Sec. 7.3 Reflection, refraction at an interface

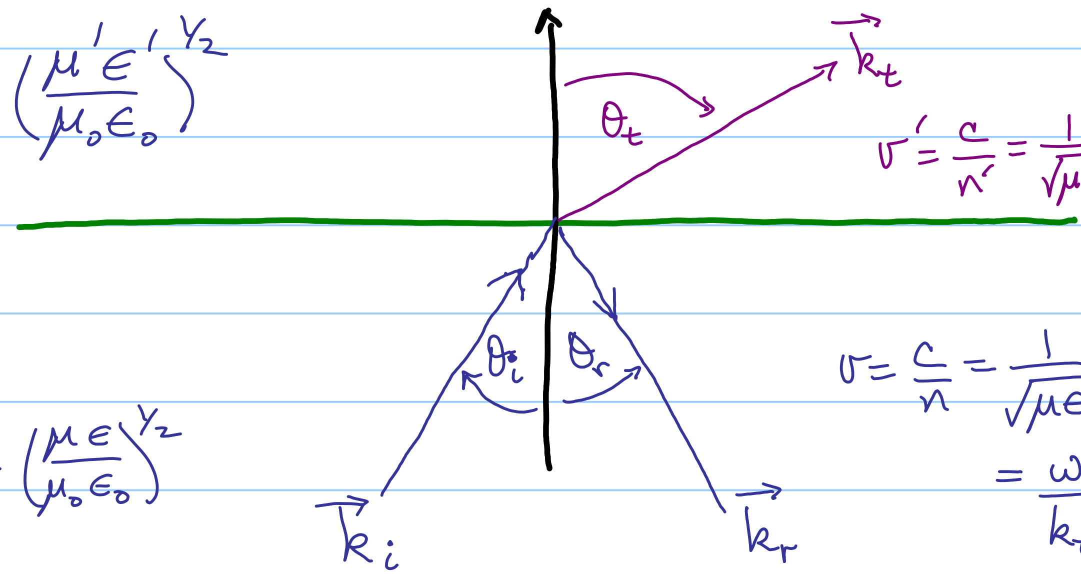
normal, $\hat{n}$ - directed into transmitted side

transmitted side

$\mu', \epsilon', n' = \left(\frac{\mu'\epsilon'}{\mu_0\epsilon_0}\right)^{1/2}$

$v' = \frac{c}{n'} = \frac{1}{\sqrt{\mu'\epsilon'}} = \frac{\omega}{k_t}$

boundary



incident side

$\mu, \epsilon, n = \left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)^{1/2}$

$v = \frac{c}{n} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{\omega}{k_i}$

$= \frac{\omega}{k_t}$

The 3 arrows here correspond to 3 plane waves:

Incident
$$\vec{E}_i = \vec{E}_{oi}\, e^{i(\vec{k}_i \cdot \vec{x} - \omega t)}$$
$$\vec{B}_i = \sqrt{\mu\epsilon}\, \hat{k}_i \times \vec{E}_i = \frac{n}{c}\, \hat{k}_i \times \vec{E}_i$$

Reflected
$$\vec{E}_r = \vec{E}_{or}\, e^{i(\vec{k}_r \cdot \vec{x} - \omega t)}$$
$$\vec{B}_r = \frac{n}{c}\, \hat{k}_r \times \vec{E}_r$$

Transmitted
$$\vec{E}_t = \vec{E}_{ot}\, e^{i(\vec{k}_t \cdot \vec{x} - \omega t)}$$
$$\vec{B}_t = \frac{n'}{c}\, \hat{k}_t \times \vec{E}_t$$

and of course, here $k_i = k_r = \frac{n\omega}{c}$

while $$k_t = \frac{n'\omega}{c} = \frac{n'}{n} k_i = \frac{n'}{n} k_r$$

- One must have the <u>same</u> ω in all 3 plane waves in order to satisfy BC's at <u>all</u> <u>times</u>.

