Retarded solutions for the potentials (Lorentz gauge)

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{[\rho(\vec{x}',t')]_{ret}}{R}$$

and

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{[\vec{J}(\vec{x}',t')]_{ret}}{R}$$

- where $[\ \ ]_{ret}$ means to evaluate at the <u>retarded time</u> $t' = t - \frac{R}{c}$

- and where $\vec{R} \equiv \vec{x} - \vec{x}'$,

$R = |\vec{x} - \vec{x}'|$, $\hat{R} = \frac{\vec{R}}{R}$

$\vec{E}$- and $\vec{B}$-fields if no media are present

Start from Maxwell's equations $\nabla\cdot\vec{E} = \rho/\epsilon_0$, $\nabla\cdot\vec{B} = 0$

and

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \nabla\times(\nabla\times\vec{E}) + \frac{\partial}{\partial t}\nabla\times\vec{B} = 0$$

$$\Rightarrow \nabla(\underbrace{\nabla\cdot\vec{E}}_{\rho/\epsilon_0}) - \nabla^2\vec{E} + \frac{\partial}{\partial t}(\underbrace{\nabla\times\vec{B}}_{\mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}) = 0$$

$$\Rightarrow \nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial\vec{J}}{\partial t} + \frac{1}{\epsilon_0}\nabla\rho$$

or

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\frac{1}{\epsilon_0}\left(-\nabla\rho - \frac{1}{c^2}\frac{\partial\vec{J}}{\partial t}\right)$$

and similarly for $\vec{B}$:

$$\nabla \times (\nabla \times \vec{B}) = \mu_0 \nabla \times \vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\underbrace{(\nabla \times \vec{E})}_{-\frac{\partial \vec{B}}{\partial t}}$$

$$\Rightarrow \nabla(\underbrace{\nabla \cdot \vec{B}}_{0}) - \nabla^2 \vec{B} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = \mu_0 \nabla \times \vec{J}$$

$$\Rightarrow \boxed{\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = -\mu_0 \nabla \times \vec{J}}$$

Since "the same equations have the same solutions" = Feynman's dictum, we can immediately solve these inhomogeneous wave equations as:

$$\boxed{\begin{aligned} \vec{E}(\vec{x},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{1}{R}\left[-\nabla'\rho - \frac{1}{c^2}\frac{\partial \vec{J}}{\partial t'}\right]_{ret} \\ \vec{B}(\vec{x},t) &= \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{R}\left[\nabla' \times \vec{J}\right]_{ret} \end{aligned}}$$

An important aside: working with retarded sources requires <u>care</u>, e.g. observe that

$$\left[\nabla' f(\vec{x}',t')\right]_{ret} \neq \nabla'\left[f(\vec{x}',t')\right]_{ret}$$

Why? Because $[\nabla' f(\vec{x}',t')]_{ret}$ means that the gradient is taken with t' held fixed whereas $[f(\vec{x}',t')]_{ret} = f\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$ implies a gradient <u>with t fixed</u>.

Hence $[\nabla'\rho]_{ret} = \nabla'[\rho]_{ret} - \left[\frac{\partial\rho}{\partial t'}\right]_{ret} \nabla'(t - \frac{R}{c})$

since, e.g. $\nabla'\rho(\vec{x}', t - \frac{R}{c}) = [\nabla'\rho(\vec{x}',t')]_{ret} + \left[\frac{\partial\rho(\vec{x}',t')}{\partial t'}\right]_{ret} \nabla'(t - \frac{R}{c})$

and $\boxed{\nabla' R = -\hat{R}}$

or $$\boxed{[\nabla'\rho]_{ret} = \nabla'[\rho]_{ret} - \frac{\hat{R}}{c}\left[\frac{\partial\rho}{\partial t'}\right]_{ret}}$$

Next we need $[\nabla' \times \vec{J}]_{ret}$:

Consider, with implied summation over repeated indices:

$$\epsilon_{ijk}[\partial_i' J_j]_{ret} = \epsilon_{ijk}\partial_i'[J_j]_{ret} - \epsilon_{ijk}\left[\frac{\partial J_j}{\partial t'}\right]_{ret} \partial_i'(t - \frac{R}{c})$$

which is rewritten in vector notation as

$$[\nabla' \times \vec{J}]_{ret} = \nabla' \times [\vec{J}]_{ret} + \frac{1}{c}(\nabla' R) \times \left[\frac{\partial \vec{J}}{\partial t'}\right]_{ret}$$

or finally

$$\boxed{[\nabla' \times \vec{J}]_{ret} = \nabla' \times [\vec{J}]_{ret} + \frac{1}{c}\left[\frac{\partial \vec{J}}{\partial t'}\right] \times \hat{R}}$$

