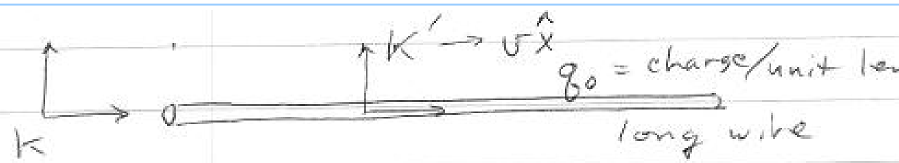
Homework 08 J: 11.13, 11.16, 12.1, 12.3

C. Greene's solutions

11.13



(a) In frame K', $\vec{J}' = 0 \Rightarrow \vec{B}' = 0$

And by Gauss' Law, $\nabla \cdot \vec{E} = 4\pi \rho_0$ in Gaussian units (see p. 779)

$$\Rightarrow \oint \vec{E} \cdot d\vec{a} = 4\pi Q_{encl}$$

$$\Rightarrow E'(2\pi \rho l) = 4\pi q_0 l$$

(adopt "cylindrical" coordinates with an axis $\hat{x}$ instead of $\hat{z}$)

$$\Rightarrow \vec{E}' = \frac{2q_0}{\rho'}\hat{\rho}' = \frac{2q_0}{(z'^2+y'^2)}(z'\hat{z} + y'\hat{y})$$

The fields in the lab frame are found from the inverse of 11.148,

$$\Rightarrow E_1 = E_1' \qquad B_1 = B_1'$$

$$E_2 = \gamma(E_2' + \beta B_3') \qquad B_2 = \gamma(B_2' - \beta E_3')$$

$$E_3 = \gamma(E_3' - \beta B_2') \qquad B_3 = \gamma(B_3' + \beta E_2')$$

$$\Rightarrow \boxed{\vec{E} = \gamma \vec{E}' = \frac{2\gamma q_0 \hat{\rho}}{\rho}} = \frac{2\gamma q_0}{z^2+y^2}(z\hat{z} + y\hat{y})$$

$$\vec{B} = \frac{2\gamma q_0 \beta}{z^2+y^2}(-z\hat{y} + y\hat{z}) = \boxed{\frac{2\gamma q_0}{\rho}\vec{\beta} \times \hat{\rho} = \vec{B}}$$

$$\text{or} \quad \boxed{\vec{B} = \frac{2\gamma q_0 \beta}{\rho}\hat{\phi}}$$

(b) In the rest frame, $\vec{J}' = 0$, $\boxed{\rho_0' = \delta(z')\delta(y') q_0}$

$$\Rightarrow J'^{\alpha} = (c\rho_0', 0) = (cq_0, 0)\,\delta(z')\,\delta(y')$$

Then obtain J^{α} by a LT:

$$J^0 = \gamma(J^{0\prime} + \beta J^{1\prime}) = \gamma c q_0 \delta(z)\delta(y)$$

$$\Rightarrow \boxed{\rho_0 = \gamma q_0 \delta(z)\delta(y)}$$

$$J^1 = \gamma(\beta J^{0\prime} + J^{1\prime}) = \gamma\beta c q_0 \delta(z)\delta(y)$$

$$\boxed{\vec{J} = \gamma q_0 v\, \delta(z)\delta(y)\,\hat{x}}$$

11.13) (d) By Gauss's Law in the lab frame,

$$\vec{E} = E\hat{\rho}$$

$$E(2\pi\rho l) = 4\pi\gamma q_0 l$$

$$\Rightarrow \boxed{\vec{E} = \frac{2\gamma q_0}{\rho}\hat{\rho}}$$ which agrees with the answer in (a)

and Ampere's Law in Gaussian units reads (see pp. 778-9)

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} \quad \text{or} \quad \oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int \vec{J}\cdot d\vec{a}$$

$$\Rightarrow \vec{B} = B\hat{\phi} \Rightarrow 2\pi\rho B(\rho) = \frac{4\pi}{c}\gamma q_0 v$$

$$\Rightarrow \boxed{\vec{B} = \frac{2\gamma q_0 v}{\rho c}\hat{\phi} = \frac{2\gamma q_0 \beta}{\rho}\hat{\phi}}$$ which agrees with (a) as expected.

Note that q_0 is increased by γ since each length unit of K' is contracted by γ when viewed in K.

