\tau_0}\left\{\frac{(x-r(\tau_0))^\alpha U^\beta(\tau_0) - (x-r(\tau_0))^\beta U^\alpha(\tau_0)}{(x-r(\tau_0))\cdot U(\tau_0)}\right\}$$

and continue to adhere to the conditions

$(x-r(\tau_0))^2 = 0$ (light cone)

and $x_0 > r_0(\tau_0)$ (causality)

Next, determine the fields:

1) $(x - r(\tau_0))^\alpha = (R, \vec{R})^\alpha$; $\vec{R} = \vec{x} - \vec{r}(\tau_0)$; $r_0(\tau_0) = c\,t_r$, $t - t_r = \frac{R}{c}$

2) $U^\alpha(\tau_0) = \gamma(\tau_0)(c, \vec{u}(\tau_0))$

3) $\frac{d}{d\tau_0}(x - r(\tau_0))^\alpha = -\frac{dr^\alpha}{d\tau}\Big|_{\tau=\tau_0} = -U^\alpha(\tau_0)$

4) $\frac{d}{d\tau_0}U^\alpha(\tau_0) = \left(c\,\frac{d\gamma(\tau_0)}{d\tau_0}, \frac{d\gamma(\tau_0)}{d\tau_0}\vec{u}(\tau_0) + \underline{\gamma^2(\tau_0)\vec{a}(\tau_0)}\right)$

since $\frac{d\vec{u}}{d\tau} = \gamma\frac{d\vec{u}}{dt} = \gamma\vec{a}$

where $\vec{a}$ = ordinary acceleration, $\dot{\vec{u}}$

and note as well that

$$\frac{d\gamma}{d\tau} = \gamma\,\frac{d\gamma}{dt} = \gamma\left[-\frac{1}{2}\gamma^3\left(-\frac{2\,\vec{u}\cdot\vec{a}}{c^2}\right)\right] = \gamma^4\vec{\beta}\cdot\dot{\vec{\beta}}$$

so

$$\frac{dU^\alpha(\tau_0)}{d\tau_0} = \left(c\gamma^4\vec{\beta}\cdot\dot{\vec{\beta}},\ c\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})\vec{\beta} + c\gamma^2\dot{\vec{\beta}}\right)$$

5) $\frac{d}{d\tau_0}\{U(\tau_0)\cdot(x - r(\tau_0))\} = \frac{dU(\tau_0)}{d\tau_0}\cdot(x - r(\tau_0)) - U^2(\tau_0)$

c^2

again using $U^2 = \gamma^2(c^2 - u^2) = c^2$

6) $U(\tau_0)\cdot(x - r(\tau)) = \gamma c R(1 - \vec{\beta}\cdot\hat{n})$ (derived on p. 261 above)

7) $\frac{dU}{d\tau_0}\cdot(x - r(\tau_0)) = (c\gamma^4\dot{\vec{\beta}}\cdot\vec{\beta})R - [c\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})\vec{\beta} + \gamma^2 c\dot{\vec{\beta}}]\cdot R\hat{n}$

And now, somewhat tediously, put all of these pieces together:

$$F^{\alpha\beta}_{(x)} = \left(\frac{q}{\gamma c R(1-\vec{\beta}\cdot\hat{n})}\right) \times \qquad (c\gamma^4\dot{\vec{\beta}}\cdot\vec{\beta})R - [c\gamma^4(\dot{\vec{\beta}}\cdot\vec{\beta})\vec{\beta} + c\gamma^2\dot{\vec{\beta}}]\cdot R\hat{n} - c^2$$

$$\Bigg\{ \frac{-1}{[\gamma c R(1-\vec{\beta}\cdot\hat{n})]^2}\frac{d}{d\tau_0}\big[(x-r(\tau_0))\cdot U(\tau_0)\big]\Big[(R,\vec{R})^\alpha(1,\vec{\beta})^\beta - (R,\vec{R})^\beta(1,\vec{\beta})^\alpha\Big]\gamma c$$

$$+ \frac{1}{\gamma c R(1-\vec{\beta}\cdot\hat{n})}\Big[\Big[(1,\vec{\beta})^\alpha(1,\vec{\beta})^\beta - (1,\vec{\beta})^\beta(1,\vec{\beta})^\alpha\Big]\gamma^2 c^2\Big]^{i,0}$$

$$+ \frac{c\gamma^4}{\gamma c R(1-\vec{\beta}\cdot\hat{n})}\left[\begin{array}{l}(R,\vec{R})^\alpha(\vec{\beta}\cdot\dot{\vec{\beta}},\ (\vec{\beta}\cdot\dot{\vec{\beta}})\vec{\beta}+\dot{\vec{\beta}}\gamma^{-2})^\beta \\ \quad -(R,\vec{R})^\beta(\vec{\beta}\cdot\dot{\vec{\beta}},\ (\vec{\beta}\cdot\dot{\vec{\beta}})\vec{\beta}+\dot{\vec{\beta}}\gamma^{-2})^\alpha\end{array}\right]\Bigg\}$$

Next we can read off the field components, e.g. pull off the elements $E_i = F^{i,0}$.

After the final steps of algebraic simplification we get the compact form (14.14):

$$\boxed{\begin{array}{l}\vec{E}(\vec{x},t) = \left[\dfrac{q(\hat{n}-\vec{\beta})}{\gamma^2R^2(1-\vec{\beta}\cdot\hat{n})}\right]_{\text{ret}} + \dfrac{q}{c}\left[\dfrac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{R(1-\vec{\beta}\cdot\hat{n})^3}\right]_{\text{ret}} \\ \text{and} \\ \vec{B}(\vec{x},t) = [\hat{n}\times\vec{E}]_{\text{ret}}\end{array}}$$

only this is radiation falling off like $\frac{1}{R}$ at $R\to\infty$

The Griffiths text gives an alternative derivation that is direct but "brute force", in terms of an auxiliary vector $\vec{w} = c\hat{R} - \vec{u}$, namely

$$\vec{E}(\vec{x},t) = \frac{qR}{(\vec{R}\cdot\vec{w})^3}\left[\vec{w}(c^2-u^2) + \vec{R}\times(\vec{w}\times\vec{a})\right]$$

and

$$\vec{B}(\vec{x},t) = \hat{n}\times\vec{E}$$

Sec. 14.2 Larmor formula for the radiation by an accelerated charge

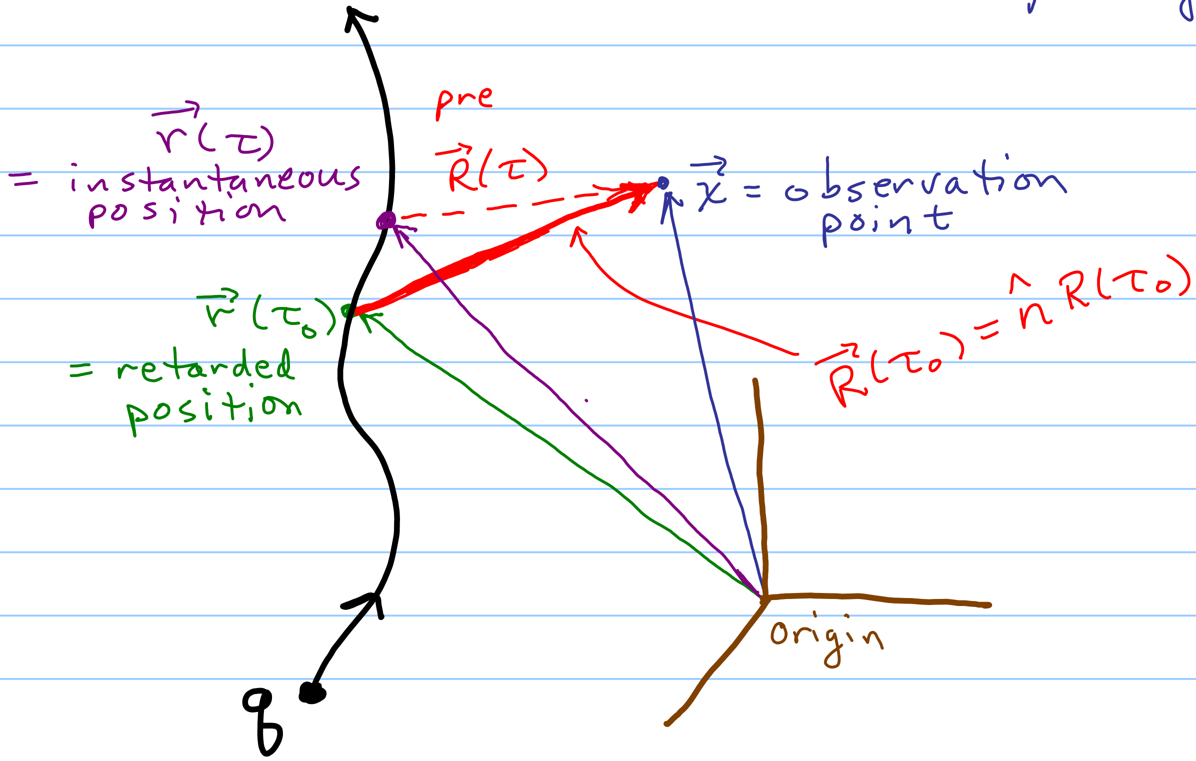
Assume now that $\dot{\vec{\beta}} \neq 0$

$\Rightarrow$ radiation occurs

Also, for now, suppose $\beta \ll 1$

$\Rightarrow$ motion is nonrelativistic in observer's frame

Then visualize the particle's trajectory as



Then in the limit $\beta \to 0$, the dominant term is

$$\vec{E}(\vec{x},t) = \frac{q}{c} \frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{R(\tau_0)}$$

$$\vec{B}(\vec{x},t) = \hat{n}\times\vec{E}(\vec{x},t)$$

Here $\vec{R}(\tau_0) = \vec{x} - \vec{r}(\tau_0) \simeq \vec{R}(\tau)$ at $\beta \ll 1$

Again, τ_0 is found from the equation

$$[x - r(\tau_0)]^2 = 0, \quad x^\alpha = (ct, \vec{x})$$

The instantaneous radiated Poynting vector is

$$\vec{S} = \frac{c}{4\pi} \vec{E}\times\vec{B}$$

since $\vec{E}, \vec{B}$ are real fields at each spacetime point x^α

$$\Rightarrow \quad \vec{S} = \frac{c}{4\pi} |\vec{E}|^2 \hat{n}$$

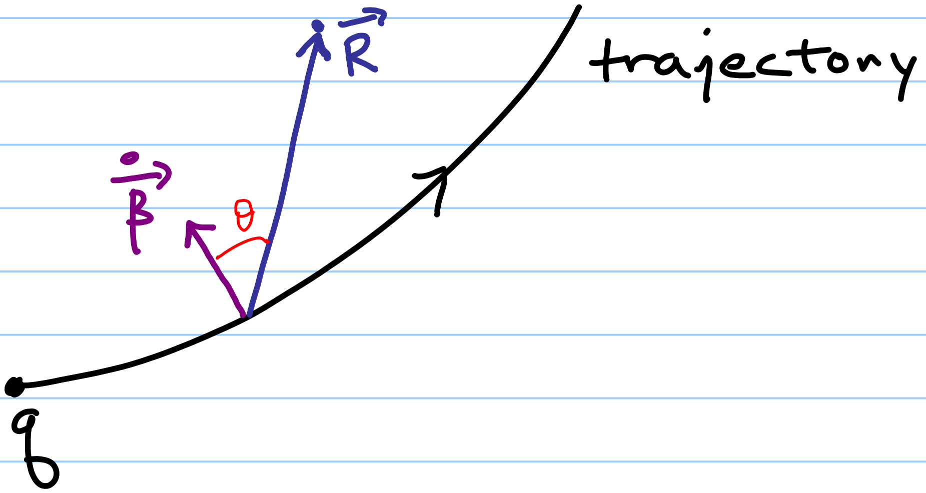
Next work out the angular distribution of radiated power. Owing to retardation, the apparent source is the position $\vec{r}(\tau_0)$.

$$\frac{dP}{d\Omega} = R(\tau_0)^2 \hat{n}\cdot\vec{S} = \frac{q^2}{4\pi c} |\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})|^2$$

and since $\vec{r}(\tau) \simeq \vec{r}(\tau_0)$, $\vec{R}(\tau) \simeq \vec{R}(\tau_0)$ in this low β limit, i.e. the apparent source of the radiation is approximately the <u>PRESENT</u> position, and so

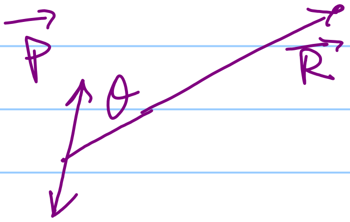
$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3}\sin^2\theta}$$

where $\vec{a} = c\dot{\vec{\beta}}$ and $\hat{R}\cdot\hat{a} = \cos\theta$



This result compares with the time-averaged power radiated by a monochromatic electric dipole, which is

$$\frac{dP^{\text{dipole}}}{d\Omega} = \frac{ck^4}{8\pi}|\vec{p}|^2\sin^2\theta$$



The total instantaneous power radiated by the accelerated nonrelativistic charge is

$$P = \int \frac{dP}{d\Omega}\,d\Omega = \frac{q^2a^2}{4\pi c^3}\,2\pi\int_{-1}^{1}(1-x^2)\,dx$$

or $$\boxed{P = \frac{2}{3}\frac{q^2a^2}{c^3}} \xrightarrow[\text{SI units}]{} \boxed{\frac{1}{4\pi\epsilon_0}\,\frac{2}{3}\,\frac{q^2a^2}{c^3}} = \text{Larmor Formula}$$

And comparing this with Jackson's Eq. 9.24 for the power radiated by a dipole,

$$P_{E1} = \frac{c^2 Z_0 k^4}{12\pi} |\vec{p}|^2 = \frac{|\vec{p}|^2 \omega^4}{4\pi\epsilon_0 (3c^3)}$$

which resembles our formula for a radiating point charge, with the identifications $p = q\ell$ for the oscillating dipole

$$z = \ell \cos\omega t$$

$$\Rightarrow v = -\omega \ell \sin\omega t$$

$$a = -\omega^2 \ell \cos\omega t$$

$$\Rightarrow \langle a^2 \rangle = \frac{1}{2}\omega^4 \ell^2$$

$$\boxed{\Rightarrow P_{E1} = \frac{2q^2}{3(4\pi\epsilon_0)} \frac{\langle a^2 \rangle}{c^3}}$$

which agrees with the accelerated charge result above.

