Sec. 6.10 Transformation properties of $\vec{E}, \vec{B}, \Phi, \vec{A}, \rho, \vec{J}, \ldots$

Observe – For the equations interrelating various quantities like $\vec{J}, \vec{B}, \frac{\partial \vec{E}}{\partial t}$, etc to make sense, the terms in an equation must ALL behave the same under the operations of

- coordinate rotations
- coordinate inversions
- time-reversal

A) Rotations ⇒ These are orthogonal transformations that replace the original coordinates x_β by transformed coordinates x'_α

$$\Rightarrow \boxed{x'_\alpha = \sum_\beta a_{\alpha\beta} x_\beta} \qquad (*)$$

where orthogonality is expressed by

$$\underline{a}^T \underline{a} = \underline{I}, \text{ or } \sum_\beta (a^T)_{\alpha\beta}\, a_{\beta\gamma} = \delta_{\alpha\gamma}$$

which ensures that $\vec{x}'^2 = \vec{x}^2$

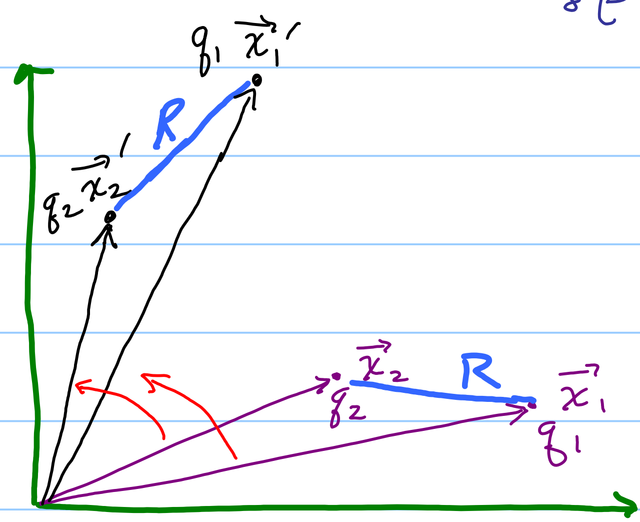
And if $\det(\underline{a}) = +1$, the rotation can be replaced by a sequence of infinitesimal steps and is a "PROPER ROTATION" while if $\det(\underline{a}) = -1$, it is an "IMPROPER ROTATION" which requires a reflection plus a rotation.

- Positions $\vec{x}_i$, Velocities $\vec{v}_i$, momenta $\vec{p}_i$ all transform according to (*) and these are VECTORS = tensors of rank 1

- Scalar products like $\vec{x}_1 \cdot \vec{x}_2$ or $\vec{x}_1 \cdot \vec{v}_2$ are invariant under rotations and are called SCALARS = tensors of rank 0

- The Maxwell Stress Tensor is an example of a rank 2 tensor, i.e.

$$\overleftrightarrow{T}'_{\alpha\beta} = \sum_{\gamma\delta} a_{\alpha\gamma} \overleftrightarrow{T}_{\gamma\delta} (\underline{a}^T)_{\delta\beta}$$



e.g. an active rotation of all physical vectors through the same angle preserves scalars such as $\dfrac{1}{|\vec{x}_1 - \vec{x}_2|}$

Behavior of <u>differential</u> <u>operators</u> under rotation:

$\nabla \Phi$ = vector, assuming Φ = scalar

$\nabla \cdot \vec{V}$ = scalar, if $\vec{V}$ = vector

$\nabla \cdot \overleftrightarrow{T}$ = vector, if $\overleftrightarrow{T}$ = 2^{nd}-rank tensor

$\nabla^2 \Phi$ = scalar

$\nabla^2 \vec{A}$ = vector, etc....

B) Spatial inversions and reflections

→ INVERSION (I) changes the sign of all 3 coordinates of a position vector, i.e.

$$\vec{x} = x\hat{x} + y\hat{y} + z\hat{z}$$

and $$I\vec{x} = -x\hat{x} - y\hat{y} - z\hat{z}$$

REFLECTION (R) reflects through one chosen plane, i.e.

$$R_y \vec{x} = x\hat{x} - y\hat{y} + z\hat{z}$$

⇒ Neither R nor I can be accomplished by simple (ie. PROPER) rotations.

<u>Polar vectors</u> (or vectors) – these transform like (*) under <u>proper</u> <u>rotations</u>, i.e. $V'_\alpha = \sum_\beta a_{\alpha\beta} V_\beta$

AND they change sign under <u>inversion</u>, i.e. $\vec{V}' = I\vec{V} = -\vec{V}$

Axial vectors (or **pseudovectors**) – these rotate the same way (*) but they are SYMMETRIC under inversion

i.e. $\vec{L}' = I\vec{L} = +\vec{L}$

e.g. a cross product of 2 polar vectors,

$$I(\vec{x}\times\vec{p}) = (-\vec{x})\times(-\vec{p}) = \vec{x}\times\vec{p}$$

Note that pseudovectors and pseudoscalars imply a "TWIST" somewhere in the physics!

Pseudoscalars are ODD under inversion:

$$I[\vec{x}_1\cdot(\vec{x}_2\times\vec{p}_2)] = [(-\vec{x}_1)\cdot(-\vec{x}_2)\times(-\vec{p}_2)]$$
$$= -\vec{x}_1\cdot(\vec{x}_2\times\vec{p}_2)$$

c) **TIME REVERSAL** – the classical microscopic laws of physics are invariant if the direction of time is reversed, i.e. if time runs backwards, $\Rightarrow T\, dt = -dt'$

Electromagnetic entities

Entity	Proper rotations	Inversions	Time Reversal	Overall
charge	scalar	even	even	true scalar
E-field ($\nabla\cdot\vec{E} = \rho/\epsilon_0$)	vector	odd	even	polar vector
B-field ($\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$)	vector	even	odd	pseudovector
Current $\vec{J} = \rho\vec{v}$	vector	odd	odd	polar vector

see Table 6.1, p 271 for more

Question: What are the most general relations among $\vec{E}, \vec{B}$ and their sources, consistent with
- rotational invariance
- inversions
- time-reversal invariance ?

Answer: suppose the relationships are linear in $\vec{E}, \vec{B}$ and first derivatives w.r.t. $\vec{x}, t$ and linear in the sources

Notation: call R = rotations
I = inversions
T = time-reversal

Behavior under

R	I	T	entities
vector	odd	even	$\vec{E}$
vector	odd	odd	$\frac{\partial \vec{E}}{\partial t}$, $\nabla \times \vec{B}$, $\vec{J}$
pseudovector	even	odd	$\vec{B}$
pseudovector	even	even	$\frac{\partial \vec{B}}{\partial t}$, $\nabla \times \vec{E}$ (and $\vec{J}_m = \rho_m \vec{v}$ magnetic monopole current density if ever found)
scalar	even	even	$\nabla \cdot \vec{E}$, ρ
pseudoscalar	odd	odd	$\nabla \cdot \vec{B}$ (and ρ_m = magnetic monopole charge if ever found)

This table gives deeper insight into why Maxwell's equations are what they are!