Next let's use these expressions in our solution for $\vec{E}$:

$$\vec{E}(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{ \frac{-\nabla'[\rho]_{ret}}{R} + \frac{\hat{R}}{cR}\left[\frac{\partial\rho}{\partial t'}\right]_{ret} - \frac{1}{c^2 R}\left[\frac{\partial\vec{J}}{\partial t'}\right]_{ret}\right\}$$

Integration by parts can simplify this a bit,

e.g. $$\int d^3x' \frac{1}{R}\partial_i'[\rho]_{ret} = -\int d^3x' [\rho]_{ret} \underbrace{\left(\partial_i' \frac{1}{R}\right)}_{= \frac{\hat{R}}{R^2}} + \underbrace{(\text{surface terms})}_{\to 0 \text{ if a localized source}}$$

Giving

$$\boxed{\vec{E}(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{ \frac{\hat{R}}{R^2}[\rho(\vec{x}',t')]_{ret} + \frac{\hat{R}}{cR}\left[\frac{\partial\rho(\vec{x}',t')}{\partial t'}\right]_{ret} - \frac{1}{c^2R}\left[\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret}\right\}}$$

= generalized Coulomb Law

And tackling $\vec{B}$ similarly, i.e.

$$\int \left(\nabla' \times [\vec{J}]_{ret}\right) \frac{1}{R}\, d^3x' = -\int \left(\nabla' \frac{1}{R}\right) \times [\vec{J}]_{ret}\, d^3x' + (\cancel{\text{surface terms}}) \to 0$$

$$= -\int \frac{\hat{R}}{R^2} \times [\vec{J}]_{ret}$$

giving

$$\boxed{\vec{B}(\vec{x},t) = \frac{\mu_0}{4\pi} \int d^3x' \left\{ [\vec{J}(\vec{x}',t')]_{ret} \times \frac{\hat{R}}{R^2} + \left[\frac{\partial \vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret} \times \frac{\hat{R}}{cR} \right\}}$$

= generalized Biot-Savart Law

Special Case: A point charge q in motion, $\vec{r}_0(t)$:

$$\Longrightarrow \quad \rho(\vec{x}',t') = q\,\delta[\vec{x}' - \vec{r}_0(t')]$$

$$\vec{J}(\vec{x}',t') = \rho(\vec{x}',t') \frac{d\vec{r}_0(t')}{dt'} = \rho\,\vec{v}(t')$$

gives (show this on your own):

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \left\{ \left[\frac{\hat{R}}{\kappa R^2}\right]_{ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{R}}{\kappa R}\right]_{ret} - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\vec{v}}{\kappa R}\right]_{ret} \right\}$$

$$\vec{B} = \frac{\mu_0 q}{4\pi} \left\{ \left[\frac{\vec{v} \times \hat{R}}{\kappa R^2}\right]_{ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\vec{v} \times \hat{R}}{\kappa R}\right]_{ret} \right\}$$

where $\kappa = 1 - \frac{\vec{v} \cdot \hat{R}}{c}$, and "Feynman's form" is

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \left\{ \left[\frac{\hat{R}}{R^2}\right]_{ret} + \frac{[R]_{ret}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{R}}{R^2}\right]_{ret} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[\hat{R}]_{ret} \right\}$$

$$\vec{B} = \frac{\mu_0 q}{4\pi} \left\{ \left[\frac{\vec{v} \times \hat{R}}{\kappa^2 R^2}\right]_{ret} + \frac{1}{c[R]_{ret}}\frac{\partial}{\partial t}\left[\frac{\vec{v} \times \hat{R}}{\kappa}\right]_{ret} \right\}$$

Poynting's Theorem - Conservation of energy & momentum

For a single charge q, the rate at which work is done by external $\vec{E}, \vec{B}$-fields is

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \vec{v} = q\,\vec{v} \cdot \vec{E}$$

and this is often stated as:

"B-fields do no work on q"

or if charge and current are distributed in space, then the rate of change of energy within any finite volume V is

$$\frac{dW}{dt} = \int_V \vec{J} \cdot \vec{E}\, d^3x$$

Next, explore explicitly how the EM-field energy within V changes, let's eliminate $\vec{J}$ using

$$\vec{J} = \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \int_V \vec{J} \cdot \vec{E}\, d^3x = \int_V \left\{ \vec{E} \cdot \left(\nabla \times \vec{H}\right) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right\} d^3x$$