- Also, assuming both media are homogeneous, <u>symmetry</u> demands that all 3 wavevectors,
$$\vec{k}_i \,,\; \vec{k}_r \,,\; \vec{k}_t \quad (+ \text{ the normal } \hat{n})$$
must lie in a <u>single</u> <u>plane</u>, taken here to be the $z=0$ plane

- In order for the imposed boundary conditions to hold along the entire planar interface, we must also have:
$$e^{i\vec{k}_i\cdot\vec{x}}\Big|_{z=0} = e^{i\vec{k}_r\cdot\vec{x}}\Big|_{z=0} = e^{i\vec{k}_t\cdot\vec{x}}\Big|_{z=0}$$

$$\Rightarrow \vec{k}_i\cdot\vec{x}\Big|_{z=0} = \vec{k}_r\cdot\vec{x}\Big|_{z=0} = \vec{k}_t\cdot\vec{x}\Big|_{z=0}$$

$$\Rightarrow \quad k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$$
(k-components <u>along</u> surface are equal)

$\Rightarrow \boxed{\theta_i = \theta_r}$ angle of incidence = angle of reflection (since $k_i = k_r$)

and since $k_i = \dfrac{n\omega}{c}$, $k_t = \dfrac{n'\omega}{c}$

$$\Rightarrow \quad n \sin\theta_i = n' \sin\theta_t$$
← Snell's law of refraction

Next, match all the BCs at interface, namely: $\vec{D}\cdot\hat{n}$ and $\vec{B}\cdot\hat{n}$ continuous,

and $\vec{E}\times\hat{n}$ and $\vec{H}\times\hat{n}$ continuous

(since no free surface charges or surface currents are assumed present)

Therefore, we see that:

$$\boxed{\begin{aligned}
&\left(\epsilon(\vec{E}_{oi}+\vec{E}_{or})-\epsilon'\vec{E}_{ot}\right)\cdot\hat{n}=0\\
&\left(\frac{n}{c}\hat{k}_i\times\vec{E}_{oi}+\frac{n}{c}\hat{k}_r\times\vec{E}_{or}-\frac{n'}{c}\hat{k}_t\times\vec{E}_{ot}\right)\cdot\hat{n}=0\\
&\left(\vec{E}_{oi}+\vec{E}_{or}-\vec{E}_{ot}\right)\times\hat{n}=0\\
&\left(\frac{1}{\mu}\frac{n}{c}\hat{k}_i\times\vec{E}_{oi}+\frac{1}{\mu}\frac{n}{c}\hat{k}_r\times\vec{E}_{or}-\frac{1}{\mu'}\frac{n'}{c}\hat{k}_t\times\vec{E}_{ot}\right)\times\hat{n}=0
\end{aligned}}$$

