As the flux through a circuit carrying current I changes from Φ_B to $\Phi_B + \delta\Phi_B$, the work done by external sources is equal to

$$\delta W = I\,\delta\Phi_B$$ ← no ohmic losses have been included here

But $\Phi_B = \int \vec{B}\cdot\hat{n}\,da = \int \nabla\times\vec{A}\cdot\vec{da} = \oint_C \vec{A}\cdot\vec{dl}$

whereby

$$\delta W = I\oint_C \delta\vec{A}\cdot\vec{dl} = \int \delta\vec{A}\cdot\vec{J}\,d^3x$$

and using $\nabla\times\vec{H} = \vec{J}$,

$$\Rightarrow \delta W = \int \delta\vec{A}\cdot(\nabla\times\vec{H})\,d^3x$$

Recall the following identity, here with $\vec{P}\to\vec{H}$, $\vec{Q}\to\delta\vec{A}$,

$$\nabla\cdot(\vec{P}\times\vec{Q}) = \vec{Q}\cdot(\nabla\times\vec{P}) - \vec{P}\cdot(\nabla\times\vec{Q})$$

$$\Rightarrow \delta W = \int\left[\vec{H}\cdot(\nabla\times\delta\vec{A}) + \underbrace{\nabla\cdot(\vec{H}\times\delta\vec{A})}\right]d^3x$$

vanishes for localized fields by the divergence theorem

or finally $\boxed{\delta W = \int \vec{H}\cdot\delta\vec{B}\,d^3x}$

which is valid for any kind of magnetic system, the MOST GENERAL formula

But if the material is linear, paramagnetic or diamagnetic, $\Rightarrow$ $\vec{H}\cdot\delta\vec{B} = \frac{1}{2}\delta(\vec{H}\cdot\vec{B})$

and in that case we have

$$\boxed{W = \frac{1}{2}\int \vec{H}\cdot\vec{B}\, d^3x}$$

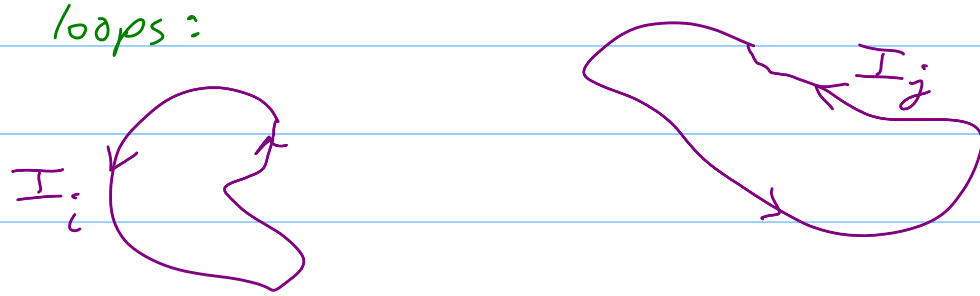
Alternatively, when $\vec{J}$ and $\vec{A}$ are linearly related, (e.g. when no magnetic media are present and $\nabla\times(\nabla\times\vec{A}) = \mu_0 \vec{J}$)

$$\Rightarrow \quad \delta\vec{A}\cdot\vec{J} = \frac{1}{2}\delta(\vec{A}\cdot\vec{J}) \Rightarrow W = \frac{1}{2}\int \vec{J}\cdot\vec{A}\, d^3x$$

So for a fixed current loop, $W = \frac{I}{2}\oint_C \vec{A}\cdot d\vec{l}$

or $\quad W = \frac{I}{2}\int_S \nabla\times\vec{A}\cdot d\vec{a} = \frac{I\,\Phi_B}{2} = W$

This leads us to the idea of INDUCTANCE, i.e. for a situation where we have some current loops:



Then the flux in that last equation includes BOTH the flux through DIFFERENT current loops $i \neq j$, AND through the SAME loop itself. Hence we can keep track of these separate contributions by writing, for N loops:

$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j>i} M_{ij} I_i I_j$$

Justification:

$$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}\, d^3x = \frac{1}{2}\frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x})\times\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\, d^3x\, d^3x'$$

$$= \frac{\mu_0}{8\pi}\sum_{i=1}^{N}\sum_{j=1}^{N} I_i I_j \underbrace{\oint_{C_i}\oint_{C_j} \frac{\vec{dl}_i\cdot\vec{dl}_j'}{|\vec{x}-\vec{x}\,'|}}$$

Observe that this integral is a purely GEOMETRICAL quantity

$\Rightarrow$ Define $L_i \equiv \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_i}\frac{\vec{dl}_i\cdot\vec{dl}_i'}{|\vec{x}-\vec{x}\,'|}$ = SELF-INDUCTANCE

and

$M_{ij} \equiv \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_j}\frac{\vec{dl}_i\cdot\vec{dl}_j'}{|\vec{x}-\vec{x}\,'|}$ = MUTUAL-INDUCTANCE