Total power radiated

Rather than working out the total radiated power from the fields of a relativistic accelerated charge, let's follow the textbook's approach. We generalize the Larmor formula, utilizing covariance arguments to obtain a formula valid even when β is not small.

⇒ First recall that Power = $\frac{\text{energy}}{\text{time}}$

and since both dE and dt are timelike components of 4-vectors, we expect that

Power = Lorentz invariant

⇒ So it is plausible that if we are able to find a Lorentz invariant expression that reduces to the Larmor formula for $\beta << 1$, it is likely to be the correct relativistic formula for radiated power.

Therefore, let's start from the correct nonrelativistic formula in the form

$$P^{\text{nonrel}} = \frac{2}{3}\frac{q^2}{m^2c^3}\, m\dot{\vec{u}} \cdot m\dot{\vec{u}}$$

and one can guess that a relativistic expression for the power might then likely be

$$\boxed{P = -\frac{2}{3}\frac{q^2}{m^2c^3}\frac{dp^\alpha}{d\tau}\frac{dp_\alpha}{d\tau}}\ {}^{*}, \quad \text{where } d\tau = \frac{dt}{\gamma_u} \text{ as usual}$$

this has the correct nonrelativistic limit because $\frac{dp^0}{d\tau} \rightarrow \frac{d}{d\tau}(\gamma mc) \underset{\substack{\gamma\to 1\\ \beta\to 0}}{\longrightarrow} O(\beta\,\dot{\beta})$

$\left(\frac{d}{d\tau}(1-\beta^2)^{-1/2} \simeq \beta\dot{\beta}\gamma \text{ at } \beta\simeq 0\right)$

and so $-\frac{dp^\alpha}{d\tau}\frac{dp_\alpha}{d\tau} \simeq \left(\frac{d\vec{p}}{d\tau}\right)^2 - \beta^2\dot{\beta}^2\gamma^2$ at low β

whereas $\frac{d\vec{p}}{d\tau} = \gamma\frac{d}{dt}(\gamma m\vec{\beta}c) = \gamma^2 mc\dot{\vec{\beta}} + \gamma\frac{d\gamma}{dt}$

$$\simeq \gamma^2 mc\dot{\vec{\beta}} + O(\beta)$$

and so in the limit $\beta \to 0$ the leading term is $-\frac{dp^\alpha}{d\tau}\frac{dp_\alpha}{d\tau} \xrightarrow[\beta\to 0]{} m^2c^2\dot{\beta}^2$

and plugging this into (*) above does indeed give the Larmor formula at $\beta \to 0$

While the result (*) is elegant, it is more useful to work out a noncovariant expression in one specific inertial frame, starting from

$$\frac{dU^\alpha}{d\tau} = \frac{d}{d\tau} = (\gamma c, \gamma\vec{u}) = c\gamma\frac{d}{dt}(\gamma, \gamma\vec{\beta})$$

and now $\frac{d\gamma}{dt} = \gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}$ where $\dot{\vec{\beta}} = \frac{1}{c}\frac{d\vec{u}}{dt}$

$$\Rightarrow \frac{dU^\alpha}{d\tau} = c\gamma^2\left(\gamma^2\vec{\beta}\cdot\dot{\vec{\beta}},\ \dot{\vec{\beta}} + \gamma^2\vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})\right)$$

and so

$$\frac{dU^\alpha}{d\tau}\frac{dU_\alpha}{d\tau} = c^2\gamma^4\left\{\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - \dot{\vec{\beta}}^2 - \gamma^4\beta^2(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - 2\gamma^2(\vec{\beta}\cdot\dot{\vec{\beta}})^2\right\}$$

$$= c^2\gamma^4\left\{\gamma^2(\gamma^2 - \beta^2\gamma^2 - 2)(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - \dot{\vec{\beta}}^2\right\}$$

$$= -c^2\gamma^4\left[\dot{\vec{\beta}}^2 + \gamma^2(\vec{\beta}\cdot\dot{\vec{\beta}})^2\right]$$

, (using $\gamma^2 - \beta^2\gamma^2 = 1$)

Now, observing that

$$(\vec{\beta}\times\dot{\vec{\beta}})^2 = (\vec{\beta}\times\dot{\vec{\beta}})\cdot(\vec{\beta}\times\dot{\vec{\beta}}) = \vec{\beta}\cdot[\dot{\vec{\beta}}\times(\vec{\beta}\times\dot{\vec{\beta}})]$$

$$= \beta^2\dot{\beta}^2 - (\vec{\beta}\cdot\dot{\vec{\beta}})^2$$

$1+\gamma^2\beta^2=\gamma^2$

we obtain:

$$-\frac{dU^\alpha}{d\tau}\frac{dU_\alpha}{d\tau} = c^2\gamma^4\left[(\gamma^2\beta^2+1)\dot{\beta}^2 - \gamma^2(\vec{\beta}\times\dot{\vec{\beta}})^2\right]$$

So finally we have both the covariant form

$$P = -\frac{2}{3}\frac{q^2}{m^2c^3}\frac{dp^\alpha}{d\tau}\frac{dp_\alpha}{d\tau}$$

and the form in a specific chosen inertial frame,

$$\boxed{P = \frac{2}{3}q^2\gamma^6\left[\dot{\beta}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right]}$$

This is the Lienard result (1898), and Jackson Eq. 14.26

IMPORTANT – Note the γ^6 dependence!

So the radiated power increases RAPIDLY as $u\to c$!

Sec. 14.3 Angular distribution of radiation from an accelerated charge

To work this out, begin by generalizing the nonrelativistic expression

$$\frac{dP^{\text{nonrel}}}{d\Omega} = \frac{q^2}{4\pi c}\left|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})\right|^2, \text{ for } \beta \ll 1$$

In general we can use the field expressions derived earlier, i.e.

falls off as $\frac{1}{R^2}$ and is negligible in the radiation zone

$$\vec{E}(\vec{x},t) = \left[\frac{q(\hat{n}-\vec{\beta})}{\gamma^2 R^2 (1-\vec{\beta}\cdot\hat{n})}\right]_{ret} + \frac{q}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{R(1-\vec{\beta}\cdot\hat{n})^3}\right]_{ret}$$

and

$$\vec{B}(\vec{x},t) = \left[\hat{n}\times\vec{E}\right]_{ret}$$

only this is radiation falling off like $\frac{1}{R}$ at $R \to \infty$

relativistically valid

We start from $\frac{dP}{d\Omega} = R^2(\tau_0)\,\hat{n}\cdot\vec{S}$

where $\vec{R}(\tau_0) = \vec{x} - \vec{r}(\tau_0) \leftarrow$ displacement vector to the observation point $\vec{x}$ from the retarded source position of q.

$$\Rightarrow \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}\hat{n}\,|\vec{E}|^2$$

which we evaluate in the radiation zone where $\vec{B} = \hat{n}_{ret}\times\vec{E}$, with $\vec{E} = \frac{q}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3 R}\right]_{ret}$

Important point: This $\vec{S}$ gives the energy flux that is observed at a fixed detector in the observer's rest frame, as a function of the time of detection, t_{obs}.

i.e. $\frac{d^2W}{dA\,dt_{obs}} = (\vec{S}\cdot\hat{n})^{detected}$

But we are often interested in the rate of energy EMISSION of the particle rather than the rate of energy detection.

The detection or observation time t is related to the emission or retarded time t_r by

$$t_r = t - \frac{R(t_r)}{c}$$

i.e. $t = t_r + \frac{R(t_r)}{c} \implies dt = dt_r + \frac{1}{c}\frac{d}{dt_r}|\vec{x} - \vec{r}(t_r)|\, dt_r$

and $$\frac{d}{dt_r}|\vec{x} - \vec{r}(t_r)| = \frac{d}{dt_r}\left[(\vec{x} - \vec{r}(t_r))^2\right]^{1/2} = \frac{\frac{-2}{2}\frac{d\vec{r}(t_r)}{dt_r}\cdot(\vec{x} - \vec{r}(t_r))}{\left[(\vec{x} - \vec{r}(t_r))^2\right]^{1/2}}$$

$$= -c\vec{\beta}\cdot\hat{n}$$

$$\implies \quad dt = (1 - \vec{\beta}\cdot\hat{n})\, dt_r$$

or $$(\vec{S}\cdot\hat{n})^{\text{emission}} = (\vec{S}\cdot\hat{n})^{\text{detection}} \frac{dt}{dt_r}$$

i.e. $$\frac{dP^{\text{detector}}}{d\Omega} = R^2\,\hat{n}\cdot\vec{S}^{\,\text{det}} = \frac{q^2}{4\pi c}\frac{|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]|^2}{(1-\vec{\beta}\cdot\hat{n})^6}$$

whereas $$\boxed{\frac{dP^{\text{emission}}}{d\Omega} = R^2\,\hat{n}\cdot\vec{S}^{\,\text{emission}} = \frac{q^2}{4\pi c}\frac{|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]|^2}{(1-\vec{\beta}\cdot\hat{n})^5}}$$

Qualitative picture – Think of someone firing bullets from a moving car. The rate of bullets striking a stationary target is NOT the same as the rate they must be fed to the gun. This is analogous to the Doppler effect.

Next consider examples of linear + circular acceleration

Example 1 Linear motion with $\vec{\beta} \parallel \dot{\vec{\beta}}$

$$\Rightarrow \quad \frac{dP^{emission}}{d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

As expected, this reduces to Larmor's result in the limit $\beta \ll 1$. But as $\beta \to 1^-$ the angular distribution is oriented more + more along $\vec{\beta}$.

Now, the total rate of energy emission by the particle is

$$P^{emission} = \int \frac{dP^{emission}}{d\Omega} d\Omega = \frac{2}{3} \frac{q^2}{c^3} \frac{\dot{u}^2}{(1-\beta^2)^3}$$

or $$\boxed{P^{emission} = \frac{2q^2 \dot{u}^2 \gamma^6}{3c^3}}$$ ← agrees with Eq. 14.26

Ultrarelativistic limit: Set $\beta \to 1-\delta$ where $\delta << 1$ $\Rightarrow$ small θ dominates

and $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$

and $\gamma = \frac{1}{\sqrt{1-(1-\delta)^2}} \simeq \frac{1}{\sqrt{2\delta}}$

so $(1-\beta\cos\theta)^{-5} \simeq \left[1-(1-\delta)(1-\frac{1}{2}\theta^2)\right]^{-5}$

$$= \left(\delta + \tfrac{1}{2}\theta^2 - \frac{\delta\theta^2}{2}\right)^{-5} \cong \delta^{-5}\left(1+\frac{\theta^2}{2\delta}\right)^{-5}$$

and of course $\sin^2\theta \simeq \theta^2$ at $\theta << 1$

and since $\gamma = \frac{1}{\sqrt{2\delta}} \Rightarrow \delta \simeq \frac{1}{2\gamma^2}$

Thus we have in this ultrarelativistic limit,

$$\boxed{\frac{dP^{emission}}{d\Omega} \cong \frac{q^2\dot{u}^2}{4\pi c^3}\, 2^5 \frac{\gamma^{10}\ \theta^2}{(1+\gamma^2\theta^2)^5}}$$

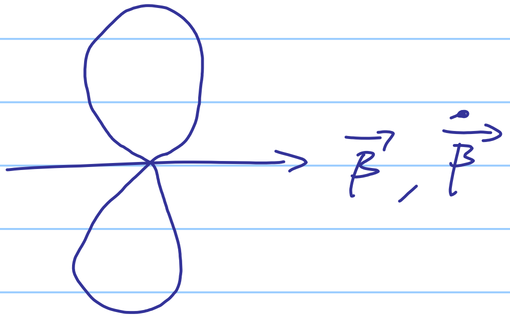
$\Rightarrow$ This function peaks at $\theta_{max} = \frac{1}{2\gamma}$

and the peak intensity approaches

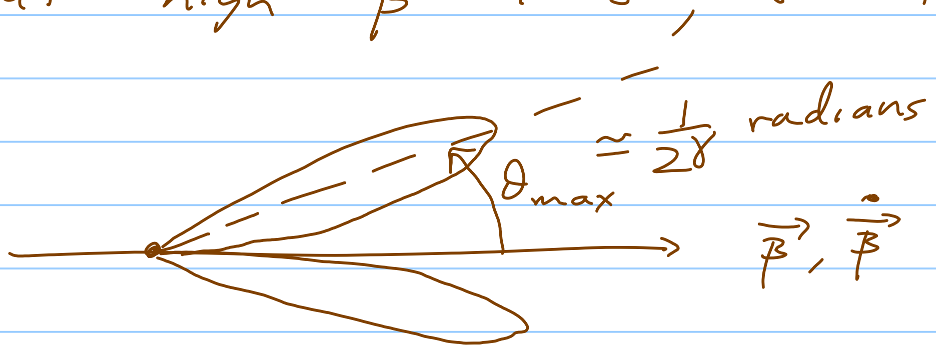
$$\left(\frac{dP^{emission}}{d\Omega}\right)_{max} \xrightarrow[\gamma>>1]{} \frac{2048}{3125}\,\frac{q^2\dot{u}^2}{\pi c^3}\,\gamma^8$$

and note that the integrated result for P^{emiss} does agree with (14.25)

