Homework Set 4
Physics 631
C. Greene's solutions

J 9.22, 9.23

#9.22 (a, b)

We have a spherical resonant cavity inside a perfect conductor

From 9.122, the relevant solutions to Maxwell's equations are written in terms of electric (E) and magnetic (M) multipole coefficients as:

$$\vec{H} = \sum_{lm} \left(a_E^{lm}\, j_l(kr) \vec{X}_{lm} - \frac{i}{k} a_M^{lm}\, \nabla \times j_l(kr) \vec{X}_{lm} \right)$$

$$\vec{E} = Z_0 \sum_{lm} \left(\frac{i}{k} a_E^{lm}\, \nabla \times j_l(kr) \vec{X}_{lm} + a_M^{lm}\, j_l(kr) \vec{X}_{lm} \right)$$

And the boundary conditions at $r = a$ are:

$$E_{\parallel} = 0 \,, \qquad H_{\perp} = 0$$

Case (1) TE = M modes, i.e set $a_E^{lm} \to 0$

Now, since $\vec{X}_{lm}$ has only angular components,

$$\Rightarrow \vec{E}_{\ell m} = Z_0\, j_\ell(kr)\, \vec{X}_{\ell m}(\theta,\phi)$$

$$\vec{H}_{\ell m} = -\frac{i}{k} \nabla \times \left[j_\ell(kr)\, \vec{X}_{\ell m}(\theta,\phi) \right]$$

Therefore $\vec{E}_{\parallel}$ will vanish at $r=a$ if $j_\ell(ka)=0$

$$\Rightarrow k_{\ell,p} = \frac{s_{\ell,p}}{a} \quad \text{where } s_{\ell,p} = p^{th} \text{ zero of } j_\ell(s).$$

Note that since $\vec{X}_{00} = \vec{L}\, Y_{00} = 0$, $\ell = 0$ could only give the trivial solution, of no interest. So the lowest 4 modes, obtained using Mathematica's command BesselJZeros[$L+\frac{1}{2}$, n], are

ℓ	p	$s_{\ell,p} = \frac{\omega_{\ell,p}}{c/a}$	$\lambda_{\ell,p}$ "$\frac{2\pi a}{s_{\ell,p}}$"
1	1	4.49341	1.40a
2	1	5.76346	1.09a
3	1	6.98793	0.90a
1	2	7.72525	0.81a

<u>Case (2)</u> TM = E modes $\Rightarrow a_M \to 0$

and $$\vec{H}_{\ell m} = j_\ell(kr)\, \vec{X}_{\ell m}$$

$$\vec{E}_{\ell m} = \frac{iZ_0}{k} \nabla \times \left[j_\ell(kr)\, \vec{X}_{\ell m}(\theta,\phi) \right]$$

Actually, let's first check whether the BC $H_\perp = 0$ is satisfied in case (1):

$$\nabla\times(\psi\vec{v}) = \nabla\psi\times\vec{v} + \psi\nabla\times\vec{v}$$
$$\nabla j_\ell = \hat{r}\frac{\partial}{\partial r} j_\ell(kr)$$

$$\hat{r}\cdot\vec{H}_{\ell m} = \hat{r}\cdot\left(\frac{-i}{k}\nabla\times\left[j_\ell(kr)\vec{X}_{\ell m}\right]\right)$$
$$= \hat{r}\left(\frac{-i}{k}\right)\cdot\left[\underbrace{\nabla j_\ell(kr)\times\vec{X}_{\ell m}}_{=0} + j_\ell(kr)\nabla\times\vec{X}_{\ell m}\right]$$

while $\nabla\times\vec{L}Y_{\ell m} = -i\vec{r}\nabla^2 Y_{\ell m} + i\nabla\left(1+\cancelto{0}{r\frac{\partial}{\partial r}}\right)Y_{\ell m}$

(using 9.125) $$= -\frac{i\vec{r}}{r^2}\left[-\ell(\ell+1)\right]Y_{\ell m} + i\nabla Y_{\ell m}$$

and so finally,
$$\hat{r}\cdot\vec{H}_{\ell m} = \frac{i}{r}\ell(\ell+1)Y_{\ell m}\,j_\ell(kr)$$

so this means that the BC $H_\perp = 0$ is satisfied at $r=a$ after we have satisfied the $E_\parallel = 0$ BC for M modes by choosing $k = \frac{s_{\ell,p}}{a}$!