Quasi-static B-fields in conductors

Observation: A changing B-field induces $\vec{E}$ according to Faraday's Law, but if

$$\left|\frac{\partial \vec{B}}{\partial t}\right| = \text{"small"}$$

then B-Field physics still dominates

$\Rightarrow$ This is called the QUASI-STATIC regime

Under these conditions we can:

(i) ignore the finite value of light speed

(ii) neglect Maxwell's displacement current (i.e. neglect the effect on $\vec{B}$ of a changing $\vec{E}$-field)

Conducting Media

– To treat a conductor in this regime, begin by assuming that Ohm's Law is valid,

$$\vec{J} = \sigma \vec{E}, \quad \sigma = \text{conductivity}$$

$$\Rightarrow \nabla \times \vec{H} = \vec{J} \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{and} \quad \vec{B} = \nabla \times \vec{A}$$

$$\Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

(Note: (a) if there is no free charge, $\vec{E}$ is produced only by $\partial \vec{B}/\partial t$.)

Hence we can introduce a potential Φ such that

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \Phi$$

And for a linear, homogeneous medium,

$$\vec{H} = \frac{\vec{B}}{\mu} \Rightarrow \nabla \times \vec{B} = \mu \vec{J} = \mu \sigma \vec{E}$$

hence we can take $\Phi = 0 \Rightarrow \vec{E} = \frac{-\partial \vec{A}}{\partial t}$,

and in the gauge where $\nabla \cdot \vec{A} = 0$ (Coulomb gauge)

and recalling that $\nabla \cdot \vec{E} = 0$ here too,

then (b):

$$\nabla \times (\nabla \times \vec{A}) = \mu\sigma\left(-\frac{\partial \vec{A}}{\partial t}\right) = \nabla(\cancelto{0}{\nabla \cdot \vec{A}}) - \nabla^2 \vec{A}$$

or

$$\boxed{\nabla^2 \vec{A} = \mu \sigma \frac{\partial \vec{A}}{\partial t}}$$

This is a diffusion equation obeyed by each Cartesian component of the vector potential.

Diffusion equation solution by separation of variables

For just one component of $\vec{A}$, this PDE has the structure

$$\nabla^2 u(\vec{x}, t) = \gamma \frac{\partial u(\vec{x}, t)}{\partial t}$$

Look for separable space/time solutions like:

$$u(\vec{x}, t) = F(\vec{x})\, T(t)$$

Plug this in and divide by $F\,T$

$$\Rightarrow \frac{\nabla^2 F(\vec{x})}{F(\vec{x})} = \frac{\gamma}{T(t)} \frac{dT(t)}{dt} = -k^2$$

$$\Rightarrow \frac{dT(t)}{dt} = -\frac{k^2}{\gamma} T(t), \text{ solution is } T(t) = e^{-\frac{k^2 t}{\gamma}}$$

and

$F(\vec{x})$ obeys the Helmholtz PDE

$$\boxed{(\nabla^2 + k^2) F(\vec{x}) = 0}$$

This equation has solutions in spherical coordinates of the form

$$F_{klm}(\vec{x}) = Y_{lm}(\hat{x}) \begin{cases} \text{either} \\ j_l(kr) \\ \text{or} \\ n_l(kr) \end{cases}$$

spherical Bessel Functions

$\Rightarrow$ See Jackson pp. 425 – 427, which has the same spatial equation, but a different time-dependent equation there.

e.g. $j_l(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} J_{l+\frac{1}{2}}(kr)$, which is regular at $r \to 0$, oscillatory at $r \to \infty$.

Hence, in the context of $\vec{A}$ diffusing in conducting media, each component can be expanded as

$$A_i(\vec{x},t) = \sum_{lm} \int_0^\infty dk \; C_{lm}^{(i)}(k) \, Y_{lm}(\hat{x}) \, j_l(kr) \, e^{-k^2 \frac{t}{\gamma}}$$

the solution <u>must</u> have this form if the volume includes $r=0$ and the full angular range

<u>TERMINOLOGY</u>: Jackson calls this "A Laplace transform representation of the solution"

How can we use this? One might be given $A_i(\vec{x}, t=0)$. Then one uses "Fourier's trick" to determine the time-independent coefficients $C_{lm}^{(i)}(k)$, which then determine the solution at all later times.

Hint: See the orthogonality relations in Jackson's problem #3.16(a).

Addendum about inductance and energy sign conventions

Recall from basic electrical circuit theory that the energy stored in an inductor is

$$W = \frac{1}{2} L I^2,$$ whereby the rate at which power is delivered to it is

$$\frac{dW}{dt} = I\left(L\frac{dI}{dt}\right)$$

But the instantaneous flux through the inductor is $\Phi_B = L I$, so the induced emf according to Faraday's Law is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

So $$\frac{dW}{dt} = -I\mathcal{E}$$ where $\mathcal{E} = \oint \vec{E}\cdot d\vec{l}$

$=$ power, P

How can relate this to the voltage change as one goes around the loop?

Recall $$V = -\int_{loop} \vec{E}\cdot d\vec{l} = -\mathcal{E}$$

whereby $$P = IV = -I\mathcal{E}$$

the usual expression from circuit theory