Chapter 6 Maxwell's equations

To this point we have 4 incomplete maxwell equations,

$$\nabla\cdot\vec{D} = \rho \qquad \boxed{\nabla\times\vec{H} = \vec{J}}$$

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad \nabla\cdot\vec{B} = 0$$

apply $\nabla\cdot(\nabla\times\vec{H}) = \underline{0 = \nabla\cdot\vec{J}}$

But we know that the general form of the continuity equation for change and current reads

$$\boxed{\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$$

not consistent! in general!

To see how to make them consistent, rewrite as:

$$\nabla\cdot\vec{J} + \nabla\cdot\frac{\partial\vec{D}}{\partial t} = 0 = \nabla\cdot\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right),$$

which suggests that we should replace $\vec{J}$ (in the Maxwell equation $\nabla\times\vec{H} = \vec{J}$) by $\vec{J} + \frac{\partial\vec{D}}{\partial t}$.

$$\Rightarrow \boxed{\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}}$$

MAXWELL's EQUATIONS

$$\boxed{\nabla\cdot\vec{D} = \rho} \qquad \boxed{\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0} \qquad \boxed{\nabla\cdot\vec{B} = 0}$$

Electromagnetic Potentials

Start by solving the 2 homogeneous Maxwell equations "automatically"

Since $\nabla \cdot \vec{B} = 0 \implies \boxed{\vec{B} = \nabla \times \vec{A}}$

and Faraday's Law $\implies \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$

which implies that $\rightarrow \boxed{\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \Phi}$

or $\boxed{\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}}$

Now, specialize to a vacuum, no media

$$\implies \vec{D} = \epsilon_0 \vec{E}$$
$$\vec{H} = \vec{B}/\mu_0$$

Next, look at the 2 inhomogeneous equations:

$\implies \nabla \cdot \left(-\nabla \Phi - \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\epsilon_0}$ (Gauss' Law)

$$\implies \boxed{\nabla^2 \Phi + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}}$$

and

$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$ (Ampere's Law + Maxwell's term)

$$\implies \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \frac{1}{c^2}\left(\nabla \frac{\partial \Phi}{\partial t} + \frac{\partial^2 \vec{A}}{\partial t^2}\right)$$

which has used $\epsilon_0 \mu_0 = \frac{1}{c^2}$

$$\implies \boxed{\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \vec{J}}$$

Gauge Choices

Observe that $\vec{B}$ will not be changed if we modify $\vec{A} \longrightarrow \vec{A} + \nabla\Lambda \equiv \vec{A}'$,
where Λ = any scalar function

However, under this change $\vec{A} \longrightarrow \vec{A}'$,
$\vec{E}$ will change UNLESS we simultaneously change $\Phi \longrightarrow \Phi - \frac{\partial \Lambda}{\partial t} \equiv \Phi'$

Throughout most of this course we will choose our gauge such that $(\vec{A}, \Phi)$ obey the LORENTZ CONDITION,

$$\boxed{\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0}$$

which is advantageous because the wave equations for $(\vec{A}, \Phi)$ then look symmetrical AND uncoupled:

$$\boxed{\begin{aligned} \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -\rho/\epsilon_0 \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} \end{aligned}}$$

These are the inhomogeneous wave equations for $(\vec{A}, \Phi)$ in the LORENTZ GAUGE

A different gauge sometimes adopted is characterized by $\boxed{\nabla \cdot \vec{A} = 0}$,

which is called by different names: the Coulomb gauge

" radiation "

" transverse "

$$\Rightarrow \quad \nabla^2 \Phi = -\rho/\epsilon_0$$

whose solution we already know to be:

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}', t)\, d^3x'}{|\vec{x} - \vec{x}'|}$$

whereas the vector potential equation is more complicated in the Coulomb gauge, namely:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \frac{1}{c^2}\nabla \frac{\partial \Phi}{\partial t}$$

Aside: It is well-known that ANY vector field can be written as the sum of a longitudinal term $\vec{J}_\ell$ obeying $\nabla \times \vec{J}_\ell = 0$ PLUS a transverse term $\vec{J}_t$ obeying $\nabla \cdot \vec{J}_t = 0$ (see Jackson, top of p. 242)

$$\Rightarrow \quad \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}_t$$

Green function for the inhomogeneous wave equation

We see that we must solve wave equations having the form:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(\vec{x},t) = -4\pi f(\vec{x},t)$$

where

$f(\vec{x},t)$ = a known, specified source

This is a linear, homogeneous equation, so it can be solved using Green's function methods!