Now, returning to the E-modes, they automatically obey $H_\perp = 0 = \hat{r}\cdot\vec{H}_{\ell m}$ for any k, since $\vec{H}_{\ell m}$ has only $\hat{\theta},\hat{\phi}$-components.

Thus we must impose $E_\parallel = 0$, using some of the above formulas, i.e.

$$\nabla \times \left[j_\ell(kr) \vec{L}\, Y_{\ell m} \right] = \frac{\partial j_\ell(kr)}{\partial r}\, \hat{r} \times (\vec{L}\, Y_{\ell m})$$

$$+ \; j_\ell(kr)\, \nabla \times \vec{L}\, Y_{\ell m}$$

$$= \frac{\partial j_\ell(kr)}{\partial r}\, \underbrace{\hat{r} \times (\vec{r} \times \vec{p})\, Y_{\ell m}} + j_\ell(kr)\left[\cancel{+\frac{i\vec{r}}{r^2}\ell(\ell+1) Y_{\ell m}} + i \nabla Y_{\ell m} \right]$$

$$= \left[\vec{r}\, \cancel{(\hat{r} \cdot \vec{p})} - r\vec{p} \right] Y_{\ell m}$$

$$\cancel{(\hat{r}\cdot\vec{p})} \to 0 \;, \quad \vec{p} = -i\nabla$$

does not contribute to $E_{\parallel}$, so ignore

$\parallel$- components in $\hat{\theta}, \hat{\phi}$

$$0 = i \left(r \frac{\partial j_\ell(kr)}{\partial r}\bigg|_{r=a} + j_\ell(kr)\bigg|_{r=a} \right) \nabla Y_{\ell m}$$

and therefore the $E_{\parallel} = 0$ BC for E modes will be satisfied if k obeys:

$$\boxed{ka\, j_\ell'(ka) + j_\ell(ka) = 0}, \quad \text{where } j_\ell'(z) \equiv \frac{d}{dz} j_\ell(z)$$

In Mathematica, find roots as follows:

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jj[L_, z_] := (π/(2z))^(1/2) BesselJ[L + 1/2, z]
jjD[L_, z_] := D[jj[L, x], x] /. x -> z
FindRoot[z jjD[L, z] + jj[L, z] == 0,
    {z, 3 + L}]
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Call $s'_{\ell,p} = p^{th}$ zero of $s' j_\ell'(s') + j_\ell(s') = 0$

i.e $\frac{d}{ds'}[s' j_\ell(s')] = 0$

which gives the following for the lowest 4 or 5 roots:

ℓ	p	$s'_{\ell,p} = \dfrac{\omega'_{\ell,p}}{c/a}$	$\lambda'_{\ell,p}/a$
1	1	2.744	2.29
2	1	3.870	1.62
3	1	4.973	1.26
4	1	6.062	1.04
1	2	6.117	1.03

(c) The lowest TM mode and the lowest TE mode both have $\ell = 1$, and 3 m-values $m = 0, \pm 1$. Here I work out just the $m = 0$ cases, for simplicity:

$\underline{TE_{p\ell m} = TE_{110}}$

$$\vec{E}^{(TE)}_{110} = Z_0\, j_1(kr)\, \vec{X}_{10}(\theta,\phi)$$

$$\vec{H}^{(TE)}_{110} = \frac{-i}{k} \nabla\times\left[j_1(kr)\, \vec{X}_{10}(\theta,\phi)\right]$$

I will solve this using the conventions of Edmonds and/or Varshalovic et al. p.217, 7.3.6.55 or Edmonds 5.9.19, p.84:

$$\nabla\times(j_\ell \vec{X}_\ell) = \nabla\times\left(j_\ell(kr)\, \vec{Y}^{\ell}_{\ell M}\right)$$

$$= \left(\frac{\ell}{2\ell+1}\right)^{1/2} i\left(\frac{d}{dr} j_\ell(kr) - \frac{\ell}{r} j_\ell(kr)\right)\vec{Y}^{\ell+1}_{\ell M} + i\left(\frac{dj_\ell}{dr} + \frac{(\ell+1)}{r} j_\ell\right)\left(\frac{\ell+1}{2\ell+1}\right)^{1/2}\vec{Y}^{\ell-1}_{\ell M}$$

and explicit expressions were obtained in Mathematica, VectorSphericalHarmonics.nb e.g. $\vec{Y}^{\,1}_{1,0}(\theta,\phi) = Y_{11}\,\hat{e}_{-1}\,\langle 11, 1-1 | 10\rangle + Y_{1,-1}\,\hat{e}_{1}\,\langle 1-1, 11 | 10\rangle$