Contrast in the angular distribution at low $\beta << 1$:

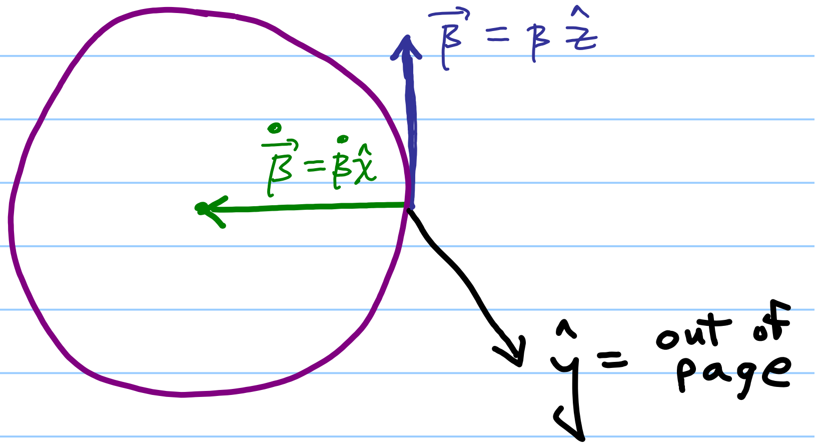


versus at high $\beta \simeq 1-\delta$, $\gamma >> 1$:



Example 2 Charge q in instantaneously circular motion

In this coordinate system depicted here → we adopt spherical polar coordinates for the observation direction $\hat{n} = (\theta, \phi)$



⇒ Now evaluate

$$\frac{dP^{emission}}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}|^2}{(1 - \vec{\beta} \cdot \hat{n})^5}$$

$$= \frac{q^2}{4\pi c} \frac{|\dot{\beta}[\hat{n}\times(\hat{n}\times\hat{x}) - \beta\hat{n}\times(\hat{z}\times\hat{x})]|^2}{(1-\beta\cos\theta)^5}$$

Now evaluate the explicit angle dependence, setting $\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$

$$\Rightarrow \hat{n}\times\hat{x} = -\sin\theta\sin\phi\,\hat{z} + \cos\theta\,\hat{y}$$

and $\hat{n}\times(\hat{n}\times\hat{x}) = (-\cos^2\theta - \sin^2\theta\sin^2\phi)\,\hat{x}$

$$+ \cos\phi\sin\phi\sin^2\theta\,\hat{y} + \cos\phi\sin\theta\cos\theta\,\hat{z}$$

and $\hat{n}\times(\hat{z}\times\hat{x}) = \hat{n}\times\hat{y} = -\cos\theta\,\hat{x} + \sin\theta\cos\phi\,\hat{z}$

So our expression simplifies to

$$\frac{dP^{emission}}{d\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\frac{1}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

Concentrate now on the limit $\gamma >> 1$

$$\Rightarrow 1 - \beta\cos\theta \simeq \frac{1}{2\gamma^2}(1+\gamma^2\theta^2)$$

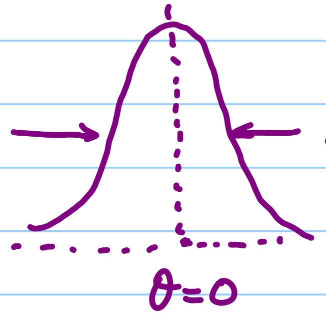
$$\Rightarrow \frac{dP^{emission}}{d\Omega} \longrightarrow \frac{q^2a^2}{4\pi c^3}\frac{8\gamma^6}{(1+\gamma^2\theta^2)^3}\left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right]$$

This peaks at $\theta \simeq 0$ with $\langle\theta\rangle_{rms} \simeq \frac{1}{\gamma}$ for $\gamma >> 1$

The total power radiated can be calculated by integrating this last expression, or else from the relativistic formula,

$$P^{emission} = \frac{2}{3}\frac{q^2\gamma^6}{c}\left\{\dot{\beta}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right\} = \frac{2q^2a^2}{3c^2}\gamma^6(1-\beta^2)$$

$$= \frac{2q^2a^2}{3c^3}\gamma^4$$



ang. distrib.

$\langle\theta\rangle_{rms} \approx \frac{1}{\gamma}$

at $\gamma << 1$

Note also that for circular motion, the rate of momentum change is

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{u}) = \gamma m\dot{\vec{u}} + m\vec{u}\cancelto{\approx 0}{\frac{d\gamma}{dt}}$$

$$\Rightarrow P^{emission}_{circ} = \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2\left(\frac{d\vec{p}}{dt}\right)^2$$

as compared to the formula for <u>linear</u> (parallel) velocity & acceleration, Eq. 14.27

$$P^{emission}_{linear} = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2$$

$$\Rightarrow P^{emiss}_{circular} = \gamma^2 P^{emiss}_{linear}$$

⇒ Circular motion produces radiated power γ^2 higher than linear acceleration, for the same applied force, $\vec{F} = \frac{d\vec{p}}{dt}$

Real-life example

The Advanced Photon Source at
Argonne National Laboratory

http://www.aps.anl.gov/

Relevant parameters of the machine:

- positrons circulate at 7 GeV
  ($mc^2 = 511$ keV)

$$\Rightarrow \gamma - 1 = \frac{7 \text{ GeV}}{511 \times 10^3} \Rightarrow \gamma \simeq 14{,}000$$

$$\Rightarrow \theta_{rms} \simeq \frac{1}{\gamma} \simeq 70\,\mu\text{radians}$$

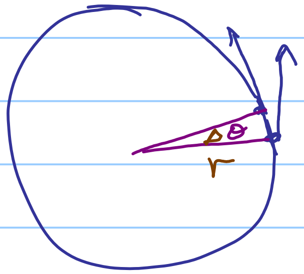
(away from insertion devices)

- beam circumference $= 768 \text{ m} = 2\pi R$

$$\Rightarrow \text{circulation frequency is} \simeq \frac{c}{2\pi R} = 390 \text{ kHz} = \nu$$

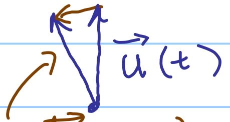
- average energy lost to radiation by 1 e^+ per second is

$$P^{\text{emission}}_{\text{circ}} = \frac{2e^2a^2}{3c^3}\gamma^4$$



$$e = (4.8 \times 10^{-10} \text{ esu})$$

$$c = 3 \times 10^{10} \text{ cm/s}$$



$$\vec{u} + d\vec{u} = \vec{u}(t+dt)$$

$$a = \left|\frac{d\vec{u}}{dt}\right| = \frac{u^2}{r} \simeq \frac{c^2}{r} \quad \text{since} \quad \frac{|\Delta\vec{u}|}{u} = \frac{|\Delta\vec{r}|}{r} \simeq \Delta\theta$$

$$\ell \simeq \frac{2\pi r}{N} ,$$

$$\text{and } |\Delta\vec{r}| = \vec{u}\,\Delta t$$

$$\Rightarrow \frac{\Delta u}{\Delta t} = a = \frac{u^2}{r}$$

$$a = \frac{\left(3\times10^{10}\,\frac{cm}{s}\right)^2}{\left(76800\,cm/2\pi\right)} = 7.4\times10^{16}\,\frac{cm}{s^2}$$

$$\Rightarrow P^{emission}_{circ} = \frac{2\left(4.8\times10^{-10}\right)^2\left(7.4\times10^{16}\,\frac{cm}{s^2}\right)^2}{3\left(3\times10^{10}\right)^2}\gamma^4 = 1.2\,\frac{ergs}{sec}$$

(1 positron)

$$1 = 6.24\times10^{5}\,\frac{MeV}{erg}\,, \qquad 1\,J = 10^{7}\,ergs$$

$\Rightarrow$ 1 positron radiates 1.9 GeV per revolution

The typical APS beam current is 300 mA $= 1.9\times10^{18}\,\frac{e^+}{sec}$

$$\Rightarrow N_{e^+} = \frac{1.9\times10^{18}}{sec} \times \frac{1}{\#\,revs/sec} = \frac{1.9\times10^{18}}{390{,}000} = 4.9\times10^{12}\,e^+$$

So when the APS is running, just to replenish the energy radiated away as photons requires

$$P^{emiss} = \left(4.9\times10^{12}\,e^+\right)\times 1.2\,\frac{ergs}{sec}\times 10^{-7}\,\frac{J}{erg} = 580{,}000\,W = 0.58\,MW$$

Note: there are ~439 nuclear power plants worldwide
- average power output = 0.85 GW per plant.

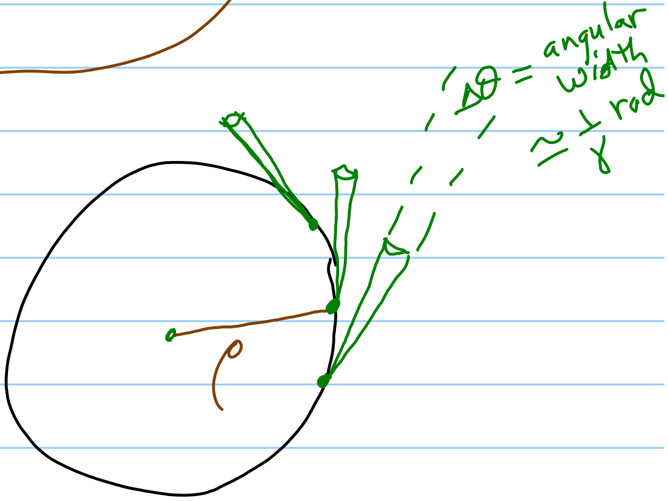
$\Rightarrow$ it only takes 0.1% of the power output of a typical nuclear power plant to run the APS!

And in 24 hours of running, APS uses about 14000 kW-hours of energy @ \$0.10/kW-hr = \$1400 worth of photons

Sec. 14.4 Frequency distribution of radiated light – qualitative analysis

Let's still consider circular motion

$$\Rightarrow \quad \boxed{\begin{aligned} a_\perp &= \omega^2 \rho = \omega u \simeq \omega c \\ u &= \omega \rho \simeq c \end{aligned}}$$



Recall that the angular width of the radiation cone is $\Delta\theta \simeq \frac{1}{\gamma}$

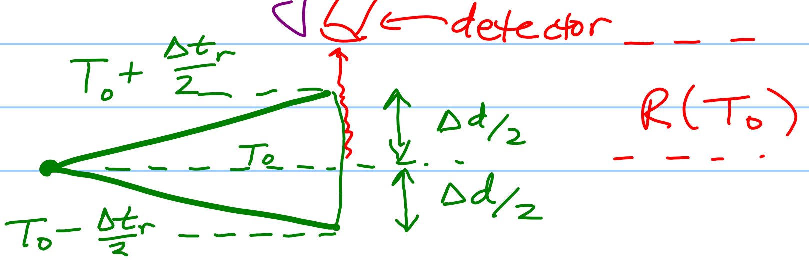
So in a time $\Delta t_{rad} \simeq \frac{\Delta\theta}{\omega} \simeq \frac{\rho}{\gamma u}$,

the distance the particle moves is approximately

$$\Delta d \simeq \rho \Delta\theta = \frac{\rho}{\gamma}$$

$\Rightarrow \quad \Delta t_{rad} \simeq \frac{\rho}{\gamma u}$ is the time duration when the radiated "searchlight" illuminates an observer

Let's expand this stretch of the particle's motion, approximating it as linear



To the observer, radiation begins to arrive at $t_1 = T_0 - \frac{\Delta t_r}{2} + \frac{R + d/2}{c}$

and it ends at

$$t_2 = T_0 + \frac{\Delta t_r}{2} + \frac{R - \frac{d}{2}}{c}$$

$$\Rightarrow \Delta t = t_2 - t_1 = \Delta t_r - \frac{d}{c} = \frac{\rho}{\gamma u} - \frac{\rho}{\gamma c}$$

Or since $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = (1-\gamma^{-2})^{1/2}$

or $\beta \simeq 1 - \frac{1}{2\gamma^2}$

$$\Rightarrow \boxed{\Delta t = \frac{\rho}{\gamma c}\left(\frac{1}{\beta} - 1\right) \simeq \frac{\rho}{2c\gamma^3}}$$

Now, from Fourier analysis, we expect that $\Delta\omega\,\Delta t \gtrsim \frac{1}{2}$, whereby $\omega_{cutoff} \simeq \Delta\omega \simeq \frac{1}{2\Delta t}$ is an estimate of the highest usable light frequency, i.e., having appreciable intensity.

$$\Rightarrow \omega_{cutoff} = \frac{c}{\rho}\gamma^3 = \frac{a_\perp \gamma^3}{c}$$

e.g. at the 7 GeV APS synchrotron, $\gamma \simeq 14{,}000$; and the bending radius is $\rho \simeq 39\,m$

$$\Rightarrow \omega_{cutoff} \simeq \frac{3\times10^8 \frac{m}{s}}{39\,m}(1.4\times10^4)^3 = 2.1\times10^{19}\,\frac{rad}{s}$$

$\Rightarrow \hbar\omega_{cutoff} \simeq 14$ keV photons

In fact, from the APS website, the photon energy at peak intensity is $\hbar\omega_{peak} \simeq 20\,\text{keV}$ although usable intensity is delivered up to even 100 keV photons, and somewhat beyond.