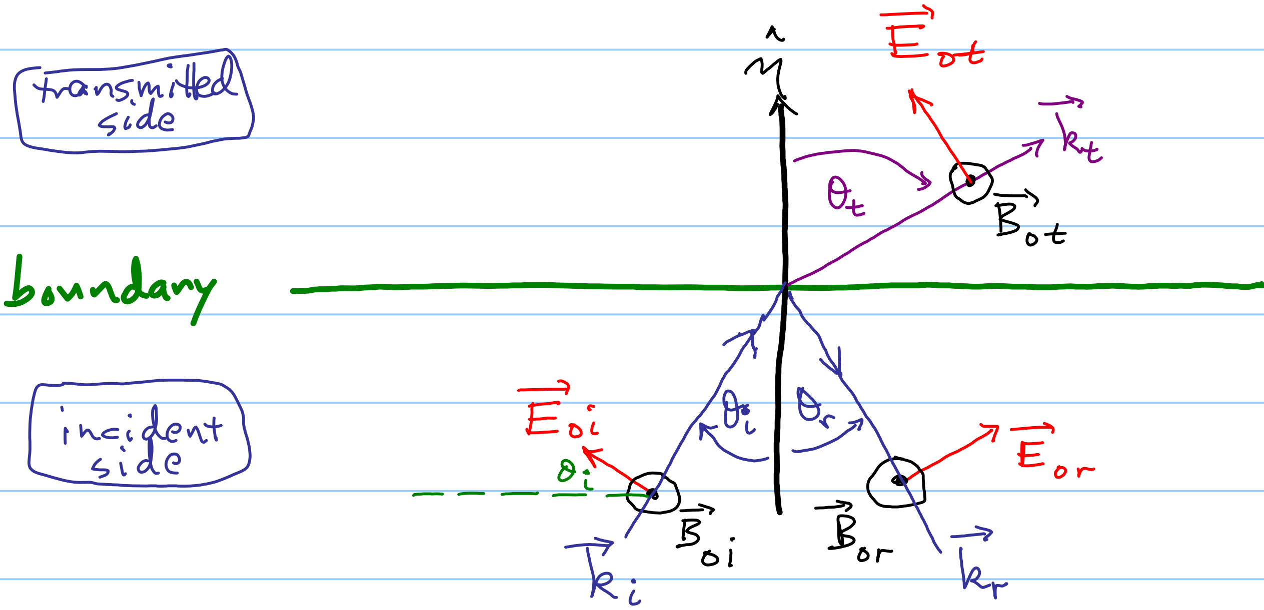
To apply these BCs, it is simplest to consider 2 separate cases:

(i) The incident linearly-polarized wave has $\vec{E}_{oi}$ IN the plane of incidence

(ii) The incident linearly-pol. wave has $\vec{E}_{oi}$ PERPENDICULAR to the plane of incidence

Of course, if the incident light has general elliptical polarization, it will just be a linear combination of these 2 cases.

Case (i) $\vec{E}_{oi}$ lies IN the plane of incidence "POI"



Now $\vec{D}\cdot\hat{n}$ = continuous $\Rightarrow$ $\epsilon E_{oi}\sin\theta_i + \epsilon E_{or}\sin\theta_i = \epsilon' E_{ot}\sin\theta_t$

$E_{ot}\dfrac{\epsilon' n \sin\theta_i}{n'}$

and $\vec{E}\times\hat{n}$ = continuous,

$$\Rightarrow \boxed{E_{oi}\cos\theta_i - E_{or}\cos\theta_i = E_{ot}\cos\theta_t}$$

and $\vec{H}\times\hat{n}$ = continuous $\Rightarrow$ $\dfrac{1}{\mu}(B_{oi}+B_{or}) = \dfrac{1}{\mu'}B_{ot}$

$$\Rightarrow \boxed{\frac{n}{\mu}\left(E_{oi}+E_{or}\right) = \frac{n'}{\mu'}E_{ot}}$$

