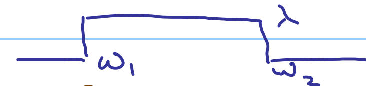
Homework Set 1, C. Greene's solutions
J: 7.22, 7.26

Problem 7.22 (a)



$$\mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0} = \lambda\left(\theta(\omega-\omega_1) - \theta(\omega-\omega_2)\right)$$

where $\omega_2 > \omega_1 > 0$

$$= \begin{cases} \lambda, & \omega_1 < \omega < \omega_2 \\ 0, & \text{elsewhere} \end{cases}$$

$$\Rightarrow \mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \,\mathrm{Im}\left(\frac{\epsilon(\omega')}{\epsilon_0}\right) d\omega'}{\omega'^2 - \omega^2}$$

$$= 1 + \frac{2\lambda}{\pi} P \int_{\omega_1}^{\omega_2} \frac{\omega' d\omega'}{\omega'^2 - \omega^2}$$

Case 1 if ω is not between ω_1 and ω_2, there is no singularity to deal with

$\Rightarrow$ if $\omega < \omega_1$, the integrand is positive everywhere and

$$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{2\lambda}{\pi}\frac{1}{2}\ln(\omega'^2 - \omega^2)\Big|_{\omega_1}^{\omega_2}$$

or

$$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\lambda}{\pi}\ln\left(\frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2}\right) \qquad \omega < \omega_1$$

Case (2) if $\omega_1 < \omega < \omega_2$

$$\Rightarrow \mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{2\lambda}{\pi}\lim_{\delta\to 0^+}\left(\int_{\omega_1}^{\omega-\delta}\frac{\omega' d\omega'}{\omega'^2 - \omega^2} + \int_{\omega+\delta}^{\omega_2}\frac{\omega' d\omega'}{\omega'^2 - \omega^2}\right)$$

$$= 1 + \frac{2\lambda}{\pi}\frac{1}{2}\lim_{\delta\to 0^+}\ln\left\{\frac{(\omega-\delta)^2 - \omega^2}{\omega_1^2 - \omega^2}\,\frac{\omega_2^2 - \omega^2}{(\omega+\delta)^2 - \omega^2}\right\}$$

$$\Rightarrow \mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\lambda}{\pi}\lim_{\delta\to 0^+}\ln\left\{\frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2}\;\frac{-2\omega\delta + \delta^2}{2\omega\delta + \delta^2}\right\}$$

or

$$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\lambda}{\pi}\ln\left(\frac{\omega_2^2 - \omega^2}{\omega^2 - \omega_1^2}\right) \quad \text{if } \omega_1 < \omega < \omega_2$$

<u>Case 3</u> $\omega > \omega_2$ $\quad \mathrm{Re}\,\dfrac{\epsilon(\omega)}{\epsilon_0} = 1 - \dfrac{2\lambda}{\pi}\displaystyle\int_{\omega_1}^{\omega_2} \frac{\omega' d\omega'}{\omega^2 - \omega'^2}$

$$\Rightarrow \quad \mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\lambda}{\pi}\ln\left(\frac{\omega^2-\omega_2^2}{\omega^2-\omega_1^2}\right) \quad \text{if } \omega > \omega_2$$

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In[2]:= w1 = 1.0; w2 = 2.0;

In[3]:= Plot[plotfunc[w], {w, 0.2, 2.8}]
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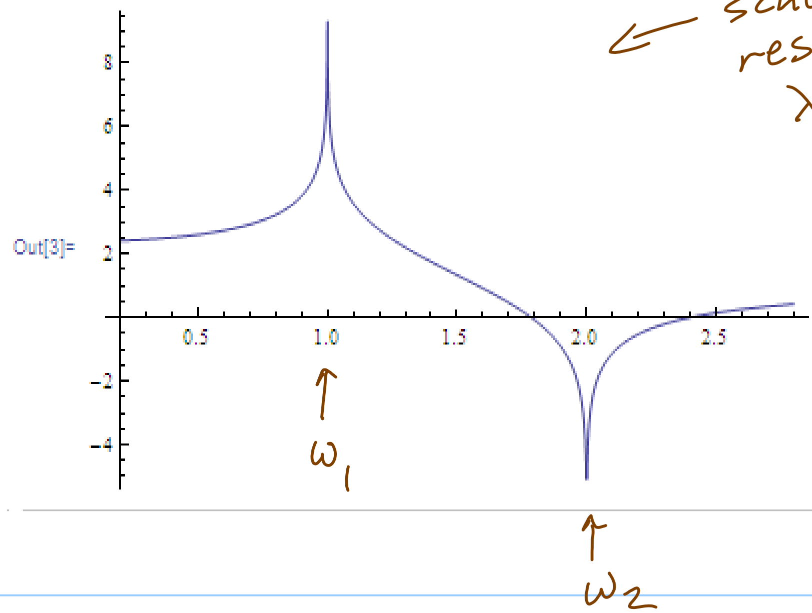