Sec. 14.5 Quantitative treatment of frequency and angle distributions.

Consider $\frac{dP(t)}{d\Omega} = |\vec{A}(t)|^2$ with $\vec{A}(t) = \sqrt{\frac{c}{4\pi}}\,[R\vec{E}]_{ret}$

and we write this in the OBSERVER's time since we are interested in the frequencies detected.

$$\Rightarrow \frac{\text{Total energy}}{\text{unit solid angle}} = \int_{-\infty}^{\infty} |\vec{A}(t)|^2 dt \equiv \frac{dW}{d\Omega}$$

The following Fourier representation is useful,

$$\vec{A}(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \vec{A}(t)\, e^{i\omega t}\, dt$$

and $$\vec{A}(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \vec{A}(\omega)\, e^{-i\omega t}\, d\omega$$

So, $$\frac{dW}{d\Omega} = \frac{1}{2\pi}\int dt \int d\omega \int d\omega' \, \vec{A}^*(\omega')\cdot\vec{A}(\omega)\, \underbrace{e^{i(\omega'-\omega)t}}_{t\text{-integral gives } \delta(\omega'-\omega)}$$

$$\Rightarrow \boxed{\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\vec{A}(\omega)|^2 \, d\omega} \leftarrow \text{this result is just Parseval's theorem}$$

Since $\vec{A}(t) = \text{real}$, $\Rightarrow \vec{A}(-\omega) = \vec{A}^*(\omega)$, this can be written as an integral over just POSITIVE frequencies, i.e. as

$$\frac{dW}{d\Omega} = \int_0^{\infty} \frac{d^2 I(\omega,\hat{n})}{d\omega\, d\Omega}\, d\omega$$

where $$\frac{d^2 I}{d\omega\, d\Omega} \equiv |\vec{A}(\omega)|^2 + |\vec{A}(-\omega)|^2$$
$$= 2|\vec{A}(\omega)|^2$$

And in our problem,

$$\vec{A}(\omega) = \left(\frac{q^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega t} \left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3}\right]_{\text{ret}} dt$$

and as usual $t = t_r + \dfrac{R(t_r)}{c}$

Then we can change variables to t_r using $\dfrac{dt}{dt_r} = 1 - \vec{\beta}\cdot\hat{n}$, giving

$$\vec{A}(\omega) = \left(\frac{q^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega\left(t_r + \frac{R(t_r)}{c}\right)} \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2}\, dt_r$$

and at large distances as we saw back in Chap. 9,

$$R(t_r) \simeq x - \hat{n}\cdot\vec{r}(t_r)$$

And the text points out that this integral will be simpler if we use the fact that:

$$\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2} = \frac{d}{dt_r}\left(\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right)$$

This suggests that we integrate by parts, giving

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{n}\times(\hat{n}\times\vec{\beta})\, e^{i\omega\left[t_r - \frac{\hat{n}\cdot\vec{r}(t_r)}{c}\right]}\right|^2$$

This expression generalizes to <u>many</u> changes by replacing:

$$q\vec{\beta}\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}(t_r)} \longrightarrow \sum_j q_j \vec{\beta}_j\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}_j(t_r)}$$

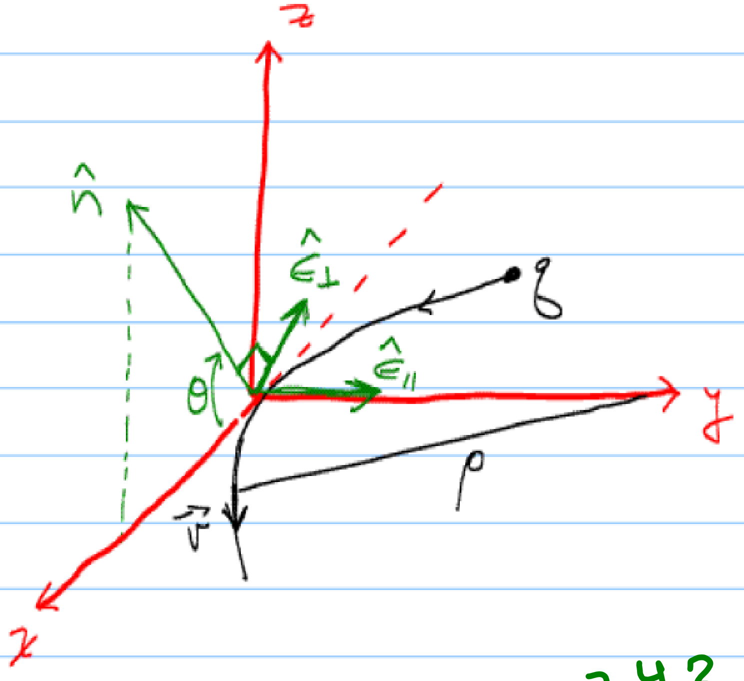
or in the case of a macroscopic current density, to

$$\longrightarrow \frac{1}{c}\int d^3x'\, \vec{J}(\vec{x}', t_r)\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{x}'}$$

Hence we have

$$\boxed{\frac{d^2 I}{d\omega\, d\Omega} = \frac{\omega^2}{4\pi^2 c}\left|\int dt' \int d^3x'\, \hat{n}\times\left[\hat{n}\times\vec{J}(\vec{x}', t')\right] e^{i\omega\left(t' - \frac{\hat{n}\cdot\vec{x}'}{c}\right)}\right|^2}$$

Sec. 14.6 Radiation spectrum for $\gamma \gg 1$, from a particle in instantaneously circular motion.



Assumptions

(1) Only a very short part of the trajectory contributes, which we choose to define the xy-plane, with the x-axis tangent to the trajectory at $y=0$, $t=0$.

(2) The path has an instantaneous radius of curvature ρ at $t=0$.

(3) The detector is taken to lie in the xz-plane, i.e. $\hat{n} = \hat{x}\cos\theta + \hat{z}\sin\theta$

(4) We further define 2 polarization unit vectors, $\hat{\epsilon}_{\parallel} = \hat{y}$ lying IN the orbit plane,

$\hat{\epsilon}_{\perp} = \hat{n} \times \hat{\epsilon}_{\parallel}$ representing polarization $\perp$ (perp.) to the orbit plane

Next write the circular approximation to the trajectory at $t \simeq 0$:

$$\vec{r}(t) = \rho\,\hat{x}\sin\frac{ut}{\rho} + \rho\,\hat{y}\left(1 - \cos\frac{ut}{\rho}\right)$$

$$\vec{u}(t) = u\,\hat{x}\cos\frac{ut}{\rho} + u\,\hat{y}\sin\frac{ut}{\rho}$$

and
$$\hat{n} \times (\hat{n} \times \vec{\beta}) = -\hat{x}\beta\cos\frac{ut}{\rho}\sin^2\theta - \hat{y}\beta\sin\frac{ut}{\rho} + \hat{z}\beta\cos\frac{ut}{\rho}\sin\theta\cos\theta$$

$$= \hat{\epsilon}_{\perp}\beta\cos\frac{ut}{\rho} - \underset{\underset{\hat{y}}{\uparrow}}{\hat{\epsilon}_{\parallel}}\sin\frac{ut}{\rho}$$

and the argument of the exponential in the integral is: $\omega\left(t - \frac{\hat{n}\cdot\vec{r}(t)}{c}\right) = \omega t - \frac{\omega\rho}{c}\sin\frac{ut}{\rho}\cos\theta$

$$\simeq \omega t - \frac{\omega\rho}{c}\left(1 - \tfrac{1}{2}\theta^2\right)\left(\frac{ut}{\rho} - \frac{1}{6}\frac{u^3t^3}{\rho^3}\right)$$

$\theta \ll 1$
$t \ll \frac{\rho}{u}$

$$= \omega\left(1 - \frac{u}{c}\right)t + \frac{\theta^2}{2}\frac{u}{c}\omega t + \frac{t^3}{6}\frac{\omega}{c}\frac{u^3}{\rho^2}$$

Moreover, $\frac{u}{c} = (1-\gamma^{-2})^{1/2} \simeq 1 - \frac{1}{2}\gamma^{-2} + \dots$

$\simeq \frac{\omega}{2}\left[(\gamma^{-2}+\theta^2)t + \frac{c^2}{3\rho^2}t^3\right]$ + ... terms of order γ^{-2} smaller

and to leading order,

$$\hat{n}\times(\hat{n}\times\vec{\beta}) \simeq \hat{\epsilon}_\perp \theta - \hat{\epsilon}_\parallel \frac{ct}{\rho}$$

So, plugging in these expressions gives our desired integral as

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\hat{\epsilon}_\perp A_\perp(\omega) - \hat{\epsilon}_\parallel A_\parallel(\omega)\right|^2$$

where $A_\perp(\omega) \simeq \theta \int_{-\infty}^{\infty} dt\; e^{i\frac{\omega}{2}\left[(\gamma^{-2}+\theta^2)t + \frac{c^2t^3}{3\rho^2}\right]}$

and $A_\parallel(\omega) = \frac{c}{\rho}\int_{-\infty}^{\infty} dt\; t\, e^{\frac{i\omega}{2}\{(\gamma^{-2}+\theta^2)t + \frac{c^2t^3}{3\rho^2}]}$

After making a variable change to call

$$\xi \equiv \frac{\omega\rho}{3c}(\gamma^{-2}+\theta^2)^{3/2}$$

these are evaluated as

$$\Rightarrow \boxed{\begin{aligned} A_\perp(\omega) &= \frac{\rho}{c\sqrt{3}}\,\theta\,(\gamma^{-2}+\theta^2)^{1/2}\,K_{\frac{1}{3}}(\xi) \\ A_\parallel(\omega) &= \frac{\rho}{c\sqrt{3}}(\gamma^{-2}+\theta^2)\,K_{\frac{2}{3}}(\xi) \end{aligned}}$$

Kewl! This answer is so simple it can be expressed as Bessel functions/Airy fcns!

Sec. 16.2 Radiative reaction force

Since acceleration of a charge q by an external force $\vec{F}_{ext}$ causes it to radiate, we know that the process of radiation must remove kinetic energy from q.

i.e. the radiation must exert a recoil force on q, which we denote by
$$\vec{F}_{rad}.$$

To keep things simple, consider first the nonrelativistic limit where
$$m\dot{\vec{u}} = \vec{F}_{ext}$$

But since the power radiated is

$$P(t) = \frac{2}{3}\frac{q^2}{c^3}\dot{\vec{u}}^2,$$

this suggests that there must be some force opposing the motion, which therefore reduces $|\vec{u}|$.

An energy conservation might seem to suggest that this force should satisfy

$$\vec{F}_{rad} \cdot \vec{u} = -\frac{2}{3}\frac{q^2}{c^3}\dot{\vec{u}}^2 ,$$

but in fact this is not true in general.

Why?

⇒ One reason is that EM fields associated with a charge moving at uniform velocity $\vec{u}$ ALSO carry energy.

⇒ It is not ONLY in the radiated fields.

⇒ A more correct statement of energy conservation would be:

(Energy lost by particle during $t_1 \to t_2$)

= (Energy carried away by radiation)

+ (change in energy stored in the velocity fields)

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But, observe that if the particle motion is periodic (a special case), then the only net energy loss is due to radiation, i.e.

$$\int_{t_1}^{t_2} \vec{F}_{rad}\cdot\vec{u}\,dt = -\int_{t_1}^{t_2} \frac{2}{3}\frac{q^2}{c^3}\dot{\vec{u}}^{\,2}\,dt$$

But $\displaystyle\int_{t_1}^{t_2} a^2\,dt = \int_{t_1}^{t_2} \frac{d\vec{u}}{dt}\cdot\frac{d\vec{u}}{dt}\,dt$

$$= \underbrace{\vec{u}\cdot\frac{d\vec{u}}{dt}\Big|_{t_1}^{t_2}}_{0} - \int_{t_1}^{t_2}\frac{d^2\vec{u}}{dt^2}\cdot\vec{u}\,dt$$

One can also consider this valid in a perturbative sense, if the change in velocity during this time interval is "small".

$$\Rightarrow \int_{t_1}^{t_2}\left[\vec{F}_{rad} - \frac{2}{3}\frac{q^2}{c^3}\ddot{\vec{u}}\right]\cdot\vec{u}\,dt = 0$$

$$\Rightarrow \boxed{\vec{F}_{rad} = \frac{2}{3}\frac{q^2}{c^3}\dot{\vec{a}}}$$

← This is called the Abraham-Lorentz equation.