WφφT!

Specifically, start by solving for G, obeying

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\vec{x},t;\vec{x}\,',t') = -4\pi\,\delta(\vec{x}-\vec{x}\,')\,\delta(t-t')$$

and once we know G, we can solve the original PDE in the form

$$\Psi(\vec{x},t) = \int\!\!\int G(\vec{x},t;\vec{x}\,',t')\,f(\vec{x}\,',t')\,d^3x'\,dt'$$

(assumes that G obeys the same BCs as Ψ)

Considerations relevant to finding G:

(1) Only RELATIVE times (not absolute) should matter $\Rightarrow$ $G(\vec{x},t;\vec{x}\,',t') \rightarrow G(\vec{x},\vec{x}\,';t-t')$

(2) And in free space with no boundaries, only RELATIVE positions should matter,

$\Rightarrow G(\vec{x},\vec{x}\,';t-t') \rightarrow G(\vec{x}-\vec{x}\,',t-t') \equiv G(\vec{R},\tau)$

where $\vec{R} \equiv \vec{x} - \vec{x}'$

$\tau \equiv t - t'$

We can solve this using Fourier analysis in time, τ by writing

$$G(\vec{R},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau}\, \mathcal{G}(\vec{R},\omega)$$

After plugging this into the PDE, we get the following equation for $\mathcal{G}(\vec{R},\omega)$:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{R},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau}\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\mathcal{G}(\vec{R},\omega)$$

$$= -4\pi\,\delta(\vec{R})\,\delta(\tau) = \frac{1}{2\pi}\int_{-\infty} d\omega\, e^{-i\omega\tau}\left(-4\pi\delta(\vec{R})\right)$$

and this is obeyed provided $\mathcal{G}$ obeys:

$$\boxed{\left(\nabla^2 + k^2\right)\mathcal{G}(\vec{R},\omega) = -4\pi\,\delta(\vec{R})}\,, \quad \left(\text{where } k = \frac{\omega}{c}\right)$$

This equation can be solved by various methods, but let's use the method of eigenfunction expansion in <u>ALL</u> coordinates. Recall that, in free space, an orthonormal set of eigenfunctions of $-\nabla^2$ are:

$$u_{\vec{k}'}(\vec{R}) = (2\pi)^{-3/2} e^{i\vec{k}'\cdot\vec{R}}$$

which obey $\quad -\nabla^2 u_{\vec{k}'}(\vec{R}) = k'^2 u_{\vec{k}'}(\vec{R})$

and $$\int u^*_{\vec{k}'}(\vec{R})\, u_{\vec{k}''}(\vec{R})\, d^3R = \delta(\vec{k}'-\vec{k}'')$$

Since these states are orthonormal & complete, we can expand as

$$G(\vec{R},\omega) = \int d^3k' \, \frac{e^{i\vec{k}'\cdot\vec{R}}}{(2\pi)^{3/2}} \, a_\omega(\vec{k}')$$

Plugging this into the PDE for G gives:

$$(\nabla^2+k^2)\int d^3k' \, \frac{e^{i\vec{k}'\cdot\vec{R}}}{(2\pi)^{3/2}} \, a_\omega(\vec{k}')$$

$$= \int d^3k' \, (-k'^2+k^2)\, \frac{e^{i\vec{k}'\cdot\vec{R}}}{(2\pi)^{3/2}} \, a_\omega(\vec{k}') = -4\pi\,\delta(\vec{R})$$

and since we know that

$$(2\pi)^{-3}\int d^3k' \, e^{i\vec{k}'\cdot\vec{R}} = \delta(\vec{R})$$

we deduce that $$a_\omega(\vec{k}') = \frac{4\pi}{k'^2-k^2}\,\frac{1}{(2\pi)^{3/2}}$$

$$\Rightarrow G(\vec{R},\omega) = \frac{4\pi}{(2\pi)^3}\int d^3k' \, \frac{e^{i\vec{k}'\cdot\vec{R}}}{k'^2-k^2}$$