schematic result for $\lambda = \pi$

(b) Now let $\mathrm{Im}\,\dfrac{\epsilon(\omega)}{\epsilon_0} = \dfrac{\gamma\lambda\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$

Then using 7.119, we have

$$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\lambda\gamma\omega'}{(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2}\,\frac{d\omega'}{\omega'-\omega}$$

Observe that a useful alternative way to write $\mathrm{Im}\,\dfrac{\epsilon(\omega)}{\epsilon_0}$ is:

$$\mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0} = \mathrm{Im}\,\frac{\lambda}{\omega_0^2-\omega^2-i\gamma\omega} \qquad \text{(check for yourself!)}$$

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So
$$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{1}{\pi}\,\mathrm{Im}\,\mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'-\omega}\;\frac{\lambda}{(\omega_0^2-\omega'^2)-i\gamma\omega'}$$
$$= 1 + \frac{1}{\pi}\,\mathrm{Im}\; i\pi\,\mathrm{Res}\left\{\frac{\lambda}{(\omega'-\omega)[(\omega_0^2-\omega'^2)-i\gamma\omega']},\omega\right\}$$
$$= 1 + \mathrm{Re}\,\frac{\lambda}{(\omega_0^2-\omega^2)-i\gamma\omega}$$

or finally
$$\boxed{\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\lambda(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$$

See the attached figures below.

The sharper structures plotted above in (a) presumably derive from the fact that $\epsilon(\omega)$ is NOT AN ANALYTIC FUNCTION!

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In[7]:= Plot[Re[1 + 1*0.1/((w0^2 - w^2) - i g w)] /. {g → 0.1, w0 → 1}, {w, 0.2, 1.8}];
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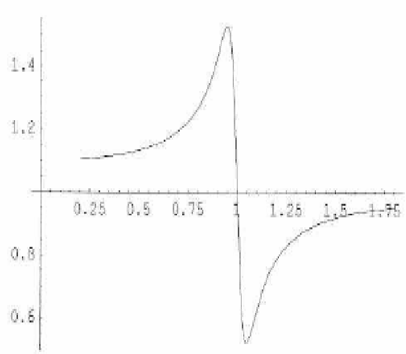


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In[8]:= Plot[Im[1 + 1*0.1/((w0^2 - w^2) - i g w)] /. {g → 0.1, w0 → 1}, {w, 0.2, 1.8}];
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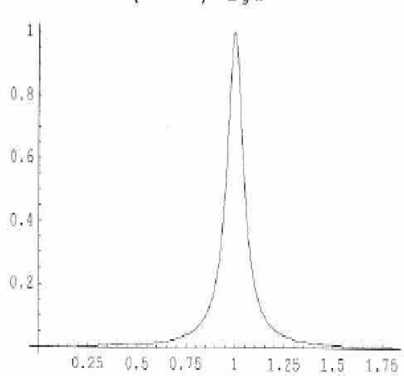

Problem 7.26

(a) Charge (Ze) moves through medium with const. $\vec{v}$

$$\sigma(\vec{q},\omega) = i\omega\left(\epsilon_0 - \epsilon(\vec{q},\omega)\right)$$

$$\text{i.e. } \epsilon(\vec{q},\omega) = \epsilon_0 - \frac{\sigma(\vec{q},\omega)}{i\omega}$$

approximation: assume $\vec{E} = -\nabla\Phi$ (i.e. neglect $\frac{\partial\vec{A}}{\partial t}$)