Or defining $\tau \equiv \dfrac{2q^2}{3mc^2}$ which has units of time,

$$\Rightarrow \boxed{\vec{F}_{rad} = m\tau\ddot{\vec{u}}}$$

Note that for an e^-,

$\tau = 6.3\times10^{-24}$ s

or $c\tau = 10^{-15}$ m

(Of course, this derivation will not determine any component of $\vec{F}_{rad}$ that is $\perp$ to $\vec{u}$.)

And it is a plausible conjecture that the modified equation of motion might be

$$\boxed{m\dot{\vec{u}} - m\tau\ddot{\vec{u}} = \vec{F}_{ext}}$$

This is called the Abraham–Lorentz equation of motion.

One interesting (though seemingly trivial) limit is when $\vec{F}_{ext} = 0$.

$$\Rightarrow \vec{a} - \tau\dot{\vec{a}} = 0$$

This simple 1st-order equation has the solution

$$\underline{\vec{a}(t) = \vec{a}(0)\, e^{t/\tau}}$$

This solution exhibits exponentially divergent self-acceleration, which is unphysical! This is unacceptable except in the special case,

$$\vec{a}(t) = \vec{a}(0) = 0.$$

Accordingly, one usually considers an alternative approach that does not suffer from this pathological runaway behavior:

$\Rightarrow$ use $\frac{d}{dt}\left(m\dot{\vec{u}} = \vec{F}_{ext}\right)$ to eliminate $m\ddot{\vec{u}}$.

i.e. $m\ddot{\vec{u}} = \frac{d\vec{F}_{ext}}{dt}$ which gives instead

$$\boxed{m\dot{\vec{u}} = \vec{F}_{ext} + \tau \frac{d\vec{F}_{ext}}{dt}}$$

or since $\vec{F}_{ext} = \vec{F}_{ext}(\vec{x}(t), t)$

$$\Rightarrow \frac{d}{dt}\vec{F}_{ext} = \frac{\partial \vec{F}_{ext}}{\partial t} + (\vec{u}\cdot\nabla)\vec{F}_{ext}$$

$$\Rightarrow \boxed{m\dot{\vec{u}} = \vec{F}_{ext} + \tau\left[\frac{\partial \vec{F}_{ext}}{\partial t} + (\vec{u}\cdot\nabla)\vec{F}_{ext}\right]}$$

This is Eq. 16.10, and it has no runaway solutions.

Example Apply this to a simple harmonic oscillator in 1D, with

$$F_{ext} = -kx = -m\omega_0^2 x$$

$$\Rightarrow m\ddot{x} = -m\omega_0^2 x + \tau\left(\cancelto{0}{\frac{\partial F_{ext}}{\partial t}} + \dot{x}\frac{\partial F_{ext}}{\partial x}\right)$$

$$\Rightarrow m\ddot{x} = -m\omega_0^2 x - \tau m\omega_0^2 \dot{x}$$

$$\Rightarrow \ddot{x} + \tau\omega_0^2\dot{x} + \omega_0^2 x = 0$$

This equation has solutions of the form (the real part of):

$$x(t) = x_0 e^{-i\omega t}$$

$$\Rightarrow \left(-\omega^2 - i\tau\omega_0^2\,\omega + \omega_0^2\right)x_0 = 0$$

$$\Rightarrow \omega = \pm\left[\omega_0^2 - \left(\frac{\omega_0^2\tau}{2}\right)^2\right]^{1/2} - \frac{i\omega_0^2\tau}{2}$$

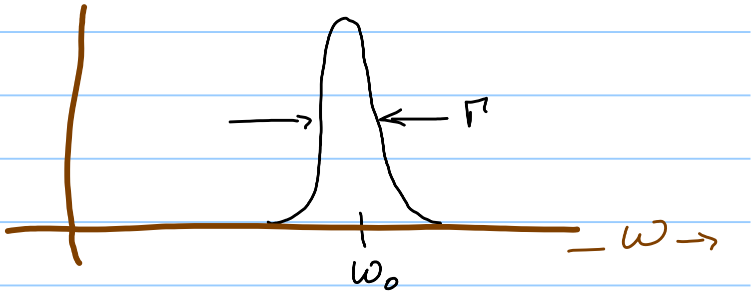
Thus, energy decays at the rate

$$e^{2(\text{Im}\,\omega)t} = e^{-(\omega_0^2\tau)t}$$

i.e. $e^{-\Gamma t}$ where $\Gamma = \omega_0^2\tau$

And if this damped oscillator is driven, monochromatically, its frequency response looks like

See Sec. 16.7



But problems remain in attempting to formulate a fully consistent theory of radiation reaction. One finds intriguing comments (e.g. Dirac, Proc. R. Soc. Lond. A 167 148 (1938)) "... the interior of the electron being a region ... allowing signals to propagate at v>c"

⇒ Dirac proposes to use a modified field $f^{\mu\nu}$ in the equations of motion of q, namely

$$\boxed{f^{\mu\nu} = F^{\mu\nu}_{actual} - \frac{1}{2}\left(F^{\mu\nu}_{ret} + F^{\mu\nu}_{adv.}\right)}$$

Wheeler + Feynman
1945 Rev. Mod. Phys.
follow + extend some
of Dirac's ideas

REVIEWS OF MODERN PHYSICS VOLUME 17, NUMBERS 2 AND 3 APRIL–JULY, 1945

Interaction with the Absorber as the Mechanism of Radiation†*

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"We must, therefore, be prepared to find that further advance into this region will require a still more extensive renunciation of features which we are accustomed to demand of the space time mode of description."—Niels Bohr[1]

To carry the analysis further requires us to find a new idea. We go back to a suggestion once made by Tetrode.[10] He proposed to abandon the conception of electromagnetic radiation as an elementary process and to interpret it as a consequence of an interaction between a source and an absorber. In his words,

Tetrode quote 1922

> "The sun would not radiate if it were alone in space and no other bodies could absorb its radiation. . . . If for example I observed through my telescope yesterday evening that star which let us say is 100 light years away, then not only did I know that the light which it allowed to reach my eye was emitted 100 years ago, but also the star or individual atoms of it knew already 100 years ago that I, who then did not even exist, would view it yesterday evening at such and such a time. . . . One might accordingly adopt the opinion that the amount of material in the universe determines the rate of emission. Still this is not necessarily so, for two competing absorption centers

---

[10] H. Tetrode, Zeits. f. Physik **10**, 317 (1922). When we gave a preliminary account of the considerations which appear in this paper (Cambridge meeting of the American Physical Society, February 21, 1941, Phys. Rev. **59**, 683 (1941)) we had not seen Tetrode's paper. We are indebted to Professor Einstein for bringing to our attention the ideas of Tetrode and also of Ritz, who is cited in this

article. An idea similar to that of Tetrode was subsequently proposed by G. N. Lewis, Nat. Acad. Sci. Proc. **12**, 22 (1926): "I am going to make the . . . assumption that an atom never emits light except to another atom, and to claim that it is as absurd to think of light emitted by one atom regardless of the existence of a receiving atom as it would be to think of an atom absorbing light without the existence of light to be absorbed. I propose to eliminate the idea of mere emission of light and substitute the idea of *transmission*, or a process of exchange of energy between two definite atoms or molecules." Lewis went nearly as far as it is possible to go without explicitly recognizing the importance of other absorbing matter in the system, a point touched upon by Tetrode, and shown below to be essential for the existence of the normal radiative mechanism.

Wheeler - Feynman →

Our picture of the mechanism of radiation is seen to be self-consistent. Any particle on being accelerated generates a field which is half-advanced and half-retarded. From the source a disturbance travels outward into the surrounding absorbing medium and sets into motion all the constituent particles. They generate a field which is equal to half the retarded minus half the advanced field of the source. In this field we have the explanation of the radiation field assumed by Dirac. The radiation field combines with the field of the source itself to produce the usual retarded effects which we expect from observation, and such retarded effects only. The radiation field also acts on the source itself to produce the force of radiative reaction. What we have said of one particle holds for every particle in a completely absorbing medium. All advanced fields are concealed by interference. Their effects show up directly only in the force of radiative reaction. Otherwise we appear to have a system of particles acting on each other via purely retarded forces.

Greene 2012, wacky idea

A conjecture – using Monte Carlo one could sample the time and direction and energy of photon emission as discrete events.

Then each time a photon is emitted with frequency ω, wavevector $\vec{k}$, the instantaneous 4-momentum of the particle q should be reduced by $\left(\frac{\hbar\omega}{c}, \hbar\vec{k}\right)$

i.e. $$p^\alpha \to p^\alpha - \left(\frac{\hbar\omega}{c}, \hbar\vec{k}\right)$$

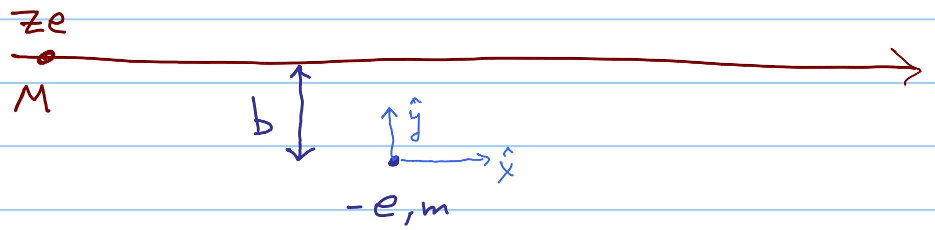
Then continue solving the equations of motion in the external field neglecting radiation emission until Monte Carlo sampling says that the next photon is emitted, etc.

Survey of some topics in Chapter 13

Collisions, energy loss, Cerenkov & Transition Radiation

13.1 Coulomb collision between heavy particle, charge Ze, mass $M \gg$ electron mass m, charge $-e$ at rest

Assume an undeflected heavy (M) trajectory (nonrelativistic)



The position of Ze is $\vec{X}(t) = b\hat{y} + vt\,\hat{x}$

The e^- experiences a force $\vec{F} = \dfrac{(vt-x)\,\hat{x} + (b-y)\,\hat{y}}{\left[(vt-x)^2 + (b-y)^2\right]^{3/2}} e^2 Z$ when the e^- is at $\vec{x}(t) = x\hat{x} + y\hat{y}$

$$\Rightarrow \quad \frac{d\vec{p}}{dt} = \vec{F}$$

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m}$$

If we neglect the small displacement of the e^- during the collision, then $\vec{x} \cong 0$ and $\vec{\Delta p} = \int_{-\infty}^{\infty} \vec{F}(t)\,dt = \dfrac{2Ze^2}{bv}\hat{y}$,

and energy transmitted to the e^- equal to

$$\Delta E = \frac{\Delta p^2}{2m} = \frac{2Z^2e^4}{mb^2v^2}$$

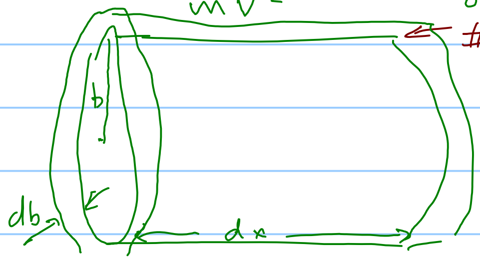
By momentum conservation the heavy particle is deflected, by small $\theta \simeq \dfrac{\Delta p}{p} = \dfrac{2Ze^2}{pvb}$

An exact nonrelativistic treatment (Rutherford cross section) gives $2\tan\dfrac{\theta}{2} = \dfrac{2Ze^2}{pvb}$ (agrees at small θ with our approximation)

And a more accurate formula valid even as $b \to 0$ is readily derived (see problem 13.1):

$$\Delta E(b) = \frac{2Z^2e^4}{mv^2}\left(\frac{1}{b^2+b_0^2}\right) \quad \text{where } b_0 \equiv \frac{Ze^2}{\gamma m v^2}$$

How much energy is lost to the e's between b, $b+db$?