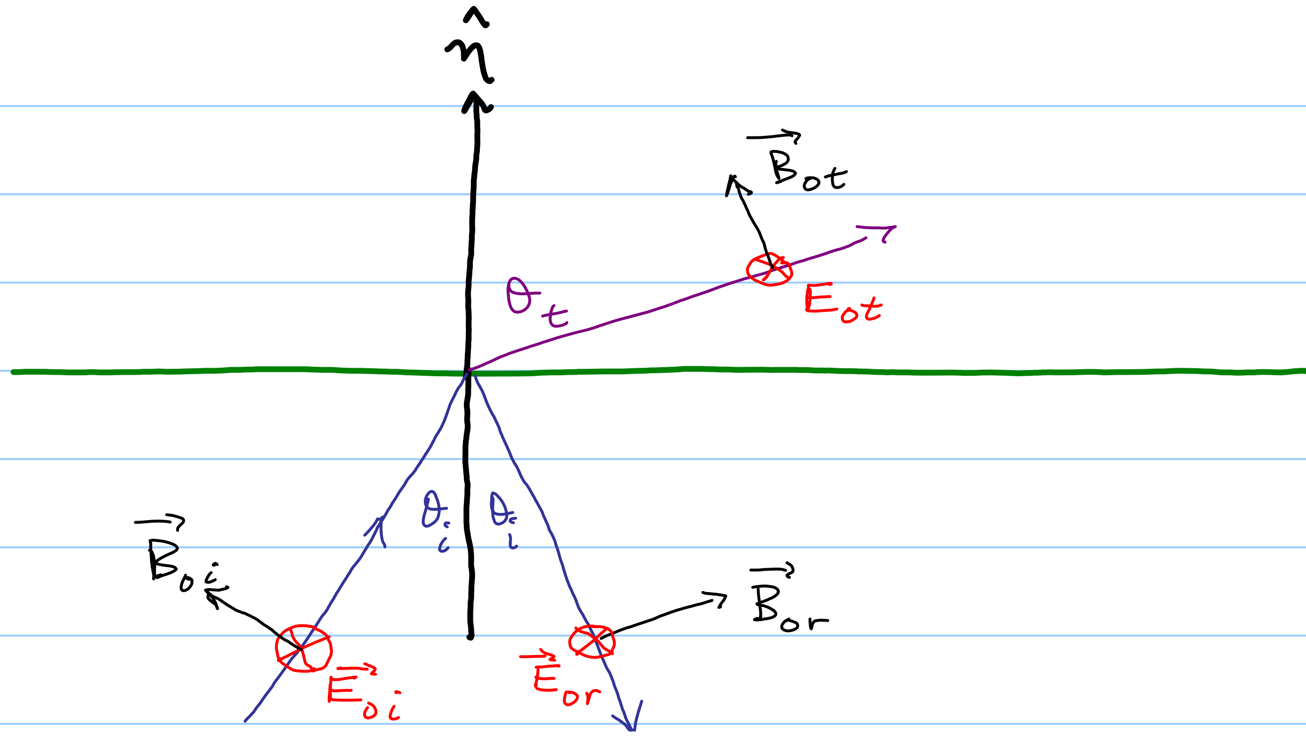
solution of these 2 equations, 2 unknowns, for an assumed known E_{oi}, gives:

$$E_{or} = E_{oi}\,\frac{n'\mu\cos\theta_i - n\mu'\cos\theta_t}{n'\mu\cos\theta_i + n\mu'\cos\theta_t}$$

for $\vec{E}_{oi}$ <u>IN</u> plane of incidence

$$E_{ot} = E_{oi}\,\frac{2n\mu'\cos\theta_i}{n'\mu\cos\theta_i + n\mu'\cos\theta_t}$$

Case (ii) $\vec{E}_{oi} \perp$ to P.O.I.



Continuity of $\vec{E} \times \hat{n} \Rightarrow$ $\boxed{E_{oi} + E_{or} = E_{ot}}$

Continuity of $\vec{H} \times \hat{n} \Rightarrow \frac{1}{\mu}\left(B_{oi}\cos\theta_i - B_{or}\cos\theta_i\right) = \frac{B_{ot}}{\mu'}\cos\theta_t$

or $\boxed{\frac{n}{\mu}\left(E_{oi} - E_{or}\right)\cos\theta_i = \frac{n'}{\mu'} E_{ot}\cos\theta_t}$

Solution is:

$$E_{or} = E_{oi}\,\frac{n\mu'\cos\theta_i - n'\mu\cos\theta_t}{n\mu'\cos\theta_i + n'\mu\cos\theta_t}$$

$\vec{E}_{oi} \perp$ to P.O.I.

$$E_{ot} = E_{oi}\,\frac{2n\mu'\cos\theta_i}{n\mu'\cos\theta_i + n'\mu\cos\theta_t}$$

and we could eliminate θ_t using Snell's law, if desired: $\cos\theta_t = \sqrt{1 - \frac{n^2}{n'^2}\sin^2\theta_i}$

