Problem 10.14

Starting point is J. 10.109:

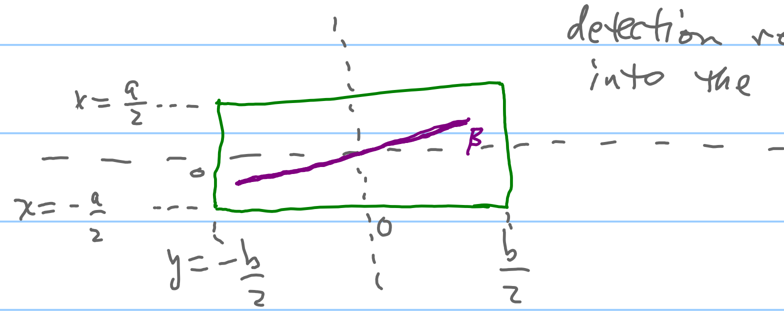
$$\vec{E}(\vec{x}) = \frac{i e^{ikr}}{2\pi r}\,\vec{k} \times \int_{S_1} \hat{n}' \times \vec{E}_{inc}(\vec{x}')\, e^{-i\vec{k}\cdot\vec{x}'}\, da'$$

(a)

Incident polarization vector is

$\hat{e}_0 = \hat{y}\cos\beta + \hat{x}\sin\beta$

detection region is into the page



$$\Rightarrow \vec{E}_{inc}(\vec{x}) = E_0\,(\hat{x}\sin\beta + \hat{y}\cos\beta)\,e^{ikz'} \qquad \hat{n}' = \hat{z}$$

so $\hat{n}' \times \vec{E}_{inc}(\vec{x}') = E_0(\hat{y}\sin\beta - \hat{x}\cos\beta)\cancel{e^{ikz'}}$ $= 1$ @ $z' = 0$

$\Rightarrow$ Integral needed is

$$E_0 \int_{-\frac{a}{2}}^{\frac{a}{2}} dx' \int_{-\frac{b}{2}}^{\frac{b}{2}} dy'\,(\hat{y}\sin\beta - \hat{x}\cos\beta)\,e^{-ik_x x' - ik_y y'}$$

$$= E_0(\hat{y}\sin\beta - \hat{x}\cos\beta)\,\frac{e^{-ik_x\frac{a}{2}} - e^{ik_x\frac{a}{2}}}{(-ik_x)}\,\frac{e^{-ik_y\frac{b}{2}} - e^{ik_y\frac{b}{2}}}{(-ik_y)}$$

$$= \left\{\frac{4E_0}{k_x k_y}(\hat{y}\sin\beta - \hat{x}\cos\beta)\,\sin\frac{k_x a}{2}\,\sin\frac{k_y b}{2}\right\}$$

Next take $\vec{k} \times \{\ \} = \dfrac{4E_0 \sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{k\sin^2\theta\sin\phi\cos\phi}$

$$\times\left\{-\cos\theta\sin\beta,\ -\cos\theta\cos\beta,\ \sin\theta\sin(\phi+\beta)\right\}$$

and so

answer, part (a)

$$\boxed{\vec{E}(\vec{x}) = \frac{ie^{ikr}}{2\pi r}\,\frac{4E_0\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{k\sin^2\theta\sin\phi\cos\phi} \times\left\{-\cos\theta\sin\beta,\ -\cos\theta\cos\beta,\ \sin\theta\sin(\phi+\beta)\right\}}$$

and $$\vec{H} = \frac{\hat{k}\times\vec{E}}{Z_0} = \frac{ie^{ikr}}{2\pi k r Z_0}\, 4E_0 \,\frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\sin\theta\cos\phi}\,\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\sin\theta\sin\phi}$$

$$\times\left\{\cos\beta\cos^2\theta+\sin^2\theta\sin\phi\sin(\beta+\phi),\ -\cos^2\theta\sin\beta-\sin^2\theta\cos\phi\sin(\beta+\phi),\ -\frac{1}{2}\sin 2\theta\cos(\beta+\phi)\right\}$$

and $$\frac{dP}{d\Omega} = \frac{r^2|\vec{E}|^2}{2Z_0} = \frac{2|E_0|^2}{k^2\pi^2 Z_0}\left[\cos^2\theta+\sin^2\theta\sin^2(\phi+\beta)\right]$$

$$\times\frac{\sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\sin^2\theta\cos^2\phi}\,\frac{\sin^2\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\sin^2\theta\sin^2\phi}$$

(b) Scalar Kirchoff approx. is 10.79,

$$\psi(\vec{x}) \xrightarrow[r\to\infty]{} -\frac{e^{ikr}}{r}\frac{1}{4\pi}\int_{S_1}\left[\hat{n}'\cdot\nabla'\psi_{inc}+i\vec{k}\cdot\hat{n}'\psi_{inc}\right]e^{-i\vec{k}\cdot\vec{x}'}da'$$

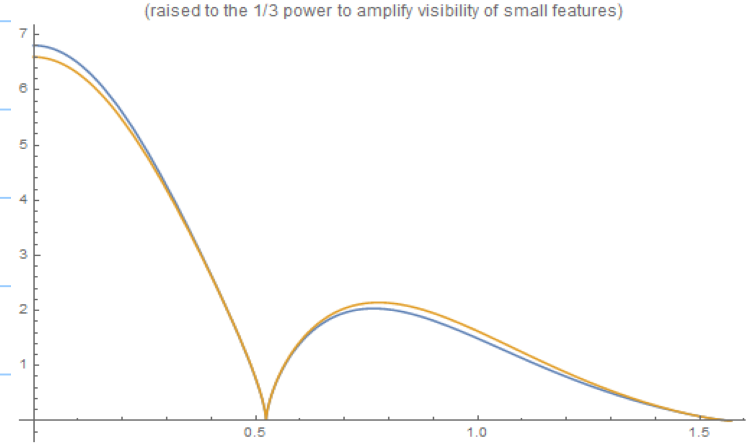
and set $\psi_{inc} \equiv \psi_0 e^{ikz'}$

$$\Rightarrow \psi(\vec{x}) \xrightarrow[r\to\infty]{} -\frac{e^{ikr}}{4\pi r}\psi_0\int dx'dy'\left[\hat{z}\cdot ik\hat{z}+i\vec{k}\cdot\hat{n}'\right]e^{-i(k_x x'+k_y y')}$$

$$= \frac{\psi_0 e^{ikr}}{\cancel{4}\pi i r}\, k\left[1+\cos\theta\right]\,\cancel{4}\,\frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{k\sin\theta\cos\phi}\,\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{k\sin\theta\sin\phi}$$

and $$\frac{dP}{d\Omega} = r^2|\psi|^2 = \frac{|\psi_0|^2}{\pi^2k^2}(1+\cos\theta)^2\,\frac{\sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\sin^2\theta\cos^2\phi}\,\frac{\sin^2\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\sin^2\theta\sin^2\phi}$$

Nearly identical shapes of scalar and vector diffraction
(raised to the 1/3 power to amplify visibility of small features)



Note: the Mathematica notebook that calculated & plotted this figure is posted online.