This integral can be evaluated by choosing spherical coordinates in k'-space, with the k'_z-axis chosen along $\hat{R}$, whereby we have axial symmetry, an integrand independent of $\varphi_{k'}$, so

$$\int d^3k' \longrightarrow 2\pi\int k'^2 dk' \int dx' \qquad \text{with } x' \equiv \cos\theta_{k'} = \hat{k}'\cdot\hat{R}$$

and $$\vec{k}'\cdot\vec{R} = k'Rx'$$

Thus $G(\vec{R},\omega) = \frac{4\pi}{(2\pi)^2}\int_0^\infty k'^2 dk' \frac{k'^2}{k'^2-k^2}\int_{-1}^{1} dx' e^{ik'Rx'}$

$$= \frac{4\pi}{(2\pi)^2}\int_0^\infty dk' \frac{k'^2}{k'^2-k^2}\left(\frac{e^{ik'R}-e^{-ik'R}}{ik'R}\right)$$

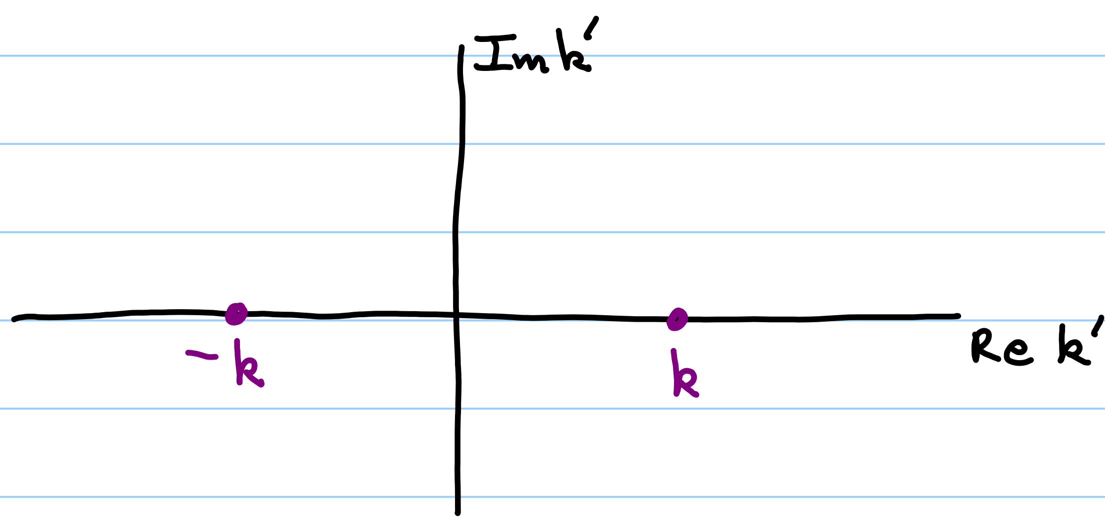
$$= \frac{1}{\pi R}\int_0^\infty dk' \frac{k'}{k'^2-k^2}\sin k'R$$

and since this integrand is an <u>even</u> function of k', we can rewrite it as:

$$\boxed{G(\vec{R},\omega) = \frac{1}{i\pi R}\int_{-\infty}^{\infty} dk' \frac{k' e^{ik'R}}{k'^2-k^2}}$$

<u>A problem</u> ..... (is it a "bug" or a "feature"?):

This integrand has two poles on the real axis, i.e. <u>on the integration contour</u>!



Hence, this integral is <u>not</u> <u>uniquely</u> <u>defined</u> as written above.

This type of ambiguity in the meaning of an integral occurs <u>whenever</u> a pole occurs exactly ON the integration path/contour.

Reference: Arfken + Weber, 5^{th} Edition
Sec 7.2, esp. pp 456–459

Note that there are <u>different</u> ways to interpret such a nonunique integral, and we need to invoke some <u>physics</u> to choose the right interpretation.

i.e. 3 standard interpretations to invoke:

- Principal value
- Move the poles infinitesimally into either the upper or lower half-plane
- Deform the integration contour to go above or below each pole

