Phys 631 Homework 02

C. Greene's solutions

Problem 8.2

First find the $\vec{E}$ and $\vec{H}$-fields associated with the TEM-mode in the region between two concentric cylinders of radii a, b:

$\Rightarrow$ Recall that $\vec{E}$ obeys the equations of 2-dim^l electrostatics, namely

$$\nabla_t \times \vec{E} = 0 \quad , \quad \nabla_t \cdot \vec{E} = 0$$

Let's use the azimuthal symmetry to write the 2D scalar potential as

$$\Phi(\rho,\phi) \longrightarrow \Phi(\rho) \quad , \text{and } \vec{E} = -\nabla_t \Phi$$

Then $\nabla_t^{\,2}\Phi(\rho) = 0 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\underbrace{\cancel{\frac{\partial^2\Phi}{\partial\phi^2}}}_{\to\,0}$

$\Rightarrow \rho\frac{\partial\Phi}{\partial\rho} = A$, for some constant A_1

and $\frac{\partial\Phi}{\partial\rho} = \frac{A_1}{\rho} \Rightarrow \Phi = A_1 \ln\rho + A_2$

Setting the potential difference equal to V_0,

i.e. $V_0 \equiv \Phi(b) - \Phi(a) = A_1 \ln\frac{b}{a}, \Rightarrow A_1 = \frac{V_0}{\ln\frac{b}{a}}$

whereby $\vec{E} = \frac{-A}{\rho}\hat{\rho} = -\frac{V_0}{\ln\frac{b}{a}}\frac{\hat{\rho}}{\rho}$

and from (8.28), $\vec{H} = \left(\frac{\epsilon}{\mu}\right)^{1/2} \hat{z} \times \vec{E}$

$\Rightarrow \vec{H} = -\left(\frac{\epsilon}{\mu}\right)^{1/2} \frac{V_0}{\ln\frac{b}{a}} \frac{1}{\rho}\hat{\phi} \equiv -H_0 \frac{a}{\rho}\hat{\phi}$

And of course both $\vec{E}$ and $\vec{H}$ have an implied factor of $e^{i(kz-\omega t)}$,

where $k = \omega\sqrt{\mu\epsilon}$

(8.2) (a) The t-averaged power flow down the waveguide (transmission line) is

$$P = \int_A \frac{1}{2}\vec{E}\times\vec{H}^{*}\, da = \frac{1}{2}\frac{|V_0|^2}{(\ln\frac{b}{a})^2}\left(\frac{\epsilon}{\mu}\right)^{1/2} 2\pi \int_a^b \frac{\rho\, d\rho}{\rho^2}$$

$$\Rightarrow P = \frac{\pi |V_0|^2}{(\ln\frac{b}{a})^2}\left(\frac{\epsilon}{\mu}\right)^{1/2} \ln\frac{b}{a}$$

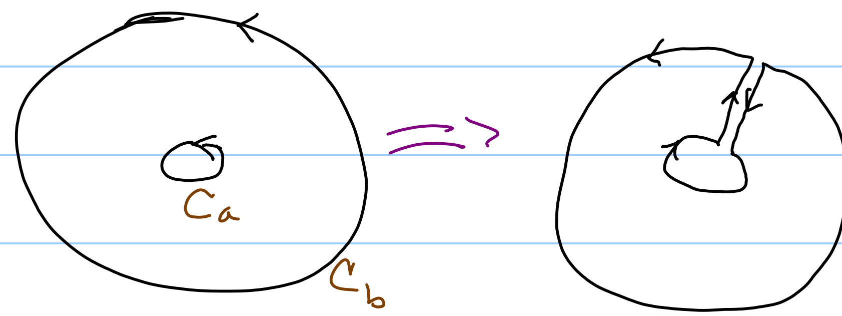
or if we denote $H_0 \equiv \left(\frac{\epsilon}{\mu}\right)^{1/2}\frac{V_0}{\ln\frac{b}{a}}\frac{1}{a}$

$$\Rightarrow P = \pi a^2 \left(\frac{\mu}{\epsilon}\right)^{1/2} |H_0|^2 \ln\frac{b}{a}$$