Specialized cases For optical frequencies it is usually accurate to set $\frac{\mu'}{\mu} \simeq 1$, and this gives a simple result for the case of normal incidence, i.e. $\theta_i = 0$:

$\vec{E}_{oi}$ $\parallel$ to P.O.I.

$$\frac{E_{or}}{E_{oi}} = \frac{n'-n}{n'+n}$$

$$\frac{E_{ot}}{E_{oi}} = \frac{2n}{n'+n}$$

$\vec{E}_{oi} \perp$ to P.O.I.

$$\frac{E_{or}}{E_{oi}} = \frac{n-n'}{n+n'}$$

$$\frac{E_{ot}}{E_{oi}} = \frac{2n}{n'+n}$$

These results for E_{or} appear to have a sign difference, a consequence of our choices of coordinate axes – not an actual difference!

Brewster's Angle – At a certain incidence angle, θ_B there is no reflected wave for the $\vec{E}$ – field component $\parallel$ to the P.O.I. To work out the value of θ_B, just set $E_{or} = 0$ above, i.e. (again take $\frac{\mu'}{\mu} \simeq 1$)

$$\frac{E_{or}}{E_{oi}} \propto \frac{\frac{n'}{n}\cos\theta_i - \sqrt{1 - \frac{n^2}{n'^2}\sin^2\theta_i}}{(\text{denominator})} \xrightarrow[\theta_i \to \theta_B]{} 0,$$

and we can solve for θ_B,

$$\left(\frac{n'}{n}\right)^2 \cos^2\theta_B = 1 - \frac{n^2}{n'^2}\sin^2\theta_B$$

$$= \cos^2\theta_B + \sin^2\theta_B\left(1 - \frac{n^2}{n'^2}\right)$$

$$\Rightarrow \tan^2\theta_B = \frac{\left(\frac{n'}{n}\right)^2 - 1}{1 - \frac{n^2}{n'^2}}$$

which simplifies to:

$$\tan\theta_B = \frac{n'}{n}$$ = Brewster angle

$\Rightarrow$ At the Brewster angle, all reflected light is polarized $\perp$ to the P.O.I., i.e. $\parallel$ to the surface itself.

e.g. – for a water surface,

$\frac{n'}{n} \simeq 1.35 \Rightarrow \theta_B(\text{water}) \simeq 53°$

while for glass, $\frac{n'}{n} \simeq 1.5 \Rightarrow \theta_B(\text{glass}) \simeq 56°$

Next: Total internal reflection

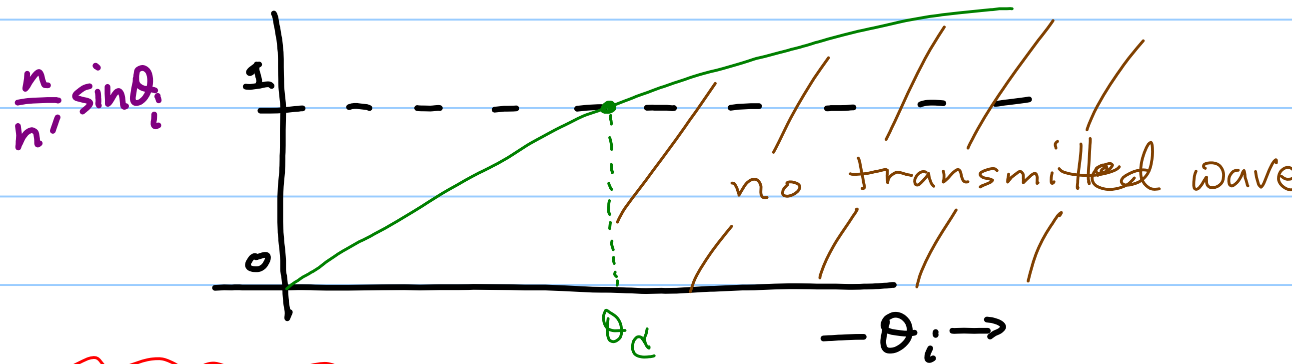
$\Rightarrow$ Suppose light is incident from a medium of <u>higher</u> refractive index, $n > n'$