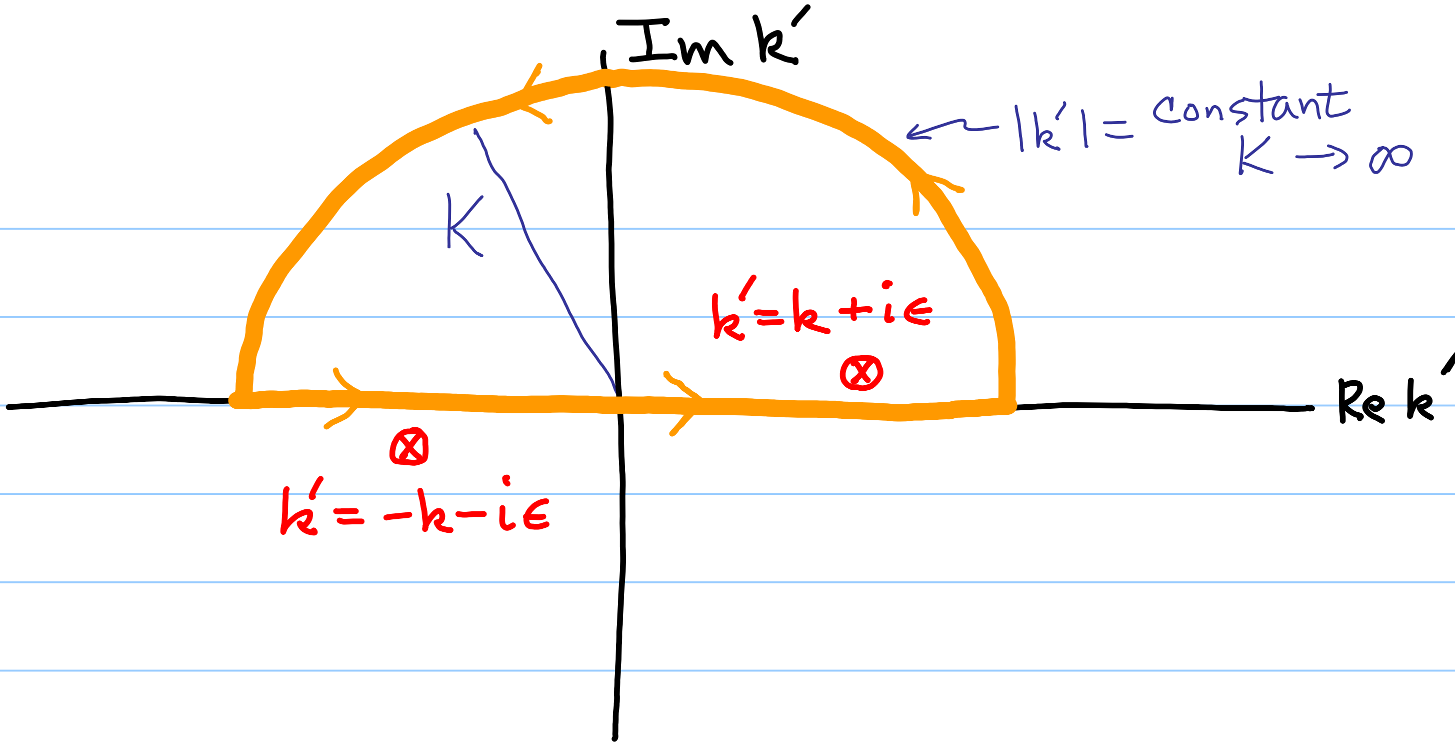
we will use THIS method

e.g. let $k \to k + i\epsilon$ and consider the above integral in the limit $\epsilon \to 0^+$

$\Rightarrow$ Thus our two poles at $k' = \pm k$ now move to

$$k' = k + i\epsilon$$

and $$k' = -k - i\epsilon$$



Thus the above integral is transformed into:

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} dk' \frac{k' e^{ik'R}}{(k'-k-i\epsilon)(k'+k+i\epsilon)} = \lim_{\epsilon \to 0^+} \lim_{K \to \infty} \oint_C \frac{dk' \, k' e^{ik'R}}{(k'-k-i\epsilon)(k'+k+i\epsilon)}$$

this equality holds because the semicircular contribution vanishes as $K \to \infty$

Next evaluate the $\oint_C$ using Cauchy's Integral Formula,

$$\oint_C \frac{f(z)\,dz}{z - z_0} = 2\pi i \,\mathrm{Res}\left(\frac{f(z)}{z-z_0}\right)_{z_0} = 2\pi i f(z_0)$$

$$= \lim_{\epsilon \to 0^+} \lim_{k' \to k+i\epsilon} \left(\frac{k' e^{ik'R}}{k'+k+i\epsilon}\right)(2i\pi) = i\pi e^{ikR}$$