(note: $\langle 10, 10 | 10\rangle = 0$)

$$= \frac{1}{\sqrt{2}}\left(-\frac{1}{2}e^{i\phi}\sqrt{\frac{3}{2\pi}}\sin\theta\right)\left(\frac{1}{\sqrt{2}}(\hat{x}-i\hat{y})\right)$$

$$-\frac{1}{\sqrt{2}}\left(\frac{1}{2}e^{-i\phi}\sqrt{\frac{3}{2\pi}}\sin\theta\right)\left(-\frac{1}{\sqrt{2}}(\hat{x}+i\hat{y})\right)$$

$$= \left(-\hat{x}\sin\phi + \hat{y}\cos\phi\right)\sqrt{\frac{3}{2\pi}}\,\frac{\sin\theta}{2}\,i = \vec{X}_{10}(\theta,\phi) \checkmark$$

The full expression for the $\nabla\times\left(j_1(kr)\vec{X}_{10}\right)$ is written out in full in the Mathematica notebook.

Then the same formulas also apply to the TM mode, with the different value of k substituted, and the following interchange:

$$\vec{H}^{TM}_{110} = \vec{E}^{TE}_{110} / Z_0$$

$$\vec{E}^{TM}_{110} = -Z_0\,\vec{H}^{TE}_{110}$$

<u>9.23</u> The cavity Q is given by:

$$Q \equiv \omega_0 \frac{\text{Stored Energy}}{\text{Rate of power loss}} = \frac{\omega_0 U}{\frac{dU}{dt}}$$

where the time-averaged energy inside for a TE mode is:

$$U^{TE} = \int d^3x \left\{ \frac{\epsilon_0}{4} |\vec{E}|^2 + \frac{\mu_0}{4} |\vec{H}|^2 \right\}$$

$$= \int_0^a r^2 dr \int d\Omega \left\{ \frac{\epsilon_0}{4} \left| Z_0 j_\ell(s_{\ell,p} \tfrac{r}{a}) \vec{X}_{\ell m} \right|^2 \right.$$

$$\left. + \frac{\mu_0}{4} \left| \frac{-i}{\frac{s_{\ell,p}}{a} Z_0} \nabla \times \left(Z_0 j_\ell(s_{\ell,p} \tfrac{r}{a}) \vec{X}_{\ell m} \right) \right|^2 \right\}$$

$$= \frac{\epsilon_0}{4} Z_0^2 \int_0^a r^2 j_\ell^2(s_{\ell,p} \tfrac{r}{a}) dr \longrightarrow \frac{\epsilon_0 Z_0^2}{4 a^{-3}} \int_0^1 x^2 j_\ell^2(s_{\ell,p} x) dx$$

$$+ \frac{\mu_0}{4} \int_0^a r^2 dr \left\{ \left(\frac{d j_\ell(s_{\ell,p} \frac{r}{a})}{dr} - \frac{\ell j_\ell(s_{\ell,p} \frac{r}{a})}{r} \right)^2 \frac{\ell}{2\ell+1} \right.$$

$$\left. + \left(\frac{d j_\ell(s_{\ell,p} \frac{r}{a})}{dr} + \frac{\ell+1}{r} j_\ell(s_{\ell,p} \tfrac{r}{a}) \right)^2 \frac{\ell+1}{2\ell+1} \right\}$$

(used last Eq. on p.05 above, + orthonormality of the vector spherical harmonics)

also recall that $\epsilon_0 Z_0^2 = \mu_0$, or calling $k = \frac{s_{\ell,p}}{a}$

$$\Rightarrow U = \frac{\mu_0}{4k^3} \int_0^{s_{\ell,p}} x^2 \left\{ \left[j_\ell'(x) - \frac{\ell j_\ell(x)}{x} \right]^2 \frac{\ell}{2\ell+1} + j_\ell^2(x) \right.$$

$$\left. + \left[j_\ell'(x) + \frac{\ell+1}{x} j_\ell(x) \right]^2 \frac{\ell+1}{2\ell+1} \right\} dx$$

$$\Rightarrow U = \frac{\mu_0 a^3}{4 s_{\ell,p}^3} \left\{ s_{\ell,p}^3 \left[j_\ell'(s_{\ell,p}) \right]^2 \right\}$$ (see next pages)

