HW Set #7 - C. Greene's solutions

J. 11.6 and 11.22

Problem 11.6 (a) Let K' be the instantaneous rest frame of the rocket, and K be the earth's rest frame

To relate the acceleration in frames K and K', consider a small velocity u' in K' changing at the rate

$$\frac{du'}{dt'} = g$$

The velocity of the ship, according to an observer in the earth frame K, is

$$u = \frac{v + u'}{1 + \frac{vu'}{c^2}}$$

$$\Rightarrow du = \frac{du'}{1 + \frac{vu'}{c^2}} - \frac{(v+u')\frac{v}{c^2}}{\left(1 + \frac{vu'}{c^2}\right)^2} du'$$

$$= \frac{\left(1 + \cancel{\frac{vu'}{c^2}}\right) - \frac{v^2}{c^2} - \cancel{\frac{u'v}{c^2}}}{\left(1 + \frac{vu'}{c^2}\right)^2} du'$$

$$\text{or } du = \frac{1 - \frac{v^2}{c^2}}{\left(1 + \frac{vu'}{c^2}\right)^2}$$

$$\Rightarrow \frac{du}{dt} = \frac{\left(1 - \frac{v^2}{c^2}\right) du'}{\gamma\left(dt' + \frac{v}{c^2}dx'\right)\left(1 + \frac{vu'}{c^2}\right)^2} = \boxed{\frac{\left(1 - v^2/c^2\right)^{3/2}}{\left(1 + \frac{vu'}{c^2}\right)^3} \frac{du'}{dt'} = \frac{du}{dt}}$$

or see Jackson #11.5

(11.6) (a) (p. 2) Or at $u' \to 0$, $\dfrac{du}{dt} = \dfrac{g}{\gamma^3}$

acceleration as seen by an observer on earth

and note that $v = u$ since $u' \to 0$

$$\Rightarrow \frac{du}{dt} = \frac{dv}{dt} \Rightarrow \int_0^v \frac{dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} = gt$$

$$\Rightarrow \quad \gamma v = gt \quad \text{or} \quad v = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}}$$

$$\text{and } \gamma = \frac{gt}{v} = \sqrt{1+\frac{g^2t^2}{c^2}}$$

To relate the clocks in K, K', use

$$dt' = \frac{dt}{\gamma} = \frac{dt}{\sqrt{1+\frac{g^2t^2}{c^2}}} \Rightarrow t' = \int_0^t \frac{dt}{\sqrt{1+\frac{g^2t^2}{c^2}}}$$

$$\Rightarrow t' = \frac{c}{g}\sinh^{-1}\left(\frac{gt}{c}\right) \quad \text{or} \quad t = \frac{c}{g}\sinh\left(\frac{gt'}{c}\right)$$

$\Rightarrow$ using $g = 9.8\ \frac{m}{s^2}$, $t' = 5$ years, $t = 83.8$ years } first stage

second stage is the same as the first stage, by symmetry $\Rightarrow t_{trip} = 4 \times 83.8 = 335$ years

$\Rightarrow$ The date when the rocket returns to earth will be $2100 + 335 = \boxed{2435\ \text{AD}}$

(11.6)(b) During the 1^{st} stage,

$$x = \int_0^t v\,dt = \int_0^t \frac{g\cancel{t}\,dt}{\sqrt{1+\frac{g^2t^2}{c^2}}}$$

or $x = g\frac{c^2}{g^2}\left[\left(1+\frac{g^2t^2}{c^2}\right)^{1/2} - 1\right]$

$$\Rightarrow \frac{x}{c} = \frac{c}{g}\left[\left(1+\frac{g^2t^2}{c^2}\right)^{1/2} - 1\right]$$

$$= 82.8 \text{ light years}$$

And the same distance is covered in stage 2, so the maximum distance from earth is $\boxed{166 \text{ light years}}$

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11.22 E_0 = energy of the background radiation, at $3°K$, i.e. $E_0 = k_B T$, $k_B = 8.6 \times 10^{-5}\ eV/K$

$\Rightarrow$ Consider a photon E_0 colliding head-on with one of energy E_1 in the "universe frame" K.

$\Rightarrow$ In K, the total 4-momentum of the system looks like:

$$p^\alpha = \left(\frac{E_0 + E_1}{c}, \frac{E_1 - E_0}{c}, 0, 0 \right)$$

since the momentum of a photon with energy E is $p = E/c$.

Call K' the CM frame in which the total momentum is zero. In this frame the $e^+ - e^-$ pair produced will have zero momentum, which corresponds to the minimum energy for formation

$\Rightarrow$ In K', $p'^\alpha = \left(\frac{E_{cm}}{c}, 0 \right)$

Now $p^\alpha p_\alpha$ = Lorentz invariant

$$\Rightarrow E_{cm}^2 = (E_0 + E_1)^2 - (E_1 - E_0)^2 = 4 E_0 E_1$$

At threshold $E_{cm} = 2 m_e c^2 = 2\,(5.11 \times 10^5\ eV)$

(a) $\Rightarrow \boxed{E_1 = \frac{(m_e c^2)^2}{E_0} = 1.0 \times 10^{15}\ eV}$ for $E_0 = (3K)\, k_B$

(b) $E_{inc} = \frac{(5.11 \times 10^5\ eV)^2}{500\ eV} = 5.2 \times 10^8\ eV$

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