But $$\nabla \cdot \left(\vec{E} \times \vec{H}\right) = \vec{H} \cdot \underbrace{\left(\nabla \times \vec{E}\right)}_{= -\frac{\partial \vec{B}}{\partial t}} - \vec{E} \cdot \left(\nabla \times \vec{H}\right)$$

whereby

$$\int_V \vec{J} \cdot \vec{E}\, d^3x = -\int_V \left[\nabla \cdot \left(\vec{E} \times \vec{H}\right) + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}\right] d^3x$$

Suppose now that the medium is linear, so the total energy density becomes

$$u = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$$

$$\Rightarrow \int_V \left[\frac{\partial u}{\partial t} + \nabla\cdot(\vec{E}\times\vec{H})\right] d^3x = -\int_V \vec{J}\cdot\vec{E}\, d^3x$$

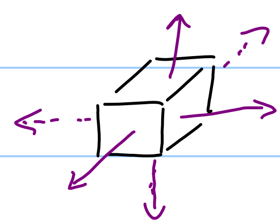
But since this holds for ANY volume V, the integrands must be equal, i.e.

$$\boxed{\frac{\partial u}{\partial t} + \nabla\cdot\vec{S} = -\vec{J}\cdot\vec{E}}$$

where $\vec{S} \equiv \vec{E}\times\vec{H}$ is called the Poynting vector

Interpretation

$\nabla\cdot\vec{S}$ represents the flow of energy OUT FROM a unit volume d^3x



$\vec{J}\cdot\vec{E}$ is the WORK DONE (per unit volume, per unit time) by the EM FIELDS, which is converted into mechanical or heat energy of the charged particles

i.e.,
$$\frac{dW^{mech}}{dt} = \int_V \vec{J}\cdot\vec{E}\, d^3x$$

$\Rightarrow W^{mech}$ = total energy of particles within V, assuming that no particles leave or enter V

And from previous analysis we know that

$$W^{fields} = \int_V u\, d^3x = \frac{1}{2}\int(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})\, d^3x$$

(for linear media)

no media $\rightarrow \dfrac{\epsilon_0}{2}\displaystyle\int_V (\vec{E}^2 + c^2\vec{B}^2)\, d^3x$

Thus we see that Poynting's theorem expresses energy conservation for the combined system:

$$\boxed{\frac{dW^{total}}{dt} = \frac{d}{dt}\left(W^{fields} + W^{mech}\right) = -\boxed{\oint_S \hat{n}\cdot\vec{S}\, da}}$$

total field energy in volume V, $W^{fields} = \int_V u\, d^3x$

total mechanical energy of particles in V

rate of EM field energy flow out of V through the surface S

Conservation of Linear Momentum

Start from the total electromagnetic force on a charge q:

$$\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B}) \longrightarrow \int_V (\rho\vec{E} + \rho\vec{v}\times\vec{B})\,d^3x$$

Then if we call $\vec{P}_{mech}$ = sum of the momenta of all particles in a volume V, we have:

$$\frac{d}{dt}\vec{P}_{mech} = \int_V (\rho\vec{E} + \vec{J}\times\vec{B})\,d^3x$$

Next, use Maxwell's equations to eliminate the sources $\rho, \vec{J}$, again assuming no media.

$$\Rightarrow \rho = \epsilon_0 \nabla\cdot\vec{E}, \quad \vec{J} = \frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}(\nabla\cdot\vec{E}) + \frac{1}{\mu_0\epsilon_0}(\nabla\times\vec{B})\times\vec{B} - \frac{\partial\vec{E}}{\partial t}\times\vec{B}\right],$$

and observe that

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

So we can write also:

we've added 0 here

$$\rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0\Big[\vec{E}(\nabla\cdot\vec{E}) + c^2\vec{B}(\nabla\cdot\vec{B}) - \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times(\nabla\times\vec{E}) - c^2\vec{B}\times(\nabla\times\vec{B})\Big]$$

Thus we get:

$$\frac{d\vec{P}_{mech}}{dt} + \frac{d}{dt}\boxed{\int_V \epsilon_0 \vec{E}\times\vec{B}\, d^3x}$$

tentatively identify this as the total EM momentum i.e. $\vec{P}_{field} = \int d^3x\, \vec{g}$, where $\vec{g} = \epsilon_0 \vec{E}\times\vec{B} = c^{-2}\vec{E}\times\vec{H}$

$$= \epsilon_0 \int_V \left\{ \vec{E}(\nabla\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E}) + c^2\vec{B}(\nabla\cdot\vec{B}) - c^2\vec{B}\times(\nabla\times\vec{B}) \right\} d^3x$$

Logic: To establish this as a momentum conservation law, we want to convert the RHS of this preceding equation into a surface integral. We would then interpret the surface integral as a flow of field momentum through the closed surface S into the volume V.