$\leftarrow$ #e's in this cylindrical shell is

$$n\,dx\,2\pi b\,db$$

where n = electron density $= N Z_1$ (if Z_1 e's per atom)

$$\Rightarrow \quad \frac{d^2E(b)}{dx\,db} = N\,2\pi b\,db\,\Delta E(b) \quad \text{or} \quad \frac{dE}{dx} = \int_{b_{min}}^{b_{max}} \frac{2z^2e^4}{mb^2v^2}\,2\pi n\,b\,db$$

giving a logarithmic dependence,

$$\boxed{\frac{dE}{dx} = 4\pi \frac{Nz^2e^4Z_1}{mv^2} \ln\frac{b_{max}}{b_{min}}}$$

This has the correct basic structure, but a more careful and complete derivation by Bethe (1930) gives:

Aside: Jackson argues that

$$b_{max} \simeq \left(\frac{2z^2e^4}{mv^2\epsilon}\right)^{1/2}$$

$$b_{min} \simeq \frac{ze^2}{pv}$$

ϵ = electron binding energy

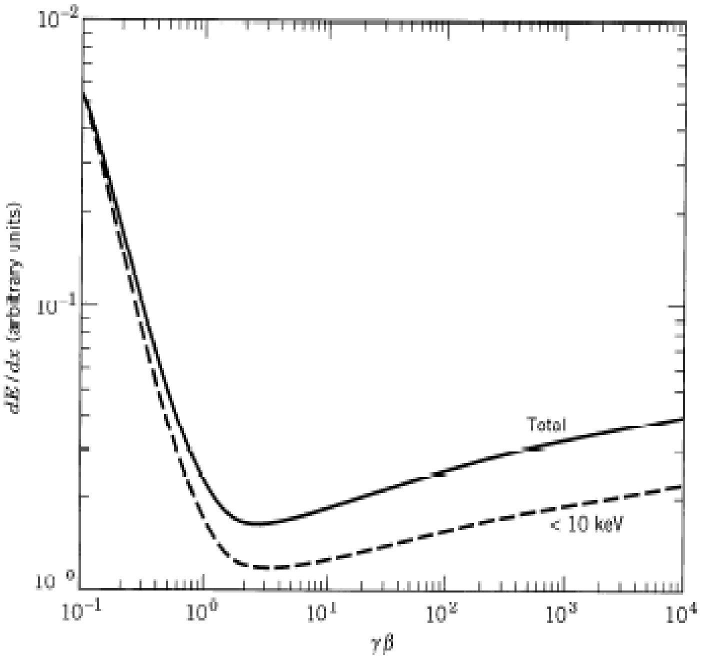
$$\frac{dE}{dx} = 4\pi N Z_1 \frac{z^2e^4}{mc^2\beta^2}\left\{\ln(B) - \beta^2\right\}, \quad \text{where } B = \frac{2\gamma\beta^2 mc^2}{\hbar\langle\omega\rangle}$$

QM correction for soft collisions with energies "small" < ϵ

relativistically valid!

and $\langle\omega\rangle$ is defined in terms of the quantum atomic oscillator strengths f_j associated with transition of frequency ω_j by

$$Z_1 \ln\langle\omega\rangle \equiv \sum_j f_j \ln\omega_j \qquad (13.14)$$



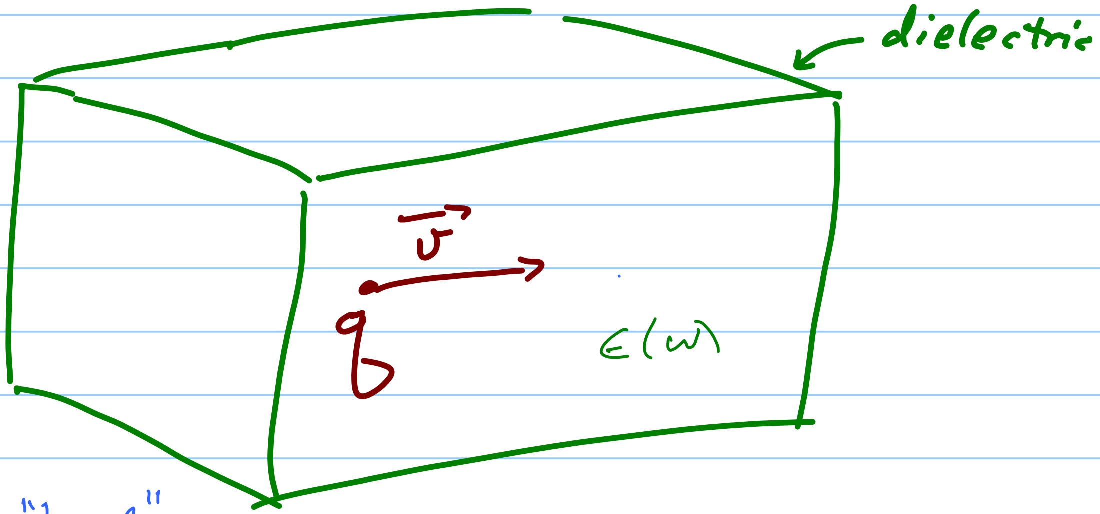
J: **Figure 13.1** Energy loss as a function of $\gamma\beta$ of the incident heavy particle. The solid curve is the total energy loss (13.14) with $\hbar\langle\omega\rangle = 160$ eV (aluminum). The dashed curve is the energy loss in soft collisions (13.12) with $\varepsilon = 10$ keV. The ordinate scale corresponds to the curly-bracketed quantities in (13.12) and (13.14), multiplied by 0.15.

Section 13.3 Density effect in collisional energy loss

The ideas that led to Bethe's formula (13.14) are inaccurate for larger impact parameters that greatly exceed the average atom-atom separation, especially for a dense material

The required correction to 13,14 was derived by Fermi, Phys. Rev. 57 p. 485 (1940)

The reason is because the dielectric material reduces the electric field at larger distances from Ze



A charged particle going through a linear dielectric medium experiences energy loss in two ways:

i) short-range "hard" collisions

ii) distant coherent interactions with the medium (Complex $\epsilon(\omega)$ causes dissipation)

Case i) is treated in Jackson Sec. 13.1

Here we will discuss case ii), i.e. Sec. 13.3; (see also problem 7.26)

Starting point: Fourier representation of potentials $A_\mu(x)$, sources $J_\mu(x)$; i.e. write for every quantity,

$$F(\vec{x},t) = (2\pi)^{-2}\int d^3k\int d\omega\, F(\vec{k},\omega)\, e^{i\vec{k}\cdot\vec{x}-\omega t}$$

Then the Fourier transformed wave equations are:

$$\left[k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)\right]\Phi(\vec{k},\omega) = \frac{4\pi}{\epsilon(\omega)}\rho(\vec{k},\omega) \qquad (1)$$

Consider charge Ze at velocity $\vec{v}$:

$$\left[k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)\right]\vec{A}(\vec{k},\omega) = \frac{4\pi}{c}\vec{J}(\vec{k},\omega)$$

Then $\rho(\vec{x},t) = Ze\,\delta(\vec{x}-\vec{v}t)$, $\vec{J}(\vec{x},t) = \vec{v}\rho(\vec{x},t)$ and their Fourier transforms are simple:

$$\rho(\vec{k},\omega) = \frac{Ze}{2\pi}\delta(\omega - \vec{k}\cdot\vec{v}) \qquad \vec{J}(\vec{k},\omega) = \vec{v}\rho(\vec{k},\omega)$$

Plugging into Eq. (1) gives the potentials:

$$\Phi(\vec{k},\omega) = \frac{2Ze}{\epsilon(\omega)}\,\frac{\delta(\omega-\vec{k}\cdot\vec{v})}{k^2-\frac{\omega^2}{c^2}\epsilon(\omega)}\,,\quad \vec{A}(\vec{k},\omega) = \epsilon(\omega)\frac{\vec{v}}{c}\Phi(\vec{k},\omega)$$

Now $\vec{E}(\vec{x},t) = \Big(-\nabla\Phi \longrightarrow -\nabla\int\frac{\Phi(\vec{k},\omega)}{2\pi}e^{i\vec{k}\cdot\vec{x}-\omega t} = \int\frac{-i\vec{k}}{2\pi}\Phi(\vec{k},\omega)e^{i\vec{k}\cdot\vec{x}-i\omega t}$

$=$

$-\frac{1}{c}\frac{\partial\vec{A}}{\partial t} \Big) \longrightarrow -\int\frac{\epsilon(\omega)\vec{v}}{2\pi\,c}\Phi(\vec{k},\omega)(-i\omega)e^{i\vec{k}\cdot\vec{x}-i\omega t}$

Thus

$$\Rightarrow \boxed{\vec{E}(\vec{k},\omega) = \left(\frac{i\epsilon(\omega)\omega}{c}\frac{\vec{v}}{c} - i\vec{k}\right)\Phi(\vec{k},\omega)\,;\quad \vec{B}(\vec{k},\omega) = i\epsilon(\omega)\frac{\vec{k}\times\vec{v}}{c}\Phi(\vec{k},\omega)}$$

Now compute the energy lost in collision with an e^- at impact param. b:

$$\Delta E = -e\int_{-\infty}^{\infty}\vec{v}\cdot\vec{E}\,dt = 2e\,\mathrm{Re}\int_0^{\infty} i\omega\,\vec{x}(\omega)\cdot\vec{E}^*(\omega)\,d\omega$$

where $\vec{E}(\omega) = (2\pi)^{-3/2}\int d^3k\,\vec{E}(\vec{k},\omega)e^{ibk_y}$ ← since we want E-field at $(0,b,0)$

e.g. let's first calculate the x-component:

k_x is easy!

$$E_x(\omega) = \frac{2iZe}{\epsilon(\omega)(2\pi)^{3/2}}\int d^3k\,e^{ik_yb}\left(\frac{\omega\epsilon(\omega)v}{c^2}-k_x\right)\frac{\delta(\omega-vk_x)}{k^2-\frac{\omega^2}{c^2}\epsilon(\omega)}$$

$$= \frac{-2iZe\omega}{(2\pi)^{3/2}v^2}\left(\frac{1}{\epsilon(\omega)}-\beta^2\right)\int_{-\infty}^{\infty}dk_y e^{ik_yb}\left[\int_{-\infty}^{\infty}\frac{dk_z}{k_y^2+k_z^2+\lambda^2}\right] = \frac{\pi}{\sqrt{\lambda^2+k_y^2}}$$

where $\lambda^2 = \frac{\omega^2}{v^2}-\frac{\omega^2}{c^2}\epsilon(\omega) = \frac{\omega^2}{v^2}\left(1-\beta^2\epsilon(\omega)\right)$

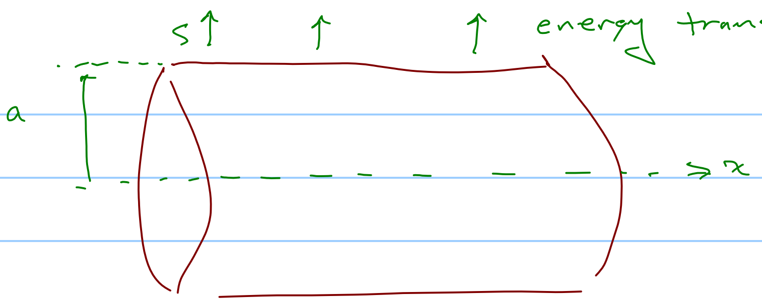
and the final integral here is

$$E_x = \frac{-iZe\omega}{(2\pi)^{1/2}v^2}\left(\frac{1}{\epsilon(\omega)}-\beta^2\right)\left[\int_{-\infty}^{\infty}\frac{e^{ibk_y}}{(\lambda^2+k_y^2)^{1/2}}dk_y\right] = 2\underline{K_0}(\lambda b)$$

modified Bessel function!

$$\boxed{\begin{aligned} &\Rightarrow \text{finally } E_x(\omega) = -\frac{iZe\omega}{v^2}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{1}{\epsilon(\omega)}-\beta^2\right)K_0(\lambda b) \\ &\text{and similarly,} \\ &E_y(\omega) = \frac{Ze}{v}\left(\frac{2}{\pi}\right)^{1/2}\frac{\lambda}{\epsilon(\omega)}K_1(\lambda b) \\ &\text{and} \quad B_z(\omega) = \epsilon(\omega)\beta E_y(\omega)\end{aligned}} \quad (*)$$

Now, the energy lost per dx ← by collisions happening at $b > a$ can either be computed by acting on all charges in a cylinder (like Bethe's), OR instead, by an energy conservation argument:



energy transferred radially outwards

(and $dx = v\,dt$)

$\Rightarrow$ energy flow involves the radial component of the Poynting vector

i.e.

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{1}{v}\frac{dE}{dt} = \frac{-c}{4\pi v}\underbrace{\int_{-\infty}^{\infty} 2\pi a\, \overset{v\,dt}{dx}\left(B_z(t)E_x(t)\right)}_{2\pi a v\,\mathrm{Re}\left[\int_0^{\infty} B_z^*(\omega)E_x(\omega)\,d\omega\right]}$$

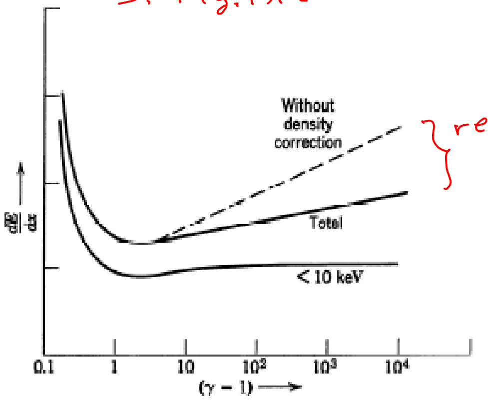
$\uparrow$ (13.35)

and giving finally Fermi's result:

$$\boxed{\left(\frac{dE}{dx}\right)_{b>a} = \frac{2}{\pi}\frac{(ze)^2}{v^2}\,\mathrm{Re}\left\{\int_0^{\infty} i\omega\lambda^* a K_1(\lambda^* a)K_0(\lambda a)\left(\frac{1}{\epsilon(\omega)} - \beta^2\right)d\omega\right.}$$

Woohoo!

J. Fig. 13.2



reduction of energy loss due to the density effect

e.g. for silver bromide

$\frac{1}{N}\frac{dE}{dx} \simeq 1.02\ \mathrm{MeV}\cdot\mathrm{cm}^2/\mathrm{g}$

approx constant at high E

See also U. Fano, Ann. Rev. Nucl. Sci. 13, 1 (1963)

Sec. 13.4 Cerenkov radiation

General statement: When an object (e.g. a charge) moves through a medium at a speed v greater than the speed of waves that travel in the medium (EM waves here) then waves are generated as the object travels.

e.g. phonons excited if you drag a particle/object through a BEC faster than sound speed

e.g. aircraft faster than sound speed in air (sonic booms!)