11.16 Call K' the rest frame of the medium, in which $\vec{J}' = \sigma \vec{E}'$

$\Rightarrow$ In K', $J'^{i} = \sigma F'^{i,0} = \sigma F'^{i,0} \frac{U'_0}{c} = \frac{\sigma}{c} F'^{i\beta} U'_\beta$

since $U'_\alpha = (c, 0)$ in the rest frame where $v = 0$, $\gamma = 1$.

this ~~holds~~ does not hold for the '0' component, we must consider it separately:
Assuming that $\rho' \neq 0$ here, since this will produce the convection current, we can rewrite this '0'-component equation in K' as

$$J'^{0} - c\rho' = \frac{\sigma}{c} F'^{0\beta} U'_\beta$$

and again using $U'^{\alpha} = (c, \vec{0})$ in K',
write $c\rho' = \frac{U'_\beta J'^{\beta} U'^{0}}{c^2}$

$$\Rightarrow J'^{0} - \frac{U'^{\beta} J'^{\beta} U'^{0}}{c^2} = \frac{\sigma}{c} F'^{0\beta} U'_\beta$$

$$\text{or } J'^{\alpha} - \frac{(U'_\beta J'^{\beta}) U'^{\alpha}}{c^2} = \frac{\sigma}{c} F'^{\alpha\beta} U'_\beta$$

(since extending '0' to 'α' still produces an identity in K')

But now in this form it is manifestly covariant, and holds in any frame, i.e.

$$\boxed{J^{\alpha} - \frac{1}{c^2}(U_\beta J^{\beta}) U^{\alpha} = \frac{\sigma}{c} F^{\alpha\beta} U_\beta}$$

11.16(b) Write $F^{\alpha\beta} \to F = \begin{pmatrix} 0 & -\vec{E} \\ \vec{E} & -\epsilon^{ijk} B_k \end{pmatrix}$,

where $i, j, k = 1, 2, 3$

$\Rightarrow$ The i^{th} spatial component of J in frame K is

$$J^i = \frac{1}{c^2}(U_\beta J^\beta) U^i + \frac{\sigma}{c} F^{i\beta} U_\beta$$

where $J^\alpha = (\rho c, \vec{J})$, $U^\alpha = \gamma c(1, \vec{\beta})$

$$\Rightarrow F^{i\beta} U_\beta = \gamma c \begin{pmatrix} 0 & -\vec{E} \\ \vec{E} & -\epsilon^{ijk} B_k \end{pmatrix} \begin{pmatrix} 1 \\ -\vec{\beta} \end{pmatrix}$$

$$= \gamma c \left(E_i + \epsilon^{ijk} \beta_j B_k \right) = \gamma c \left(\vec{E} + \vec{\beta} \times \vec{B} \right)_i$$

and $U_\beta J^\beta = \gamma c^2 \rho - \gamma c \vec{\beta} \cdot \vec{J}$

$$\Rightarrow \vec{J} = \frac{1}{c^2}\left(\gamma c^2 \rho - \gamma c \vec{\beta} \cdot \vec{J} \right) \gamma c \vec{\beta} + \gamma \sigma \left(\vec{E} + \vec{\beta} \times \vec{B} \right)$$
$$= \gamma^2 (c\rho - \vec{\beta} \cdot \vec{J}) \vec{\beta} + \gamma \sigma (\vec{E} + \vec{\beta} \times \vec{B})$$

Consider next the time-like component

$$J^0 = \rho c = \frac{1}{c^2}\left(\gamma c^2 \rho - \gamma c \vec{\beta} \cdot \vec{J} \right) \gamma c + \frac{\sigma}{c} \gamma c \vec{\beta} \cdot \vec{E}$$

or $\rho c - \gamma \sigma \vec{\beta} \cdot \vec{E} = \gamma^2 (\rho c - \vec{\beta} \cdot \vec{J})$

$$\Rightarrow \boxed{\vec{J} = \gamma \sigma \left[\vec{E} + \vec{\beta} \times \vec{B} - (\vec{\beta} \cdot \vec{E}) \vec{\beta} \right] + \underbrace{c \vec{\beta} \rho}_{= \rho \vec{v}}}$$

11.16 (c) If $\rho' = 0 \Rightarrow J'^{\alpha} = (0, \vec{J}')$

whereby $U'^{\beta} J'_{\beta} = 0 = U^{\beta} J_{\beta}$

since the scalar product of 2 4-vectors is invariant

$$\Rightarrow J^{\alpha} = \frac{\sigma}{c} F^{\alpha\beta} U_{\beta}$$

$$\Rightarrow \boxed{\vec{J} = \gamma\sigma(\vec{E} + \vec{\beta} \times \vec{B})}$$

agrees with expected nonrel. limit where $\gamma \simeq 1$.