$\Rightarrow$ $\rho(\vec{x},t) = Ze\,\delta(\vec{x}-\vec{v}t)$ and the Fourier transform is:

$$\rho(\vec{q},\omega) = \frac{Ze}{(2\pi)^4}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\,\delta(\vec{x}-\vec{v}t)$$

$$= \frac{Ze}{(2\pi)^4}\int_{-\infty}^{\infty} dt\, e^{i\omega t - i\vec{q}\cdot\vec{v}\,t} = \boxed{\frac{Ze}{(2\pi)^3}\,\delta(\omega-\vec{q}\cdot\vec{v}) = \rho(\vec{q},\omega)}$$

(b) $\nabla\cdot\vec{D} = \rho$ can be rewritten in Fourier space as:

$$\vec{D}(\vec{x},t) = \int e^{i\vec{q}\cdot\vec{x}-i\omega t}\,\vec{D}(\vec{q},\omega)\,d^3q\,d\omega$$

$$\text{and } \vec{E}(\vec{x},t) = \int e^{i\vec{q}\cdot\vec{x}-i\omega t}\,\vec{E}(\vec{q},\omega)\,d^3q\,d\omega$$

$$\rho(\vec{x},t) = \int e^{i\vec{q}\cdot\vec{x}-i\omega t}\,\rho(\vec{q},\omega)\,d^3q\,d\omega$$

So $\nabla\cdot\vec{D} = \rho$ reads, using $\vec{D}(\vec{q},\omega) = \epsilon(\vec{q},\omega)\vec{E}(\vec{q},\omega)$:

$$0 = \int d^3q\,d\omega\, e^{i\vec{q}\cdot\vec{x}-i\omega t}\left[i\vec{q}\cdot\left(\epsilon(\vec{q},\omega)\vec{E}(\vec{q},\omega)\right) - \rho(\vec{q},\omega)\right]$$

But using $\Phi(\vec{x},t) = \int d^3q\,d\omega\, e^{i\vec{q}\cdot\vec{x}-i\omega t}\,\phi(\vec{q},\omega)$

$$\Rightarrow -\nabla\Phi = \vec{E}, \text{ so } \vec{E}(\vec{q},\omega) = -i\vec{q}\,\phi(\vec{q},\omega)$$

Then $\underbrace{i\vec{q}\cdot\vec{D}(\vec{q},\omega)}_{``\rho(\vec{q},\omega)} = i\vec{q}\cdot\left(\epsilon(\vec{q},\omega)\vec{E}(\vec{q},\omega)\right) = q^2\epsilon(\vec{q},\omega)\,\phi(\vec{q},\omega)$

Therefore $\boxed{\phi(\vec{q},\omega) = \frac{\rho(\vec{q},\omega)}{q^2\,\epsilon(\vec{q},\omega)}}$

(c) Start from

$$\frac{dW}{dt} = \int \vec{J}\cdot\vec{E}\,d^3x$$

$$= \int d^3x\left(\int \vec{J}(\vec{q},\omega)\,e^{i\vec{q}\cdot\vec{x}-i\omega t}\,d^3q\,d\omega\right)\cdot\left(\int \vec{E}(\vec{q}',\omega')^{*}\,e^{-i\vec{q}'\cdot\vec{x}+i\omega' t}\,d^3q'\,d\omega'\right)$$

$$= \int d^3q\, d\omega\, d^3q'\, d\omega'\, (2\pi)^3 \delta(\vec{q}-\vec{q}\,')\, e^{i(\omega'-\omega)t}\, \sigma(\vec{q},\omega)\, \vec{E}(\vec{q},\omega)\cdot \vec{E}^*(\vec{q}\,',\omega')$$

But recall that $\vec{E}(\vec{q},\omega) = -i\vec{q}\,\phi(\vec{q},\omega) = -i\vec{q}\,\dfrac{\rho(\vec{q},\omega)}{q^2\epsilon(\vec{q},\omega)}$