8.2 (b) Now, from Sec. 8.1 and Eq. 8.58,

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \oint_C |\hat{n}\times\vec{H}|\, dl$$

$\Rightarrow$ Here the closed contour boundary of the area A has two disconnected parts:



$$\Rightarrow -\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\oint_{C_a} |\hat{n}\times\vec{H}|^2 dl + \oint_{C_b} |\hat{n}\times\vec{H}|^2 dl\right)$$

$$= \frac{1}{2\sigma\delta}\left(2\pi a |H_0|^2 \frac{a^2}{a^2} + 2\pi b |H_0|^2 \frac{a^2}{b^2}\right)$$

$$\Rightarrow -\frac{dP}{dz} = \frac{\pi |H_0|^2}{\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)a^2$$

Thus in the loss equation $P(z) = P_0\, e^{-2\gamma z}$, we have

$$\gamma = \frac{-dP/dz}{2P} = \frac{\pi |H_0|^2 \left(\frac{1}{a}+\frac{1}{b}\right)\frac{a^2}{\sigma\delta}}{2\pi a^2 |H_0|^2 \left(\frac{\mu}{\epsilon}\right)^{1/2} \ln\frac{b}{a}}$$

or

$$\gamma = \frac{1}{2\sigma\delta}\left(\frac{\epsilon}{\mu}\right)^{1/2} \frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\frac{b}{a}}$$

8.2c The current flowing on the conductor surfaces is, from (8.14):

$$\vec{K}_{eff} = \hat{n} \times \vec{H}_{\parallel} = \begin{cases} (-\hat{\rho}) \times H_\phi \hat{\phi}, & \rho = b \\ (+\hat{\rho}) \times H_\phi \hat{\phi}, & \rho = a \end{cases}$$

$$= -H_\phi(\rho=b)\,\hat{z}, \quad \text{at outer surface}$$

$$= H_\phi(\rho=a)\,\hat{z}, \quad \text{at inner surface}$$

$$\Rightarrow \vec{I}_{outer} = \int_{\rho=b} \vec{K}_{eff}\, dl = \hat{z}\int_0^{2\pi} - \left[-H_0 \frac{a}{\cancel{\rho}\ a}\right] a\, d\phi = 2\pi a H_0 \hat{z}$$

whereas

$$\vec{I}_{inner} = \int_{\rho=a} \vec{K}_{eff}\, dl = \hat{z}\int_0^{2\pi} + \left[-H_0 \frac{a}{\cancel{\rho}\ b}\right] b\, d\phi$$

$$= -2\pi H_0 a\, \hat{z}$$

So the inner and outer conductors carry equal and opposite currents, as is expected and desired, namely $I = 2\pi a H_0$

The transmission line impedace is then

$$Z_0 = \frac{V}{I} = V_0 \left[\left(\frac{\epsilon}{\mu}\right)^{1/2} \frac{V_0}{\ln\frac{b}{a}} \frac{2\pi a}{a} \right]^{-1}$$

or

$$Z_0 = \left(\frac{\mu}{\epsilon}\right)^{1/2} \frac{\ln \frac{b}{a}}{2\pi}$$

(8.2 d) The time-averaged power loss per unit length can be related to the resistance per unit length $\frac{dR}{dz}$:

$$\frac{dP_{loss}}{dz} = \frac{|I_0|^2}{2}\frac{dR}{dz}$$

$$\Rightarrow \frac{dR}{dz} = \frac{2}{|I_0|^2}\frac{dP_{loss}}{dz} = \frac{2\pi |H_0|^2\left(\frac{1}{a}+\frac{1}{b}\right)\frac{1}{\sigma\delta}}{\frac{\epsilon}{\mu}|V_0|^2\frac{(2\pi)^2}{\left(\ln\frac{b}{a}\right)^2}}$$

or plugging in $H_0 = \left(\frac{\epsilon}{\mu}\right)^{1/2}\frac{V_0}{a\ln\frac{b}{a}}$

$$\Rightarrow \frac{dR}{dz} = \frac{\frac{\epsilon}{\mu}\frac{1}{a^2}\left(\frac{1}{a}+\frac{1}{b}\right)a^2}{\frac{\epsilon}{\mu}\sigma\delta\,(2\pi)} = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{dR}{dz}$$