To see how to evaluate this integral, return to the first equation for U^{TE} above on p.07, and set $k = s_{\ell,p}/a$:

$$U^{TE} = \int d^3x \left\{ \frac{\mu_0}{4} j_\ell^2(kr) |\vec{X}_{\ell m}|^2 + \frac{\mu_0}{4k^2} |\nabla \times j_\ell(kr) \vec{X}_{\ell m}|^2 \right\}$$

and through some vector spherical harmonic identities, e.g. Jackson 10.48, p. 472, this simplifies to:

$$U^{TE} = \frac{\mu_0}{4} \int_0^a r^2 dr \left\{ \underbrace{2 j_\ell^2(kr)}_{\text{integral}_1} + \underbrace{\frac{1}{k^2 r^2} \frac{\partial}{\partial r} \left[r j_\ell(kr) \frac{\partial}{\partial r} (r j_\ell(kr)) \right]}_{\text{integral}_2} \right\}$$

First, notice that integral$_2$ is easily done because the integrand is a perfect derivative, i.e.

$$\text{integral}_2 = \frac{1}{k^2} r j_\ell(kr) \frac{\partial}{\partial r} (r j_\ell(kr)) \Big|_{\cancel{0}}^{a}$$

$\rightarrow$ lower limit $\rightarrow 0$ because $r j_\ell(kr) \xrightarrow[r\to 0]{} 0$

$$= \frac{1}{k^3} \underbrace{(x j_\ell(x))}_{\substack{\text{vanishes for} \\ \text{TE modes}}} \underbrace{(x j_\ell(x))'}_{\substack{\text{vanishes for} \\ \text{TM modes}}} \Big|_{x = ka} = 0 \,!$$

Thus $U^{TE} = 2 \frac{\mu_0}{4} \int_0^a dr \, (r j_\ell(kr))^2$

To evaluate this, consider the slightly more general integral $I(k,k') = \int_0^a dr \, [r j_\ell(kr)] [r j_\ell(k'r)]$

we will take the limit $k \to k'$ later

Now recall the differential equation obeyed, namely

$$\hat{\mathcal{L}}\left(r j_\ell(kr)\right) = k^2\left(r j_\ell(kr)\right)$$

where $\hat{\mathcal{L}} = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}$, and hence:

$$I(k,k') = \frac{\int_0^a dr \left\{ r j_\ell(kr)\, \hat{\mathcal{L}}[r j_\ell(k'r)] - [\hat{\mathcal{L}}\, r j_\ell(kr)]\, r j_\ell(k'r) \right\}}{k'^2 - k^2}$$

$$= (k'^2 - k^2)^{-1} \int_0^a dr \left\{ \frac{d^2}{dr^2}\left(r j_\ell(kr)\right) r j_\ell(k'r) - r j_\ell(kr) \frac{d^2}{dr^2}\left(r j_\ell(k'r)\right) \right\}$$

$$= (k'^2 - k^2)^{-1} \int_0^a \frac{d}{dr} W\left(r j_\ell(k'r), r j_\ell(kr)\right) dr$$

$$= (k'^2 - k^2)^{-1}\, W[r j_\ell(k'r), r j_\ell(kr)]_{r=a}$$

(the $r=0$ contribution vanishes at all k, k')
(recall, the Wronskian definition is $W(y,z) = yz' - y'z$)

$\to$ to evaluate this $\frac{0}{0}$ limit as $k' \to k$, set $k' = k + \Delta$ and expand the numerator and denominator to 1st order in Δ, which is akin to de l'Hospital's rule.