In the case considered here, a charge moves at $v > \overset{\nearrow}{c/n}$
index of refraction

Back in Sec. 13.3 we considered the regime where $|\lambda a| \ll 1$, but now we look at the opposite limit where $|\lambda a| \gg 1$, and use the fields derived in Sec. 13.3 (see * above) except now in the asymptotic expansions of the Bessel functions:

$$\Rightarrow \quad E_x(\omega, b) \to i \frac{ze\omega}{c^2}\left(1 - \frac{1}{\beta^2 \epsilon(\omega)}\right) \frac{e^{-\lambda b}}{\sqrt{\lambda b}} \qquad (**)$$

$$E_y(\omega, b) \longrightarrow \frac{ze}{v\epsilon(\omega)} \left(\frac{\lambda}{b}\right)^{1/2} e^{-\lambda b}$$

$$B_z(\omega, b) \to \beta \epsilon(\omega) E_y(\omega, b)$$

And so the integral 13.37 becomes

$$\frac{dE}{dx} = -\frac{ca}{2} \operatorname{Re} \int_0^\infty B_z^*(\omega) E_x(\omega) \, d\omega$$

$$= \operatorname{Re} \int_0^\infty \frac{z^2 e^2}{c^2} \left(-i \left(\frac{\lambda^*}{\lambda}\right)^{1/2}\right) \omega \left(1 - \frac{1}{\beta^2 \epsilon(\omega)}\right) e^{-(\lambda + \lambda^*) a}$$

where, again, $\lambda = \frac{w}{v}\left(1 - \beta^2 \epsilon(\omega)\right)^{1/2} = \frac{w}{v}\sqrt{1 - \frac{v^2 \epsilon(\omega)}{c^2}}$

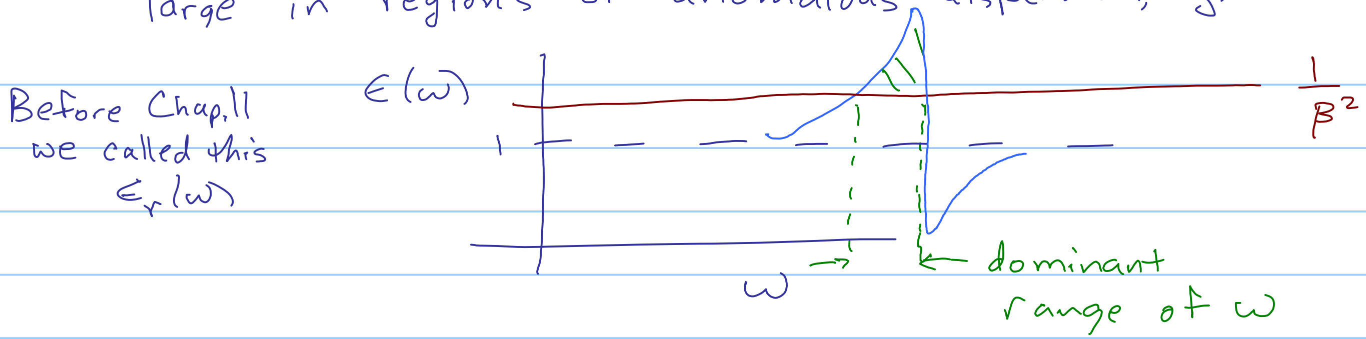
For simplicity in the following analysis, assume $\epsilon(\omega) = \text{real}$, meaning that λ will be purely IMAGINARY if $\boxed{v > \frac{c}{\sqrt{\epsilon(\omega)}}}$ and then there is no exponential decay in $e^{-(\lambda + \lambda^*) a}$

This is the case of Cerenkov radiation that escapes to ∞, i.e.

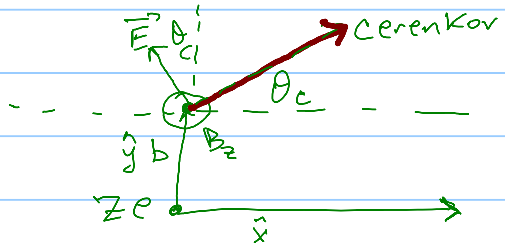
$$\frac{dE}{dx} = \frac{z^2 e^2}{c^2} \int_{\substack{\omega \\ \text{where} \\ \epsilon(\omega) > \frac{1}{\beta^2}}} \left[\omega \left(1 - \frac{1}{\beta^2 \epsilon(\omega)}\right)\right] d\omega$$

And the integrand [---] yields the distribution of Cerenkov radiation versus frequency ω

Recall from Chapter 7 that $\epsilon(\omega)$ can get quite large in regions of anomalous dispersion, e.g.



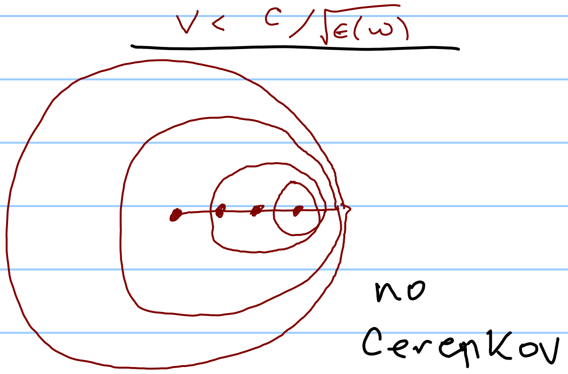
Direction of propagation

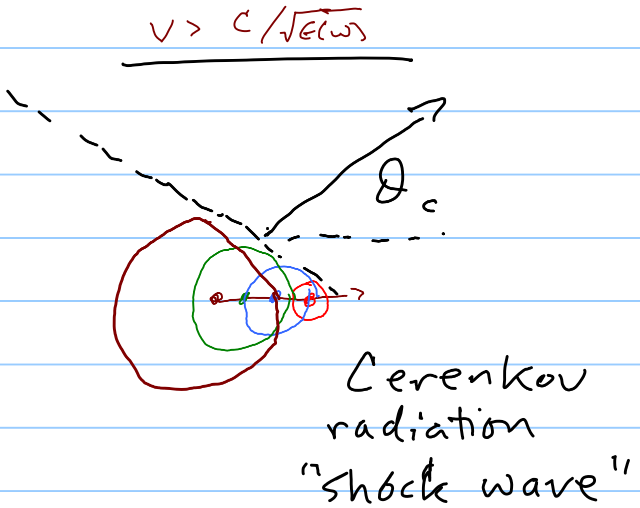


$$\boxed{\tan\theta_c = -\frac{E_x}{E_y}}$$

$$\Rightarrow \quad \cos\theta_c = \frac{1}{\beta\sqrt{\epsilon(\omega)}} \quad \text{using } (**)$$

Qualitative idea

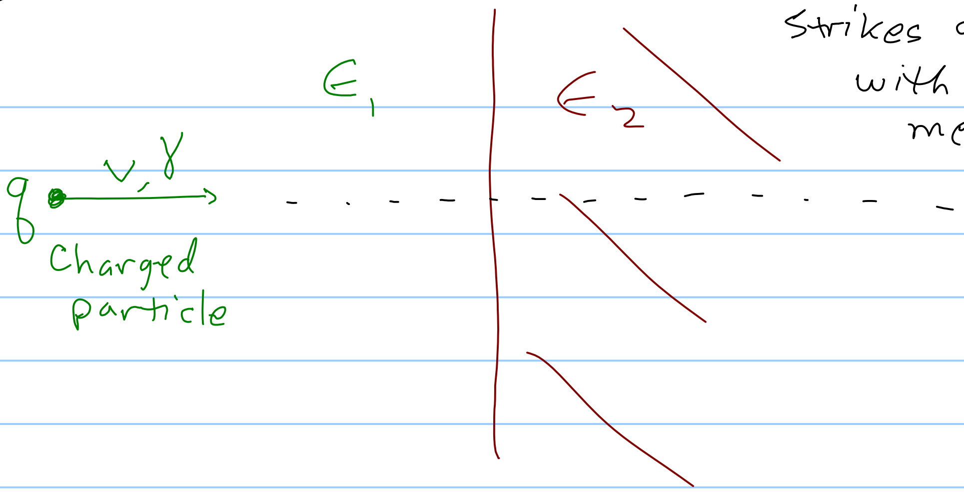




Transition Radiation, Sec. 13.7

I am showing this topic for general information without going through the detailed and rather complicated derivation (see Jackson pp 646-654 or Landau-Lifshitz, "Electrodynamics of Cont. Media", Sec. 116)

The basic idea: A charged particle in medium 1 (ϵ_1) strikes a boundary with a second medium (ϵ_2)



The disruption of the fields of q caused by the boundary produces radiation.

This is used in cosmic ray and particle physics experiments to detect very high energy particles.

Landau - Lifshitz (above Ref.) derives (for $\epsilon_1 = 1$, $\epsilon_2 = \epsilon$):

$$\frac{dI}{d\omega\, d\Omega} = \frac{e^2 \beta^2}{\pi^2 c (1-\beta^2\cos^2\theta)^2} \left| \frac{(\epsilon-1)[1-\beta^2+\beta(\epsilon-\sin^2\theta)^{1/2}]}{[1+\beta(\epsilon-\sin^2\theta)^{1/2}][\epsilon\cos\theta+(\epsilon-\sin^2\theta)^{1/2}]} \right|^2$$

And the radiation is linearly polarized, with $\vec{E}$ in the $\vec{k}-\vec{v}$ plane

In the non-relativistic limit, this simplifies to

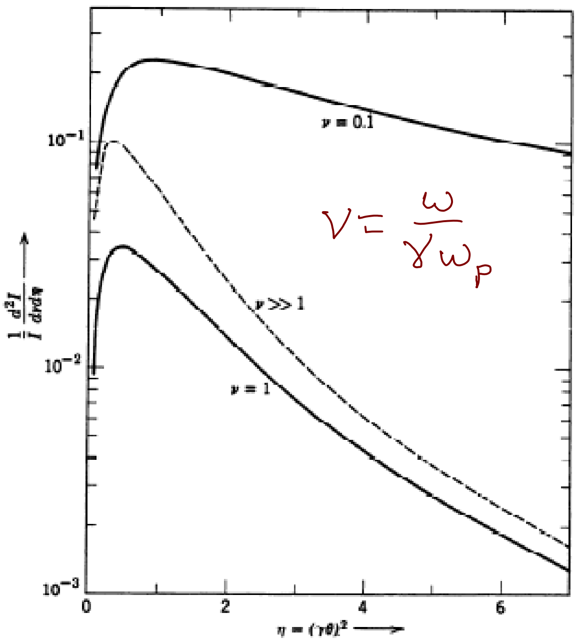
$$\frac{dI}{d\omega\, d\Omega} = \frac{e^2 v^2}{\pi^2 c^3} \frac{\sin^2\theta}{(1-\beta^2\cos^2\theta)^2}$$

But practical detectors are usually for $\gamma \gg 1$

e.g. for $\gamma \sim 10^3$ typical photon energies are about 2-20 keV

J: Fig. 13.10 Angle Distrib.

13.11 Freq. Distrib.



Transition Radiation properties

$$\nu = \frac{\omega}{\gamma \omega_p}$$

Figure 13.10 Angular distributions of transition radiation at $\nu = 0.1$, $\nu = 1$ and $\nu \gg 1$. The solid curves are the normalized angular distributions, that is, the ratio of (13.84) to (13.87). The dashed curve is ν^4 times that ratio in the limit $\nu \to \infty$.

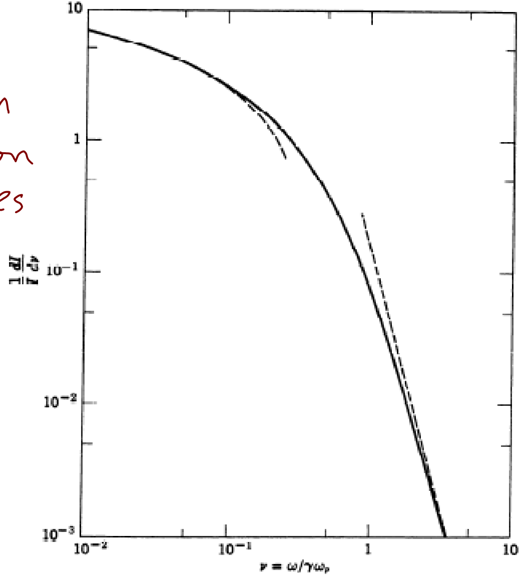


Figure 13.11 Normalized frequency distribution $(1/I)(dI/d\nu)$ of transition radiation as a function of $\nu = \omega/\gamma\omega_p$. The dashed curves are the two approximate expressions in (13.86).

Note: $\hbar\omega_p \simeq 20$ eV

In practice, transition radiation detectors use very many parallel planes of different dielectrics.