And the time-averaged magnetic energy per unit length is

$$\frac{dW_B}{dz} = \frac{1}{4}\int \vec{B}\cdot\vec{H}^*\,da$$

We must evaluate this both between the conductors and inside each conductor.

i.e. $$\frac{dW_B^{(\text{between})}}{dz} = \frac{\mu}{4} 2\pi \int_a^b \rho\, d\rho\, |\vec{H}|^2 = \frac{\mu\pi}{2} \int_a^b \frac{\epsilon |V_0|^2}{\mu (\ln\frac{b}{a})^2} \frac{d\rho}{\rho}$$

$$= \frac{\epsilon\pi}{2} \frac{|V_0|^2}{(\ln\frac{b}{a})^2} \ln\frac{b}{a} = \frac{\epsilon\pi}{2} \frac{|V_0|^2}{\ln\frac{b}{a}}$$

while the energy stored within the INNER conductor is found using:

$$\vec{H}_c^{\,in} = -H_0 \hat{\phi}\, e^{-\frac{(a-\rho)}{\delta}} e^{i(a-\rho)/\delta}$$

$$\Rightarrow \frac{dW_B^{\text{inner}}}{dz} = \frac{\mu_c}{4} 2\pi |H_0|^2 \underbrace{\int_0^a \rho\, e^{-2(a-\rho)/\delta} d\rho}_{= \frac{\delta}{4}\left(2a - \delta + \delta e^{-2a/\delta}\right) \simeq \frac{\delta a}{2},\ \text{if } \delta \ll a}$$

$$= \frac{\mu_c \pi \delta a}{4} |H_0|^2$$

$$= \frac{\mu_c \pi \delta a}{4} \frac{\epsilon}{\mu} \frac{|V_0|^2}{(\ln\frac{b}{a})^2} \frac{1}{a^2} = \frac{\pi\epsilon\delta}{4a} \frac{|V_0|^2}{(\ln\frac{b}{a})^2} \frac{\mu_c}{\mu}$$

while that stored in the outer conductor is:

$$\frac{dW_B^{\text{outer}}}{dz} = \frac{\mu_c}{4} |H_0|^2 \frac{a^2}{b^2} \underbrace{\int_b^\infty (2\pi\rho)\, e^{-2(\rho-b)/\delta} d\rho}_{= \frac{2\pi}{4}(2b\delta + \delta^2) \simeq \pi b\delta,\ \text{if } \delta \ll b}$$

$$= \frac{\mu_c \pi b \delta}{4} \frac{a^2}{b^2} |H_0|^2$$

$$= \frac{\pi\epsilon\delta}{4b} \frac{|V_0|^2}{(\ln b/a)^2} \frac{\mu_c}{\mu}$$

$$\Rightarrow \frac{1}{4}\frac{dL}{dz}|I_0|^2 = |V_0|^2\left\{\frac{\epsilon\pi}{2\ln\frac{b}{a}} + \frac{\pi\epsilon\delta}{4\left(\ln\frac{b}{a}\right)^2}\left(\frac{1}{a}+\frac{1}{b}\right)\right\}$$

$$\Rightarrow \frac{dL}{dz} = \frac{4}{(2\pi)^2}\frac{\left\{\frac{\epsilon\pi}{2\ln\frac{b}{a}} + \frac{\pi\epsilon\delta}{4\left(\ln\frac{b}{a}\right)^2}\left(\frac{1}{a}+\frac{1}{b}\right)\frac{\mu_c}{\mu}\right\}}{\frac{\epsilon}{\mu}\left(\ln\frac{b}{a}\right)^2}$$

or

$$\boxed{\frac{dL}{dz} = \frac{\mu}{2\pi}\ln\frac{b}{a} + \frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

Problem 8.4

We have a hollow cylinder of inner radius R.