e.g. water $\longrightarrow$ air

Then since $\sin\theta_t = \frac{n}{n'}\sin\theta_i$,

or $\theta_t = \sin^{-1}\left(\frac{n}{n'}\sin\theta_i\right)$

$\Rightarrow$ beyond a certain angle $\theta_i > \theta_c$, clearly $\frac{n}{n'} \sin\theta_i > 1$ so it has no real arcsin,



$$\Rightarrow \theta_c = \sin^{-1}\left(\frac{n'}{n}\right) \equiv \text{critical angle}$$

$\Rightarrow$ no transmitted light and the reflection is total

We can explore the nature of the transmitted wave at $\theta_i > \theta_c$, by analytically continuing our previous derived solution, which simply means inserting $k_t \rightarrow$ imaginary!

To see this, recall that Snell's law was derived from $\vec{k}_i \cdot \vec{x}\big|_{z=0} = \vec{k}_t \cdot \vec{x}\big|_{z=0}$

$$\Rightarrow k_i \sin\theta_i = k_t \sin\theta_t$$

$$\text{or} \quad \frac{n\omega}{c}\sin\theta_i = \frac{n'\omega}{c}\sin\theta_t$$

i.e. the component of $\vec{k}$ $\parallel$ to interface is continuous, $\Rightarrow k_{ix} = k_{tx}$

So when $\frac{n}{n'}\sin\theta_i > 1$, for $\theta_i > \theta_c$,

$\Rightarrow \sin\theta_t > 1$, whereby $\cos^2\theta_t = 1 - \sin^2\theta_t < 0$

$\Rightarrow \cos\theta_t =$ purely imaginary

But the normal component of the transmitted wave vector $\vec{k}_t$ is $k_{tz} = k_t \cos\theta_t \rightarrow$ imaginary

$$\equiv i\kappa$$

where $\kappa = \sqrt{\frac{n^2}{n'^2}\sin^2\theta_i - 1}$

Then the transmitted wave looks like

$$\vec{E}_t = \vec{E}_{ot} e^{i\vec{k}_t\cdot\vec{x}} = \vec{E}_{ot} e^{i(k_{tz}z + k_{tx}x)}$$

$$\Rightarrow \boxed{\vec{E}_t = \vec{E}_{ot} e^{-\kappa z} e^{ik_t \sin\theta}}$$

and we note this exponential damping of the transmitted wave in the direction normal to the surface.

Let's calculate the time-averaged energy flow through the surface:

$$\langle \vec{S}\cdot\hat{z}\rangle = \frac{1}{2}\,\mathrm{Re}\left(\hat{z}\cdot\vec{E}_t \times \vec{H}^*\right)$$

where $\vec{H}_t = (\mu'\epsilon')^{1/2}\frac{\hat{k}_t \times \vec{E}_t}{\mu'} = \frac{c}{n'\omega}\frac{(\mu'\epsilon')^{1/2}}{\mu'}\vec{k}_t \times \vec{E}_t$

$$= \frac{1}{\mu'\omega}\vec{k}_t \times \vec{E}_t$$

Thus $\langle \vec{S} \cdot \hat{z} \rangle = \frac{1}{2} \mathrm{Re}\left[\hat{z} \cdot \vec{E}_t \times (\vec{k}_t \times \vec{E}_t)^* \right] \frac{1}{\mu' \omega}$

$$= \frac{1}{2\mu'\omega} \mathrm{Re}\left(\hat{z} \cdot \vec{k}_t \, |\vec{E}_t|^2 \right)$$

$= 0$ since $\mathrm{Re}(k_{tz}) = 0$!

$\Rightarrow$ ALL ENERGY is Reflected!

On your own, read about:

- Evanescent waves
- light tunneling across a total internal reflection gap