$\Rightarrow$ Consider the Cartesian components, use the notations $\{1,2,3\} \longleftrightarrow \{x,y,z\}$, i.e. $\alpha,\beta,\gamma = 1,2,3$, etc.

$$\Rightarrow \left[\vec{E}(\nabla\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E})\right]_\gamma = E_\gamma \sum_\beta \partial_\beta E_\beta - \sum_{\alpha\beta} \epsilon_{\alpha\beta\gamma} E_\alpha (\nabla\times\vec{E})_\beta$$

$$= E_\gamma \sum_\beta \partial_\beta E_\beta - \sum_{\alpha\beta}\sum_{ij} \epsilon_{\alpha\beta\gamma}\epsilon_{ij\beta} E_\alpha \partial_i E_j$$

(Now use $\sum_\beta \epsilon_{\beta\gamma\alpha}\epsilon_{\beta ij} = \delta_{\gamma i}\delta_{\alpha j} - \delta_{\gamma j}\delta_{\alpha i}$)

$$= E_\gamma \sum_\beta \partial_\beta E_\beta - \sum_{\alpha ij} \delta_{\gamma i}\delta_{\alpha j} E_\alpha \partial_i E_j + \sum_{\alpha ij}\delta_{\gamma j}\delta_{\alpha i} E_\alpha \partial_i E_j$$

$$= E_\gamma \sum_\alpha \partial_\alpha E_\alpha - \sum_\alpha E_\alpha \partial_\gamma E_\alpha + \sum_\alpha E_\alpha \partial_\alpha E_\gamma$$

or $\left[\vec{E}(\nabla\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E})\right]_\gamma$

$$= \sum_\alpha \frac{\partial}{\partial x_\alpha}(E_\alpha E_\gamma) - \frac{1}{2}\sum_\alpha \frac{\partial}{\partial x_\gamma}(E_\alpha^2)$$

$$= \sum_\alpha \frac{\partial}{\partial x_\alpha}\left[E_\alpha E_\gamma - \frac{1}{2}\delta_{\alpha\gamma}\vec{E}\cdot\vec{E}\right]$$

Observe that for <u>each</u> component γ, this $[\;]_\alpha$ is a 3-component quantity i.e. a <u>vector</u> Q_α

In other words the RHS, for each γ, is the DIVERGENCE of the vector $\vec{Q}$, (i.e. $\nabla\cdot\vec{Q}$)

We can put this into the standard form by changing the index $\gamma \to \beta$, and now define the symmetric <u>MAXWELL STRESS TENSOR</u> as

$$T_{\alpha\beta} \equiv \epsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})\delta_{\alpha\beta}\right]$$

We have thus proven that

$$\frac{d}{dt}\left(\vec{P}^{\,mech} + \vec{P}^{\,field}\right) = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\beta\alpha}\, d^3x$$

$$= \int_V (\nabla\cdot\overleftrightarrow{T})_\alpha\, d^3x = \oint_S n_\beta T_{\beta\alpha}\, da$$

components of outward normal $\hat{n}$

Or finally we have simply:

$$\boxed{\frac{d}{dt}\left(\vec{P}^{\,mech} + \vec{P}^{\,field}\right) = \oint_S da\,\hat{n}\cdot\overleftrightarrow{T}}$$

WφφT!

(This looks slightly different than Jackson's 6.122, but owing to the symmetry of $T_{\alpha\beta}$, they are equivalent)

Notational Comments

$\vec{a}\cdot\overleftrightarrow{T}$ is a vector: $(\vec{a}\cdot\overleftrightarrow{T})_\alpha = \sum_\beta a_\beta T_{\beta\alpha}$

whereas

$\vec{a}\cdot\overleftrightarrow{T}\cdot\vec{b}$ is a scalar, $\sum_{\alpha\beta} a_\alpha T_{\alpha\beta} b_\beta$

Physical Interpretation

$(\hat{n}\cdot\overleftrightarrow{T})_\alpha = \sum_\beta n_\beta T_{\beta\alpha}$ is the α-th component of the

EM momentum flow per unit area

across the surface S and into volume V.

Moreover, since $\vec{F} = \frac{d\vec{p}}{dt}$, we can also view $(\hat{n}\cdot\overleftrightarrow{T})_\alpha$ as the α^{th} component of the FORCE PER UNIT AREA that is transmitted onto the particles and fields within V.