The 2D wave equation to be satisfied is

$$(\nabla_t^2 + \gamma_\lambda^2)\psi_\lambda = 0, \quad \text{where } \gamma_\lambda^2 = \mu\epsilon\omega^2 - k^2$$

$$\text{and} \quad \omega_\lambda^2 = \frac{\gamma_\lambda^2}{\mu\epsilon}$$

The solutions must be:

$$R(\rho)e^{im\phi} = J_m(\gamma_\lambda \rho)e^{im\phi}$$

(a) TM modes - these obey $\psi_\lambda(R,\phi) = 0$,

so we can write $\gamma_\lambda = \frac{x_{mn}}{R}$

where x_{mn} is the n^{th} root of $J_m(x_{mn}) = 0$

By consulting a table of Bessel zeros, or Mathematica, the lowest 5 TM modes are seen to be:

m	n	x_{mn}	$\omega_{mn} = \frac{c}{R} x_{mn}$	$\frac{\omega_{mn}}{\omega_{01}} = \frac{x_{mn}}{x_{01}}$
0	1	2.405		1
1	1	3.832		1.593
2	1	5.136		2.136
0	2	5.520		2.295
3	1	6.380		2.653

Similarly, the TE modes obey $\frac{\partial\psi}{\partial n}(R,\phi) = 0$

$$\Rightarrow J_m'(\gamma_\lambda R) = 0$$

Denote $x'_{mn} = n^{th}$ zero of $J_m'(x'_{mn}) = 0$

m	n	x'_{mn}	$\frac{x'_{mn}}{x'_{11}}$
1	1	1.841	1
2	1	3.054	1.659
0	1	3.832	2.081
3	1	4.201	2.282
4	1	5.3175	2.888

Thus, the dominant mode has cutoff frequency equal to

$$\boxed{\omega_{11}^{TE} = 1.841 \frac{c}{R}}$$

(8.4b) The relevant equations needed here are 8.59, 8.51, 8.57

Now, for the lowest TM mode $(m,n)=(0,1)$

$$\Rightarrow \beta_{01}^{TM} = -\frac{dP/dz}{2P} = \frac{\frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_\lambda}\right)^2 \frac{1}{\mu^2\omega_\lambda^2} \oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl}{\frac{\epsilon}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\int_A |\psi|^2 da} \cdot \frac{1}{2}$$

or

$$\beta_{01}^{TM} = \frac{c \oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl}{2\sigma\delta\mu\,\omega_\lambda^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\int_A |\psi|^2 da}$$

and

$$\psi_{mn} = J_m\left(\frac{x_{mn}\rho}{R}\right)e^{im\phi}$$

so

$$\oint_C \left|\frac{\partial\psi}{\partial n}\right|^2 dl = 2\pi R \left|\frac{\partial}{\partial\rho} J_m\left(\frac{x_{mn}\rho}{R}\right)\right|^2\Bigg|_{\rho=R}$$

$$= \frac{2\pi R}{R^2 x_{mn}^{-2}} \left\{\left|\frac{R}{x_{mn}}\frac{\partial}{\partial\rho} J_m\left(\frac{x_{mn}\rho}{R}\right)\right|^2\right\}_{\rho=R}$$

$$= \frac{2\pi x_{mn}^2}{R} \left| \frac{\partial}{\partial X} J_m(X) \right|^2_{X \to X_{mn}}$$

whereas,

$$\int_A |\psi|^2 da = 2\pi \int_0^R \rho\, d\rho \left| J_m\left(\frac{x_{mn}\rho}{R}\right)\right|^2$$