ρ is found from $U^{\beta} J_{\beta} = 0$

$$\Rightarrow \gamma c^2 \rho - \gamma c \vec{\beta} \cdot \vec{J} = 0$$

$$\text{or } \rho = \frac{\vec{\beta} \cdot \vec{J}}{c} = \boxed{\frac{\gamma\sigma}{c} \vec{\beta} \cdot \vec{E} = \rho}$$

12.1 (a) Consider $\tilde{L} = -\frac{m}{2} U_\alpha U^\alpha - \frac{q}{c} U_\alpha A^\alpha$ and write the principle of least action as an integral over the particle's proper time τ,

i.e.

$$A = \int_{\tau_1}^{\tau_2} \tilde{L}\, d\tau$$

Then the Euler-Lagrange equations become

$$\frac{d}{d\tau} \frac{\partial \tilde{L}}{\partial U_\delta} - \frac{\partial \tilde{L}}{\partial x_\delta} = 0$$

$$\Rightarrow \frac{d}{d\tau} \frac{\partial}{\partial U_\delta}\left(-\frac{m}{2} U_\alpha g\ U_\beta - \frac{q}{c} U_\alpha A^\alpha \right) = -\frac{q}{c} U_\beta \frac{\partial}{\partial x_\delta} A^\beta$$

or

$$-\frac{m}{2} \frac{d}{d\tau}\left(\delta^\delta_{\ \alpha} g^{\alpha\beta} U_\beta + U_\alpha g^{\alpha\beta} \delta^\delta_{\ \beta} \right) - \frac{q}{c} \frac{d}{d\tau} \delta^\delta_{\ \alpha} A^\alpha$$

$$= -\frac{m}{2} g^{\delta\beta} \frac{d}{d\tau} U_\beta - \frac{m}{2} \frac{dU_\alpha}{d\tau} g^{\alpha\delta} - \frac{q}{c} \frac{dA^\delta}{d\tau} = -\frac{q}{c} U_\beta \frac{\partial A^\beta}{\partial x_\delta}$$

Recall that $A^\delta = A^\delta(\vec{x}(t), t)$ varies with time both because the A^δ has an explicit time dependence and because the particle position changes with time, whereby we can use:

$$U_\beta = \gamma(c, -\vec{u}), \quad \partial^\beta = \left(\frac{\partial}{\partial x_0}, -\nabla\right),$$

and $$\frac{d}{d\tau} = \gamma_u \frac{d}{dt} = \gamma\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right) = \gamma c \frac{\partial}{\partial x_0} + \vec{U}\cdot\vec{\nabla}$$

or simply $$\frac{d}{d\tau} = U_\beta \partial^\beta = U^\beta \partial_\beta$$

$$\Rightarrow m\frac{dU^\delta}{d\tau} = \frac{q}{c} U_\beta \partial^\delta A^\beta - \frac{q}{c} U_\beta \partial^\beta A^\delta$$

and finally, $$\boxed{\frac{dU^\delta}{d\tau} = \frac{q}{mc} F^{\delta\beta} U_\beta \quad \text{where} \quad F^{\delta\beta} = \partial^\delta A^\beta - \partial^\beta A^\delta}$$

in agreement with Eq. 12.3!

12.1 (b) For this choice of Lagrangian, the 4-momentum p conjugate to x is given by Eq. 12.33, namely

$$p^\alpha = -\frac{\partial \tilde{L}}{\partial\left(\frac{dx_\alpha}{d\tau}\right)} = -\frac{\partial \tilde{L}}{\partial U_\alpha} = \boxed{mU^\alpha + \frac{q}{c}A^\alpha = p^\alpha}$$

i.e. $$U^\alpha = \frac{1}{m}\left(p^\alpha - \frac{q}{c}A^\alpha\right)$$

Then the Hamiltonian, from 12.34, is

$$\tilde{H} = p^\alpha U_\alpha + \tilde{L} = \frac{1}{m}p^\alpha\left(p_\alpha - \frac{q}{c}A_\alpha\right)$$

$$-\frac{m}{2}\frac{1}{m^2}\left(p^\alpha - \frac{q}{c}A^\alpha\right)\left(p_\alpha - \frac{q}{c}A_\alpha\right) - \frac{q}{mc}\left(p_\alpha - \frac{q}{c}A_\alpha\right)A^\alpha$$

or simplifying, $$\boxed{\tilde{H} = \frac{1}{2m}\left(p_\alpha - \frac{q}{c}A_\alpha\right)\left(p^\alpha - \frac{q}{c}A^\alpha\right)}$$

To work with $\tilde{H}$ as a Hamiltonian in mechanics, we must use this preceding form, but note that its "value" is more easily found from the form $\tilde{H} = \frac{m}{2} U_\alpha U^\alpha = \frac{1}{2} mc^2$

which is obviously a Lorentz invariant, half of the rest energy.

