Homework Set 05 10.9 (a)
Physics
C. Greene's solutions

Problem 10.9 (a) (only part(a) was assigned)

In the Born approximation, with $\delta\mu = 0$
and $\vec{q} = k(\hat{n}_0 - \hat{n})$, we start from:

$$\frac{d\sigma}{d\Omega}(\hat{e},\hat{n};\hat{e}_0,\hat{n}_0) = \left| \frac{k^2}{4\pi} \int d^3x \, \hat{e}^* \cdot \hat{e}_0 \, \frac{\delta\epsilon(\vec{x})}{\epsilon_0} \, e^{i\vec{q}\cdot\vec{x}} \right|^2$$

$$= \frac{k^4}{16\pi^2} |\hat{e}^*\cdot\hat{e}_0|^2 \left| 2\pi \int_{-1}^{1} d(\cos\theta) \int_0^a r^2 dr \, e^{iqr\cos\theta} \right|^2 \left|\frac{\delta\epsilon}{\epsilon_0}\right|^2$$

$$= \frac{k^4}{4} \left|\frac{\delta\epsilon}{\epsilon_0}\right|^2 |\hat{e}^*\cdot\hat{e}_0|^2 \left| \int_0^a r^2 \frac{e^{iqr} - e^{-iqr}}{iqr} dr \right|^2$$

$$= k^4 \left|\frac{\delta\epsilon}{\epsilon_0}\right|^2 |\hat{e}^*\cdot\hat{e}_0|^2 \left| \frac{1}{q^3} \int_0^{qa} \sin qx \; x\,dx \right|^2$$

$$= k^4 \left|\frac{\delta\epsilon}{\epsilon_0}\right|^2 |\hat{e}^*\cdot\hat{e}_0|^2 \left(\frac{\sin qa - qa\cos qa}{q^3} \right)^2$$

Now recall that $q = 2k\sin\frac{\theta}{2}$
where θ = angle between $\hat{n}$ and $\hat{n}_0$.

We consider small angle scattering, $\theta \ll 1$

$$\Rightarrow \quad q \simeq 2k\frac{\theta}{2} \simeq k\theta$$

$$\Rightarrow \quad \left(\;\;\right)^2 \xrightarrow[qa \gg 1]{} \frac{a^2\cos^2 qa}{q^4} \propto \frac{1}{\theta^4} \quad \text{at small } \theta, \quad \frac{1}{ka} \ll \theta \ll 1$$

and the total cross section in this limit is

$$\sigma = 2\pi \int_0^{\cancel{\pi} \to \infty \text{ (good approx)}} \sin\theta \left(\frac{\sin(ka\theta) - ka\theta \cos(ka\theta)}{(k\theta)^3} \right)^2 \frac{a^6}{a^6}\, d\theta$$

$$\times\, k^4 \left| \frac{\delta \epsilon}{\epsilon_0} \right|^2 \left\{ \frac{1}{2}\left(1+\cos^2\theta\right) \right\}$$

$|\hat{e}^* \cdot \hat{e}_0|^2 \simeq 1$ since $\theta \simeq 0$ is where $\frac{d\sigma}{d\Omega}$ peaks

Now use

$$\int_0^\infty x \left(\frac{\sin x - x\cos x}{x^3} \right)^2 dx = \frac{1}{4}$$

(from Mathematica)

$$\Rightarrow \boxed{\sigma = \frac{\pi}{2} k^2 a^4 \left| \epsilon_r - 1 \right|^2}$$