Using $j_\ell[(k+\Delta)r] = j_\ell(kr) + r\Delta\, j_\ell'(kr) + O(\Delta^2)$,

$$\Rightarrow I(k,k) = \lim_{\Delta \to 0} \left(\frac{1}{2k\Delta}\right) \left\{ r j_\ell[(k+\Delta)r] \frac{d}{dr}\left(r j_\ell(kr)\right) - \frac{d}{dr}\left[r j_\ell((k+\Delta)r)\right] r j_\ell(kr) \right\}_{r=a}$$

or simplifying,

$$\boxed{I(k,k) = \frac{1}{2k}\left\{ a\, j_\ell'(ka)\frac{d}{dx}\left(x\, j_\ell(kx)\right) - a\, j_\ell(ka)\frac{d}{dx}\left(x\, j_\ell'(x)\right)\right\}_{x=ka}}$$

$\Longrightarrow$ Through this device, all integrals needed for this problem can be evaluated analytically.

e.g. for TE modes,

$$U^{TE} = \frac{\mu_0}{2} I(k,k)\Big|_{k=s_{\ell,p}/a} \qquad \text{where } j_\ell(s_{\ell,p}) = 0$$

$$= \frac{\mu_0}{4} a^3 \left(\frac{d}{dx}\left[j_\ell(x)\right]\right)^2_{x\to s_{\ell,p}}$$

Now, to complete this problem we need to evaluate the rate of power loss,

$$P_{loss} = \frac{\mu_0 \omega \delta}{4} \int_S da\, |\hat{n}\times\vec{H}|^2$$

where for TE modes, $\vec{H}^{TE}_{p\ell m} = \frac{-i}{k}\nabla\times j_\ell(kr)\vec{X}_{\ell m}$

and $k = s_{\ell,p}/a$

and

$$\nabla\times[j_\ell\vec{X}_{\ell m}] = i\left(\frac{dj_\ell}{dr} - \frac{\ell j_\ell}{r}\right)\left(\frac{\ell}{2\ell+1}\right)^{1/2}\vec{Y}^{\ell+1}_{\ell m}$$

$$+ i\left(\frac{d}{dr}j_\ell + j_\ell\frac{\ell+1}{r}\right)\left(\frac{\ell+1}{2\ell+1}\right)^{1/2}\vec{Y}^{\ell-1}_{\ell m}$$

and using Varshalovich 7.3.73 on p 220:

$$\hat{n} \times \vec{Y}^{\,l+1}_{lm} = i \left(\frac{l}{2l+1}\right)^{1/2} \vec{X}_{lm}$$

$$\hat{n} \times \vec{Y}^{\,l-1}_{lm} = i \left(\frac{l+1}{2l+1}\right)^{1/2} \vec{X}_{lm} \quad , \quad \text{giving}$$

$$P^{TE}_{loss} = \frac{\mu_0 \omega \delta}{4} a^2 \frac{1}{k^2} \left\{ \frac{-l}{2l+1} \left(\frac{dj_l}{dr} - \frac{l j_l}{r} \right) - \frac{(l+1)}{2l+1} \left(\frac{dj_l}{dr} + \frac{l+1}{r} j_l \right) \right\}^2$$

$$= \frac{\mu_0 \omega \delta}{4} a^2 \left[j_l'(s_{l,p}) \right]^2$$

so $$\boxed{Q^{TE} = \frac{\omega U^{TE}}{P^{TE}_{loss}} = \frac{a}{\delta}}$$ for all l, m!

amazing simplification!

Next work out Q^{TM}, and we see at once that $U^{TM}_E = \underbrace{\left(Z_0^2 \frac{\epsilon_0}{\mu_0}\right)}_{=1} U^{TE}_B$, $U^{TM}_B = \underbrace{\left(\frac{\mu_0}{Z_0^2 \epsilon_0}\right)}_{=1} U^{TE}_E$

except $s_{l,p} \to s'_{l,p}$

and $$P^{TM}_{loss} = \frac{\mu_0 \omega \delta}{4} a^2 \left[j_l(s'_{l,p}) \right]^2$$

and $$U^{TM} = \frac{\mu_0 a^3}{4 s'^{\,3}_{l,p}} \left\{ s'^{\,3}_{l,p} \left[j_l(s'_{l,p}) \right]^2 \left(1 - \frac{l(l+1)}{s'^{\,2}_{l,p}} \right) \right\}$$

$$\Rightarrow \boxed{Q^{TM}_l = \frac{\omega U^{TM}}{P^{TM}_{loss}} = \left(1 - \frac{l(l+1)}{s'^{\,2}_{l,p}} \right) \frac{a}{\delta}}$$ $\Rightarrow$ again, amazingly simple in the end!