Problem 9.10

As stated in the problem, for a $2p\phi \to 1s\phi$ transition in atomic hydrogen, we are given that:

$$\rho(r,\theta,\phi,t) = \frac{2e}{a_0^4\sqrt{6}}\, r\, e^{-3r/2a_0}\, Y_{00} Y_{10}\, e^{-i\omega_0 t}$$

$$\vec{J}(r,\theta,\phi,t) = -\frac{i v_0}{2}\left(\frac{\hat{r}}{2} + \frac{a_0}{z}\hat{z}\right)\rho(r,\theta,\phi,t)$$

where $a_0 = \dfrac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529\times 10^{-10}\,\text{m} =$ Bohr radius

$$\frac{E_{2p}-E_{1s}}{\hbar} = \omega_0 = \frac{3e^2}{32\pi\epsilon_0\hbar a_0}, \quad v_0 = \frac{e^2}{4\pi\epsilon_0\hbar} = \alpha c \simeq \frac{c}{137}$$

Note that
$$\hat{r} = \hat{z}\cos\theta + \hat{\rho}\sin\theta$$
$$\hat{\theta} = -\hat{z}\sin\theta + \hat{\rho}\cos\theta$$
so
$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

so
$$\frac{\hat{r}}{2} + \frac{a_0}{z}\hat{z} = \frac{\hat{r}}{2} + \frac{a_0}{r\cos\theta}\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)$$

$$= \hat{r}\left(\frac{1}{2} + \frac{a_0}{r}\right) - \hat{\theta}\,\frac{a_0}{r}\,\frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \vec{J} = \frac{i v_0}{2}\left[\hat{r}\left(\frac{1}{2} + \frac{a_0}{r}\right) - \hat{\theta}\,\frac{a_0}{r}\tan\theta\right]\rho(r,\theta,\phi,t)$$

Then the magnetic dipole moment per unit volume $\equiv$ "Magnetization" $= \vec{M} = \frac{1}{2}\vec{r}\times\vec{J}$

$$\Rightarrow \vec{M} = \frac{r}{2}\begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 1 & 0 & 0 \\ \frac{1}{2}+\frac{a_0}{r}, & -\frac{a_0\tan\theta}{r} & 0 \end{vmatrix}\rho(r,\theta,\phi,t)\left(-\frac{i v_0}{2}\right) = \frac{-i v_0 r}{4}\rho(r,\theta,\phi,t)\left(\frac{-a_0\tan\theta}{r}\right)\hat{\phi}$$

and since $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$

so

$$\vec{M} = -\underbrace{i\alpha c a_0}_{4} \tan\theta\,(\hat{x}\sin\phi - \hat{y}\cos\phi)\,\rho(r,\theta,\phi,t)$$

so by inspection, $\nabla\cdot\vec{M} = 0$

Next, calculate the multipoles:

Observe that $\vec{m}$ and $\overleftrightarrow{Q}$ are even parity while ρ is odd parity in this problem, while the electric dipole moment operator $\vec{p}$ is odd parity, so we conclude that only $p_z = \int \rho\, z\, d^3x$ is nonzero, giving a simple integral equal to

$$\vec{p} = \hat{z}\, e\, a_0 \sqrt{2}\,\frac{256}{243} \simeq 1.49\, e\, a_0\, \hat{z}$$

(b) The time-averaged power radiated is

$$P = \frac{c^2 Z_0 k^4}{12\pi}|\vec{p}|^2 = \frac{c^2 Z_0 k^4}{12\pi}\, 2\left(\frac{256}{243}\right)^2 e^2 a_0^2$$

so

$$\frac{P}{\hbar\omega_0\left(\frac{\alpha^4 c}{a_0}\right)} = \frac{256}{6561} \approx .0390$$, when simplified in Mathematica

$$\Rightarrow P = \frac{256}{6561}\hbar\omega_0\frac{\alpha^4 c}{a_0}, \quad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$$

(c) Equate the average power radiated by this electric dipole to

$$P = \hbar \omega_0 \gamma$$

where we interpret γ as the probability per second to emit one photon of energy $\hbar\omega_0$

$$\Rightarrow \gamma = \frac{P}{\hbar\omega_0} = \frac{256}{6561}\left(\frac{1}{137}\right)^4 \left(\frac{3\times10^8 \text{ m/s}}{0.529\times10^{-10} \text{ m}}\right)$$

giving $\gamma = 6.28\times10^8 \text{ s}^{-1}$

This corresponds to a mean lifetime equal to:

$$\tau_{2p} = \frac{1}{\gamma} = 1.59\times10^{-9} \text{ s} = 1.6 \text{ ns}$$

which agrees with experiment.

(d) Now consider an electron in a circular orbit, in the xy-plane, with frequency ω_0 and radius $2a$

Cartesian components

$$\Longrightarrow \vec{p} = -e\vec{x} = -e(2a_0)\begin{pmatrix}\cos\omega_0 t\\ \sin\omega_0 t\\ 0\end{pmatrix}$$

or $\vec{p} = -2ea_0 e^{-i\omega_0 t}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}$

To determine the average power radiated, use (9.21):

$$\frac{dP^{classical}}{d\Omega} = \frac{1}{2} \mathrm{Re}\left[r^2 \hat{n} \cdot \vec{E} \times \vec{H}^*\right]$$

$$= \frac{1}{2} \mathrm{Re}\left\{\hat{n} \cdot \left[(\hat{n} \times \vec{p}) \times \hat{n}\right] \times (\hat{n} \times \vec{p}^*)\right\}\left(\frac{ck^2}{4\pi}\right)^2 Z_0$$

$$= \left(\frac{c\,\omega_0^2}{4\pi c^2}\right)^2 Z_0 \,\frac{4e^2a_0^2}{2} \,\mathrm{Re}\left\{\frac{3+\cos 2\theta}{2}\right\}$$

used Mathematica for this step and to integrate $\int d\Omega \{\ \} = \frac{16\pi}{3}$

$$\Rightarrow P^{classical} = \frac{c^2 k_0^4 Z_0}{8\pi^2} \,\frac{16\pi}{3}\, e^2 a_0^2$$

or $$P^{classical} = \frac{2c^2k_0^4 Z_0}{3\pi} e^2 a_0^2$$

Rewriting this last expression in terms of $|\vec{p}|^2 = 4e^2a_0^2$,

$$\Rightarrow P^{classical} = \frac{c^2 k_0^4 Z_0}{6\pi} |\vec{p}|^2,$$ which is twice what we would have found if we had incorrectly applied Eq. 9.24

(d)

$$\Rightarrow \frac{P^{classical}}{P(\text{part b})} = \frac{2c^2k_0^4 Z_0 e^2 a_0^2/3\pi}{\left[\frac{c^2 Z_0 k_0^4}{12\pi}\, 2\left(\frac{256}{243}\right)^2 e^2 a_0^2\right]} = \frac{59049}{16384} \simeq 3.60$$

i.e. the classical system would radiate 3.6 times faster than in part (a)