To write out the time + space components explicitly, use $p^\alpha - \frac{q}{c} A^\alpha = \left(\frac{E - q\Phi}{c}, \vec{p} - \frac{q}{c}\vec{A} \right)$

So we get readily,

$$\tilde{H} = \frac{1}{2mc^2}\left(E - q\Phi\right)^2 + \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2$$

12.3 (a)

Start from $\dfrac{d\vec{p}}{dt} = qE_0\hat{x}$, since $\vec{E} = E_0\hat{x}$ (choose $\vec{E}$ along the x-axis)

initial momentum = constant

$$\Rightarrow \quad \vec{p}(t) = \vec{p}_0 + qE_0 t\,\hat{x} = \gamma(t)\, m\vec{u}(t)$$

given: $\vec{p}_0 = m\gamma_0\vec{V}_0$.

$$\Rightarrow \left(1 - \frac{u(t)^2}{c^2}\right)^{-1/2} m\vec{u}(t) = \vec{p}_0 + qE_0 t\,\hat{x}$$

Then squaring both sides,

$$\frac{u^2(t)}{1 - \frac{u^2(t)}{c^2}} = \frac{1}{m^2}\left(\vec{p}_0 + q\vec{E}_0 t\right)^2$$

$$\text{or} \quad \vec{u}(t) = \frac{1}{m}\left(\vec{p}_0 + q\vec{E}_0 t\right)\left[1 + \left(\frac{\vec{p}_0 + q\vec{E}_0 t}{mc}\right)^2\right]^{-\frac{1}{2}}$$

We must integrate again to obtain $\vec{x}(t)$, i.e.

$$\Rightarrow \quad \vec{x}(t) = \vec{x}(0) + \int_0^t dt'\,\vec{u}(t')$$

and since $\vec{V}_0 \cdot \vec{E}_0 = 0 = \vec{p}_0 \cdot \vec{E}_0$,

let's focus on the only nontrivial component, namely $x(t)$:

$$x(t) = \cancelto{0}{x(0)} + \int_0^t \frac{qE_0 t'/m}{\left[1 + \frac{p_0^2}{m^2c^2} + \frac{q^2E_0^2 t'^2}{m^2c^2}\right]^{1/2}}$$

or setting $W^2 = 1 + p_0^2/m^2c^2$

$$\Rightarrow \quad x(t) = c\int_0^t \frac{(qE_0/mcW)\,t'\,dt'}{\left(1 + q^2E_0^2/m^2c^2W^2\right)^{1/2}}$$

i.e. $$x(t) = \frac{c}{Z_1}\int_0^t \frac{Z_1^2 t'\,dt'}{\sqrt{1+Z_1^2 t'^2}}\ , \text{ with } Z_1 = \frac{qE_0}{mcW}$$

$$\boxed{Z_1 = \frac{qE_0}{mc}\left(1+\frac{p_0^2}{m^2c^2}\right)^{-1/2}}$$

which is trivial to evaluate,

$$x(t) = \frac{c}{Z}\left(1+Z^2t'^2\right)^{1/2}\Big|_0^t = \boxed{\frac{c}{Z_1}\left[\sqrt{1+Z_1^2t^2} - 1\right] = x(t)}$$

Similarly, $$y(t) = \cancel{y(0)}^{\,0} + \int_0^t \frac{(p_0/m)\,dt'}{W(1+Z_1^2t'^2)^{1/2}}$$ ← chose $\vec{p}_0 = p_0\hat{y}$ for simplicity

or $$\boxed{y(t) = \frac{p_0}{m}\frac{mc}{qE_0} W \sinh^{-1}\left(\frac{qE_0 t}{mcW}\right)}$$

(b) To find the trajectory in space, solve for t and eliminate it:

$$t = \frac{mcW}{qE_0}\sinh\left(\frac{qE_0 y}{p_0 c}\right)$$

plug into $x(t)$

$$\Rightarrow \boxed{x(y) = \frac{mc^2W}{qE_0}\left\{\left[1+\sinh^2\left(\frac{qE_0 y}{p_0 c}\right)\right]^{1/2} - 1\right\}}$$

For short times, $t \ll \frac{mcW}{qE_0}$, $x(y) \propto y^2$ as in a gravitational constant field

For long times, $t \gg \frac{mcW}{qE_0}$, $\Rightarrow x(y) \sim \frac{mc^2W}{qE_0}e^{qE_0y/p_0c}$