$$= 2\pi \frac{R^2}{2} \left\{ J_m(x_{mn})^2 - J_{m-1}(x_{mn}) J_{m+1}(x_{mn}) \right\}$$

so

$$\beta_{01}^{TM} = \frac{c\ 2(x_{01}^2/R^3)}{2\sigma\delta\mu\ \omega_{01}^2 \left(1 - \frac{\omega_{01}^2}{\omega^2}\right)^{1/2}} \boxed{\frac{|J_m'(x_{mn})|^2}{\left\{J_m(x_{mn})^2 - J_{m-1}(x_{mn})J_{m+1}(x_{mn})\right\}}}$$

so

$$\beta_{01}^{TM} = \frac{c}{\sigma\delta\mu R} \frac{x_{01}^2}{R^2} \frac{1}{\omega_{01}^2\left(1 - \frac{\omega_{01}^2}{\omega^2}\right)^{1/2}}$$

but

$$\frac{x_{01}^2}{R^2} = \gamma_{01}^2 = \frac{\omega_{01}^2}{c^2}$$

$$\Rightarrow \boxed{\beta_{01}^{TM} = \frac{1}{\sigma\delta\mu c R}\left(1 - \frac{\omega_{01}^2}{\omega^2}\right)^{-1/2}}$$

or to make the full dependence on ω explicit, use

$$\delta \equiv \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2} = \delta_\lambda \left(\frac{\omega_\lambda}{\omega}\right)^{1/2} \to \delta_{01}\left(\frac{\omega_{01}}{\omega}\right)^{1/2}$$

$$\Rightarrow \boxed{\beta_{01}^{TM} = \frac{\sqrt{\mu\epsilon}}{\sigma\delta_{01}\mu R} \frac{(\omega/\omega_{01})^{1/2}}{\left(1 - \frac{\omega_{01}^2}{\omega^2}\right)^{1/2}}}$$

Now for the lowest TE mode (1,1), where the attenuation constant is given by

$$\beta_{11}^{TE} = \frac{\left(\frac{1}{2\sigma\delta}\right)\left(\frac{\omega}{\omega_\lambda}\right)^2 \oint_C \left[\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)\frac{1}{\mu\epsilon\omega_\lambda^2}\left|\hat{n}\times\nabla_t\psi\right|^2 + \frac{\omega_\lambda^2}{\omega^2}|\psi|^2\right] d\ell}{\frac{\mu}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\int_A |\psi|^2 da} \qquad (2)$$

The only new nontrivial integral still to evaluate is in the numerator, with $\psi = J_m\left(\frac{x'_{mn}\rho}{R}\right)e^{im\phi}$:

$$\oint_C d\ell\,\left|\hat{n}\times\nabla_t\psi\right|^2 = \int_0^{2\pi} R\,d\phi \left|\frac{i m e^{im\phi} J_m(x'_{mn})}{R}\right|^2$$

$$= \frac{2\pi m^2}{R} J_m^2(x'_{mn}) \xrightarrow[(1,1)]{} \frac{2\pi}{R} J_1^2(x'_{11})$$

and thus

$$\beta_{11}^{TE} = \frac{\left(1-\frac{\omega_{11}^2}{\omega^2}\right)\frac{1}{\mu\epsilon\omega_{11}^2}\frac{2\pi}{R}J_1^2(x'_{11}) + \frac{\omega_{11}^2}{\omega^2}J_1^2(x'_{11})(2\pi R)}{2\mu c\left(1-\frac{\omega_{11}^2}{\omega^2}\right)^{1/2}\sigma\delta\,\pi R^2\left\{J_1^2(x'_{11}) - J_0(x'_{11})J_2(x'_{11})\right\}}$$