So

$$\frac{dW}{dt} = \int d^3q\, d\omega\, d\omega'\, (2\pi)^3 e^{i(\omega'-\omega)t}\, \frac{(+q^2)\rho(\vec{q},\omega)\rho^*(\vec{q},\omega')\,\sigma(\vec{q},\omega)}{(q^2)^2\epsilon(\vec{q},\omega)\epsilon^*(\vec{q},\omega')}$$

$$= \int d^3q\, d\omega\, d\omega'\, (2\pi)^3 \frac{e^{i(\omega'-\omega)t}}{q^2}\, \frac{z^2e^2}{(2\pi)^6}\, \frac{\delta(\vec{q}\cdot\vec{v}-\omega)\,\delta(\vec{q}\cdot\vec{v}-\omega')\,\sigma(\vec{q},\omega)}{\epsilon(\vec{q},\omega)\,\epsilon^*(\vec{q},\omega')}$$

$$= \int \frac{d^3q}{q^2}\, d\omega\, \frac{z^2e^2}{(2\pi)^3}\, e^{i(\vec{q}\cdot\vec{v}-\omega)t}\, \frac{\delta(\vec{q}\cdot\vec{v}-\omega)\,\sigma(\vec{q},\omega)}{\epsilon(\vec{q},\omega)\,\epsilon^*(\vec{q},\omega)}$$

But $\epsilon(\vec{q},\omega) = \epsilon_0 + \dfrac{i\sigma(\vec{q},\omega)}{\omega}$

$$\Rightarrow \operatorname{Im}\frac{1}{\epsilon(\vec{q},\omega)} = \operatorname{Im}\frac{1}{\epsilon_0 + \frac{i\sigma}{\omega}} = \frac{-\sigma/\omega}{|\epsilon_0 + \frac{i\sigma}{\omega}|^2} = \frac{-\sigma/\omega}{\epsilon^*\epsilon}$$

$$\Rightarrow -\frac{dW}{dt} = \int_{-\infty}^{\infty} d\omega \int \frac{d^3q}{q^2}\, \frac{z^2e^2}{(2\pi)^3}\, \omega\left(\operatorname{Im}\frac{1}{\epsilon(\vec{q},\omega)}\right)\delta(\omega-\vec{q}\cdot\vec{v})$$

Now, observe that for any velocity $\vec{v}$ there will be two points in the integral over ω and $d\Omega_q$ where $\omega = \vec{q}\cdot\vec{v}$, one where $\omega > 0$ and one where $\omega < 0$, so we can rewrite that part of the integral as

$$\underbrace{\int d\Omega_q \int_{-\infty}^{0} d\omega'\, \omega' \operatorname{Im}\left(\frac{1}{\epsilon(\vec{q},\omega')}\right)\delta(\omega'-\vec{q}\cdot\vec{v})}_{= I_1}$$

$$+ \underbrace{\int d\Omega_q \int_{0}^{\infty} d\omega\, \omega \operatorname{Im}\left(\frac{1}{\epsilon(\vec{q},\omega)}\right)\delta(\omega-\vec{q}\cdot\vec{v})}_{= I_2}$$

Now change $\omega' \to -\omega$ in I_1:

$$\Rightarrow \int d\Omega_q \int_{\infty}^{0} d\omega\, \omega\, \mathrm{Im}\left(\frac{1}{\epsilon(\vec{q},-\omega)}\right) \delta(-\omega - \vec{q}\cdot\vec{v})$$

$$= -\int d\Omega_q \int_0^{\infty} \omega\, \mathrm{Im}\left(\frac{1}{\epsilon^*(\vec{q},+\omega)}\right) \delta(\omega + \vec{q}\cdot\vec{v})$$

$$" - \mathrm{Im}\frac{1}{\epsilon(\vec{q},\omega)}$$

So in fact $I_1 = I_2$ and our final result becomes

$$\boxed{\Rightarrow\ -\frac{dW}{dt} = \int_0^{\infty} d\omega \int \frac{d^3q}{q^2}\, \frac{Z^2e^2}{(2\pi)^3}\, 2\,\omega \left(\mathrm{Im}\frac{1}{\epsilon(\vec{q},\omega)}\right) \delta(\omega - \vec{q}\cdot\vec{v})}$$

This agrees with the final result in J...