Check out the link below, to see a transition radiation detector used in CERN's ATLAS detector!

https://www.youtube.com/watch?v=HH1dMz288KA

And here are a few graphics from an article on particle astrophysics experiments on balloons + satellites:

Nuclear Instruments and Methods in Physics Research A 522 (2004) 9–15. Dietrich Müller

Transition radiation detectors in particle astrophysics

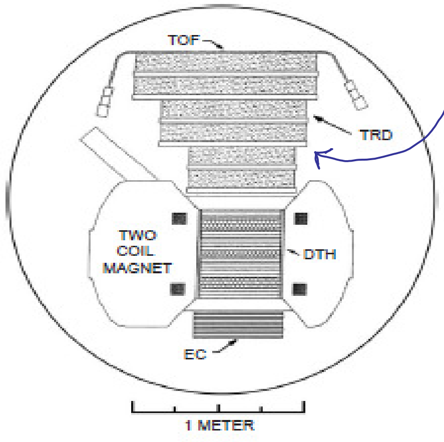


Fig. 1. Cross-section of the HEAT spectrometer for measurements of electrons and positrons. TOF: top time-of-flight scintillator; TRD: transition radiation detector; DTH: drift-tube hodoscope; EC: electromagnetic calorimeter.

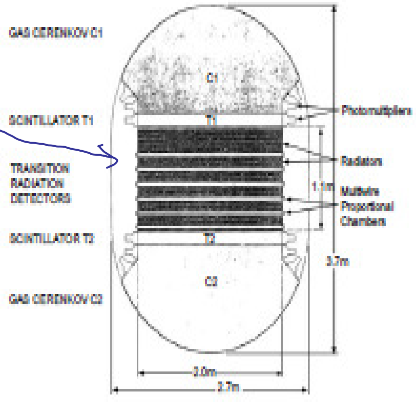


Fig. 5. Cross-section of the CRN detector.

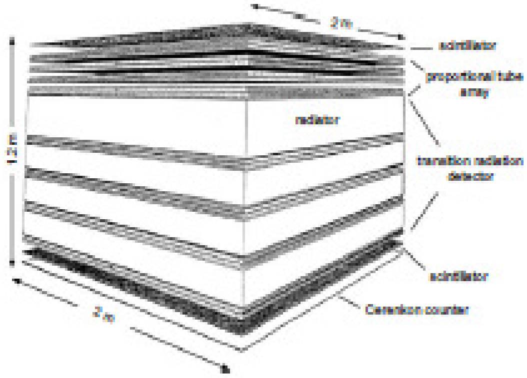


Fig. 6. Cross-section of the TRACER balloon instrument.

more from the Müller reference above:

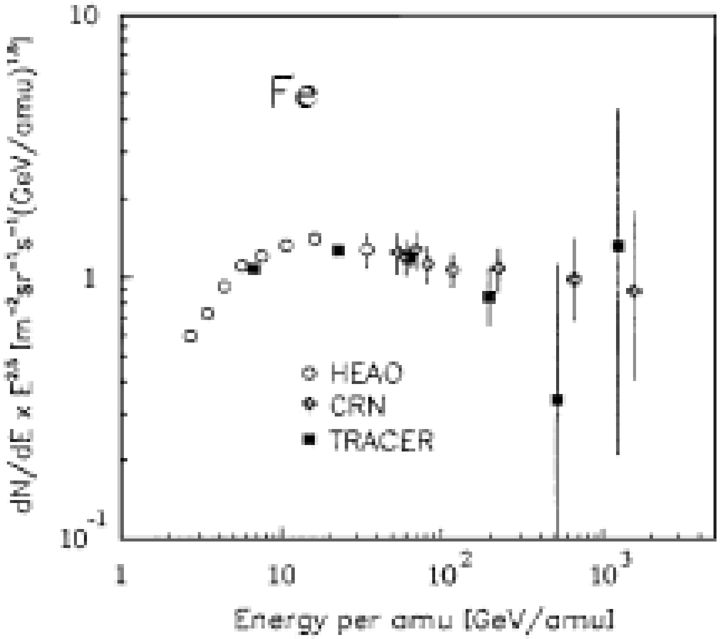


Fig. 7. Differential energy spectra of iron nuclei measured with HEAO-3, CRN and TRACER. Note that E is the energy per amu, and that the intensities are multiplied with $E^{2.5}$.

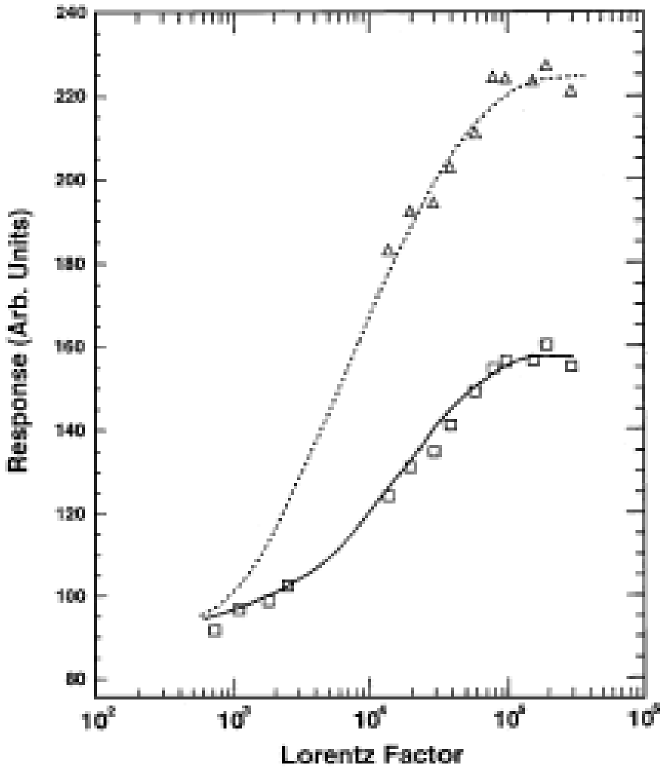


Fig. 8. Average signal versus Lorentz factor for a composite radiator/detector configuration consisting of plastic foils, foam, and fibers (triangles), and for a radiator of parallel Mylar foils of 76 μm thickness (squares). Note that the signal reaches saturation around $\gamma \approx 10^5$.

For some wacky fun, an intro to CERN is at:

https://www.youtube.com/watch?v=j50ZssEojtM

And for an audio summary of everything you learned this semester, check out

https://app.box.com/s/cq15ipcslwlnmcs93n1bhwaz1l1zhkt4

(call me nutty.mp3)

Final topic for this semester

– A simple introduction to EM Field quantization

(in vacuum, no sources)

You will likely see a more systematic treatment of this subject in your advanced QM course in the future, but here we can see the basic idea in just a few pages

Note: I will return to SI units now

Reference: U. Fano and A.R.P. Rau, p. 26
Atomic Collisions & Spectra (1986)

Start from Maxwell's eqns in vacuum, no sources:

$$\nabla\cdot\vec{E}=0 \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \nabla\times\vec{B} = \underbrace{\mu_0\epsilon_0}_{\frac{1}{c^2}}\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\cdot\vec{B}=0$$

And these have plane wave solutions,

$$\vec{E}(\vec{x},t) = \vec{E}_0 e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

$$\vec{B}(\vec{x},t) = \vec{B}_0 e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

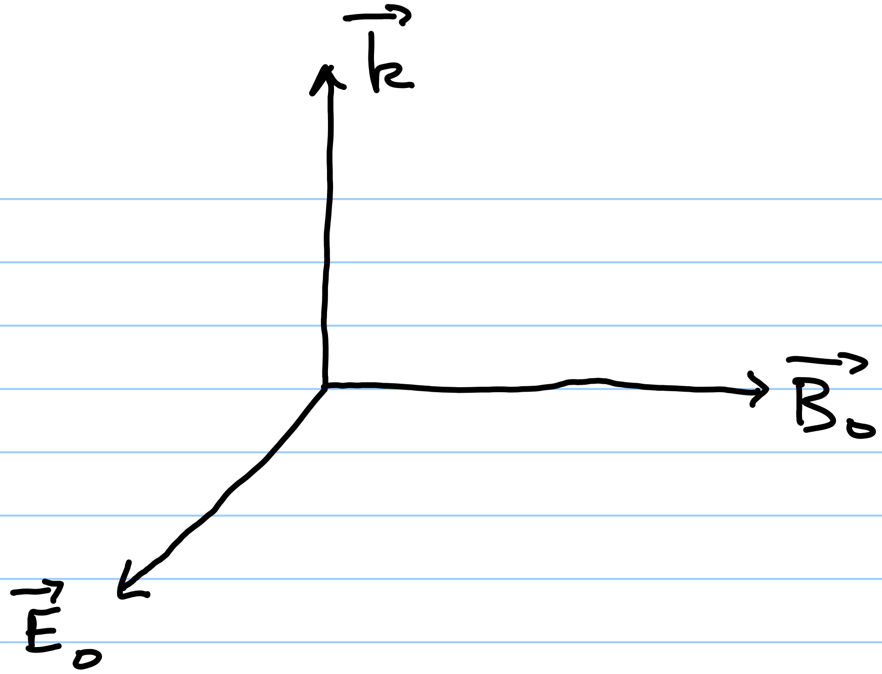
which only obey Maxwell's equations if also,

$$i\vec{k}\cdot\vec{E}_0 = 0 \quad , \quad i\vec{k}\cdot\vec{B}_0 = 0 \quad \text{(transversality)}$$

AND if $\quad i\vec{k}\times\vec{E}_0 = i\omega\vec{B}_0 \quad , \quad i\vec{k}\times\vec{B}_0 = -\frac{i\omega}{c^2}\vec{E}_0$

which are satisfied only if $\frac{\omega}{c} = k$

and $\quad |\vec{B}_0| = \frac{1}{c}|\vec{E}_0|$



We will consider a specific, simple geometry with a wave traveling along $\hat{k} = +\hat{z}$ with

$$\vec{E}_0 = E_0\,\hat{x} \qquad \vec{B}_0 = B_0\,\hat{y} = \frac{E_0}{c}\,\hat{y}$$

The full EM wave then looks like

$$\boxed{\vec{E}(\vec{x},t) = \hat{x}\,\mathcal{E}(t)\,e^{ikz}} \quad \text{where} \quad \boxed{\mathcal{E}(t) \equiv E_0\,e^{-i\omega t}}$$

and $$\vec{B}(\vec{x},t) = \hat{y}\left(\frac{E_0}{c}\,e^{-i\omega t}\right)e^{ikz}$$

which we can recast as

$$\vec{B}(\vec{x},t) = \hat{y}\left(\frac{\omega E_0}{kc^2}\,e^{-i\omega t}\right)e^{ikz}$$

or $$\boxed{\vec{B}(\vec{x},t) = \hat{y}\,\frac{i}{\omega c}\,\frac{\partial}{\partial t}\,\mathcal{E}(t)\,e^{ikz}}$$

$\Rightarrow$ Now write out the Hamiltonian, as the energy in such a wave within a large but finite quantization volume V

i.e. energy density is $u = \frac{\epsilon_0 |\vec{E}|^2}{2} + \frac{1}{2\mu_0} |\vec{B}|^2$

= constant in space here

$\Rightarrow$ energy is

$$H = \underbrace{\frac{1}{2}\int u \, d^3x}_{V} = \frac{V\epsilon_0}{4}\left(|\vec{E}|^2 + c^2 |\vec{B}|^2\right)$$

or

$$\boxed{H = \frac{V\epsilon_0}{4}\left(\mathcal{E}(t)^2 + \frac{1}{\omega^2}\dot{\mathcal{E}}(t)^2\right)}$$

which is the classical Hamiltonian for the particular mode we chose

<u>IDEA</u>: This looks just like the Hamiltonian of a classical harmonic oscillator.

The analogy is clearest if we define a "generalized momentum" as $P \equiv \frac{V\epsilon_0}{\omega\sqrt{2}} \dot{\mathcal{E}}$

and a "generalized coordinate" as $Q \equiv \frac{V\epsilon_0}{\omega\sqrt{2}} \mathcal{E}$

which recasts our Hamiltonian as

$$\boxed{H = \frac{P^2}{2} + \frac{1}{2}\omega^2 Q^2}$$

which is the Hamiltonian of a <u>unit mass</u> oscillator!

So next, simply quantize it!

Assume the usual quantization rules,

e.g. $[Q, P] = i\hbar$

We all learn in elementary QM how to solve this oscillator problem with raising and lowering operators:

$$a^{\pm} \equiv \mp \frac{i}{\sqrt{2\hbar\omega}} (P \pm i\omega Q)$$

and all the properties of quantum oscillators can be used immediately, such as the energy levels associated with this one mode (ω)

$$\Longrightarrow \quad E_n = (n + \tfrac{1}{2})\hbar\omega \quad , n = 0, 1, 2, \ldots$$

Observations

① This field oscillator is present even in the GROUND STATE ($n=0$), i.e. all fields have at least the zero point energy $\frac{\hbar\omega}{2}$

② We can interpret $n = \#$photons in this EM mode of frequency ω, and each photon has energy equal to $\hbar\omega$

③ These quantized fields can be used to compute (using the "Fermi golden rule") the spontaneous radiative decay of an excited atomic state

e.g. for atomic hydrogen, $|2p, n=0\rangle \to |1s, n=1\rangle$ in which a photon ($n=1$) appears

④ Weirdness: there are of course an infinite number of frequencies ω_s, and if you sum all the zero point energies $E^{rad}_{tot} = \sum_s \frac{\hbar\omega_s}{2} \to \infty$

A puzzling result! Infinite vacuum energy!

WHAT COULD IT MEAN... ??

To summarize, there are still unsolved vistas in classical electrodynamics. And for descriptions of actual real-world phenomena, these puzzles lead one into the world of quantum electrodynamics, quantum optics, renormalization. You will hopefully learn these beautiful subjects in your future studies.