Now note that

$$\frac{J_1^2(x'_{11})}{J_1^2(x'_{11}) - J_0(x'_{11})J_2(x'_{11})} = 1.418\ldots$$

and as before

$$\delta = \delta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^{1/2}$$

$$\Rightarrow \beta_{11}^{TE} = \frac{\sqrt{\mu\epsilon}\;1.418}{\sigma\delta_\lambda\mu R}\frac{(\omega/\omega_{11})^{1/2}}{\left(1-\frac{\omega_{11}^2}{\omega^2}\right)^{1/2}}\left\{\underbrace{\left(\frac{R^{-2}c^2}{\omega_{11}^2}\right)}_{=\,x'^{-2}_{11}}\left(1-\frac{\omega_{11}^2}{\omega^2}\right) + \frac{\omega_{11}^2}{\omega^2}\right\}$$

Plot of the lowest TM and TE mode decay parameter:

```
TMplot = Plot[Sqrt[w/w0] 1/Sqrt[1 - w0^2/w^2], {w, w0*1.01, w0*5}, PlotStyle → Thickness[0.01]] /. w0 → 1;

TEplot = Plot[Sqrt[w/w0] 1/Sqrt[1 - w0^2/w^2] 1.418 (1/(1.84118)^2 (1 - w0^2/w^2) + w0^2/w^2), {w, w0*1.01, w0*5},
    PlotStyle → {Thickness[0.01], Hue[0.7]}] /. w0 → 1;

Show[TMplot, TEplot, PlotRange → {0.9, 3}];
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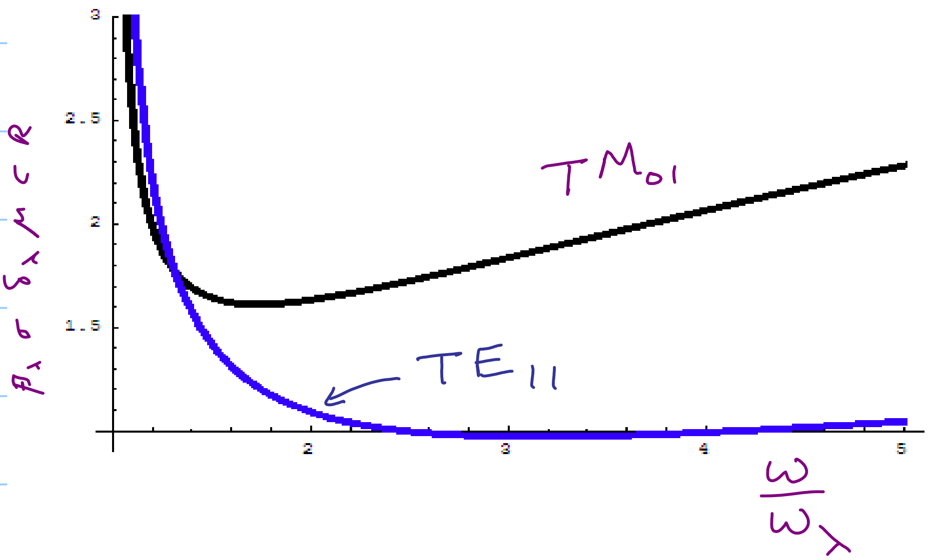

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Evaluation of $\int_0^R \rho\, d\rho\; J_m\left(x'_{11}\frac{\rho}{R}\right) J_m\left(x'_{11}\frac{\rho}{R}\right) \equiv \Lambda$

let $y \equiv \rho/R$

$$\Rightarrow \Lambda = \lim_{\bar{x} \to x'_{11}} R^2 \int_0^1 y\, dy\; J_m(x'_{11} y) J_m(\bar{x}\, y)$$

where $J'_m(x)\Big|_{x = x'_{11}} = 0$

But let's call $\mathcal{L}_y$ the Bessel function's Sturm-Liouville operator, i.e.

$$\mathcal{L}_y = -\frac{1}{y}\frac{d}{dy} y \frac{d}{dy} + \frac{m^2}{y^2},$$ then the Bessel fcns obey

$$\mathcal{L}_y J_m(xy) = x^2 J_m(xy)$$

Then the following identity is useful:

$$\Lambda = \lim_{\bar{x} \to x'_{11}} R^2 \frac{\int_0^1 y\, dy \left[J_m(x'_{11} y) \mathcal{L}_y J_m(\bar{x} y) - J_m(\bar{x} y) \mathcal{L}_y J_m(x'_{11} y) \right]}{\bar{x}^2 - x'^2_{11}}$$

and plugging in the differential operator here gives:

$$\Lambda = \lim_{\bar{x} \to x'_{11}} R^2 \frac{\int_0^1 \cancel{y}\, dy \left\{ J_m(\bar{x} y) \frac{1}{\cancel{y}} \frac{d}{dy}\left(y \frac{d J_m(x'_{11} y)}{dy}\right) - \frac{J_m(x'_{11} y)}{\cancel{y}} \frac{d}{dy}\left(y \frac{d J_m(\bar{x} y)}{dy}\right) \right\}}{\bar{x}^2 - x'^2_{11}}$$

Now integrate both terms by parts, giving

$$\Lambda = \lim_{\bar{x}\to x_{11}'} R^2 \frac{\int_0^1 dy \left[\frac{dJ_m(x_{11}'y)}{dy}\, y \cancel{\frac{dJ_m(\bar{x}y)}{dy}} - \frac{dJ_m(\bar{x}y)}{dy}\, y \cancel{\frac{dJ_m(x_{11}'y)}{dy}} \right]}{\bar{x}^2 - x_{11}'^2}$$

$$+ \lim_{\bar{x}\to x_{11}'} R^2 \frac{y J_m(\bar{x}y)\frac{d}{dy}J_m(x_{11}'y) - y J_m(x_{11}'y)\frac{d}{dy}J_m(\bar{x}y)}{\bar{x}^2 - x_{11}'^2}\Bigg|_0^1$$

$\Rightarrow$ the surface term is all that remains, and we must evaluate this $0/0$ limit: set $\bar{x} = x_{11}' + \delta$ with $\delta \ll 1$, then

$$J_m(\bar{x}y) \simeq J_m(x_{11}'y) + \frac{d}{d\bar{x}} J_m(\bar{x}y)\Big|_{\bar{x}=x_{11}'} \delta$$

and $\displaystyle \frac{dJ_m(\bar{x}y)}{dy} \simeq \frac{dJ_m(x_{11}'y)}{dy} + \delta \frac{d}{d\bar{x}} \frac{dJ_m(\bar{x}y)}{dy}\Big|_{\bar{x}=x_{11}'}$

$$\Lambda = R^2 \lim_{\delta\to 0} \frac{\cancelto{1}{\delta}}{\delta}\frac{1}{2x_{11}'}\left[\frac{dJ_m(x_{11}'y)}{dx_{11}'}\frac{dJ_m(x_{11}'y)}{dy} - J_m(x_{11}'y)\frac{d}{dx_{11}'}\frac{d}{dy}J_m(x_{11}'y)\right]_{y=1}$$

So call $J_m'(u) \equiv \frac{d}{du} J_m(u)$

$$\Rightarrow \frac{dJ_m(x_{11}'y)}{dx_{11}'}\Bigg|_{y=1} = y \frac{dJ_m(u)}{du}\Bigg|_{u=x_{11}'} = 0$$

$$\text{and } \frac{d}{dx_{11}'}\frac{d}{dy} J_m(x_{11}'y) = \frac{d}{dx_{11}'}\, x_{11}' \frac{d}{d(x_{11}'y)} J_m(x_{11}'y)$$

$$= x_{11}' \cancelto{1}{y} \frac{d^2}{du^2} J_m(u)\Big|_{u=x_{11}'} \qquad \text{since } \frac{d}{du}J_m(u)\Big|_{x_{11}'} = 0$$

$$\Rightarrow \boxed{\Lambda = \frac{-R^2}{2}\left[J_m(u) J_m''(u)\right]_{u=x_{11}'}}$$