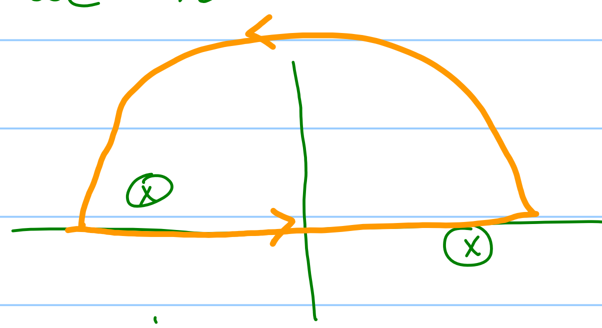
Putting this into our formula,

$$\mathcal{G}(\vec{R},\omega) = \frac{1}{i\pi R}\int_{-\infty}^{\infty} dk'\, \frac{k' e^{ik'R}}{k'^2 - k^2}$$

and our choice of moving the poles to $k \to k + i\epsilon$ gives finally a remarkably simple result:

$$\boxed{\mathcal{G}(\vec{R},\omega) = \frac{e^{ikR}}{R}}$$

If instead we had chosen $k \to k - i\epsilon$ with $\epsilon \to 0^+$



we would obtain

$$\oint_C dk'\, \frac{k' e^{ik'R}}{(k'-k+i\epsilon)(k'+k-i\epsilon)} = \frac{2\pi i(-k)}{(-2k)} e^{-ikR}$$

$$= \pi i e^{-ikR}$$

Thus we have obtained two different possible solutions,

$$\boxed{\mathcal{G}^{(\pm)}(\vec{R},\omega) = \frac{e^{\pm ikR}}{R}} \quad \text{with } k = \frac{\omega}{c}$$

Recalling that the time dependence for each ω is $e^{-i\omega\tau}$, we can see that

the $G^{(+)}$ solution corresponds to an <u>OUTGOING</u> spherical wave propagating outward from a source disturbance.

And the $G^{(-)}$ solution represents an <u>INCOMING</u> spherical wave.

To see this in more detail, evaluate the final ω-integral,

$$G^{(\pm)}(\vec{R},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega\tau \pm i\omega\frac{R}{c}}}{R}$$

which is trivially evaluated, giving

$$G^{(\pm)}(\vec{R},\tau) = \frac{1}{R}\,\delta\left(\tau \mp \frac{R}{c}\right)$$

or recalling the earlier variable change, we finally have the infinite space Green function for the wave equation in 3D:

$$G^{(\pm)}(\vec{x}-\vec{x}\,', t-t') = \frac{\delta\left(t-t' \mp \frac{1}{c}|\vec{x}-\vec{x}\,'|\right)}{|\vec{x}-\vec{x}\,'|}$$

sometimes convenient to rewrite as

$$\boxed{G^{(\pm)}(\vec{x}-\vec{x}\,',t-t') = \frac{\delta\left[t'-\left(t \mp \frac{|\vec{x}-\vec{x}\,'|}{c}\right)\right]}{|\vec{x}-\vec{x}\,'|}}$$

Using this, the resulting solutions of the original inhomogeneous wave equation are:

$$\Psi(\vec{x},t) = \iint d^3x'\,dt'\, f(\vec{x}\,',t')\,\frac{\delta\left[t'-\left(t \mp \frac{|\vec{x}-\vec{x}\,'|}{c}\right)\right]}{|\vec{x}-\vec{x}\,'|}$$

or

$$\Psi(\vec{x},t) = \int d^3x'\, \frac{f\left(\vec{x}\,', t \mp \frac{|\vec{x}-\vec{x}\,'|}{c}\right)}{|\vec{x}-\vec{x}\,'|}$$

Interpretation

The $G^{(+)}$ solution at $t' = t - \frac{|\vec{x}-\vec{x}\,'|}{c}$ implies that a wave disturbance that is FELT at time t must have originated from a SOURCE disturbance at an EARLIER time, $t' = t - \frac{|\vec{x}-\vec{x}\,'|}{c}$

It is earlier by exactly the light transit time from $\vec{x}\,'$ to $\vec{x}$.