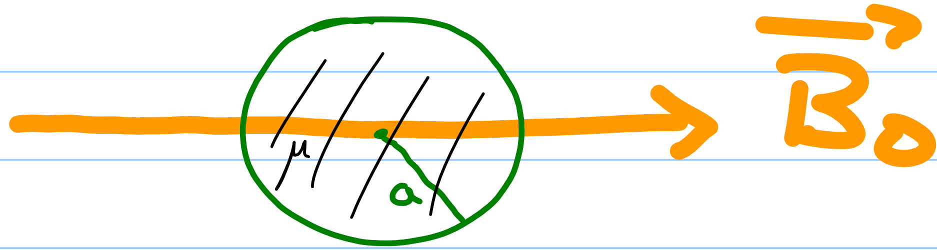
Sphere of linear magnetic permeability μ in an external $\vec{B}$-Field, $\vec{B}_0 = B_0 \hat{z}$



There are no free currents, so

$$\nabla \times \vec{H} = 0 \qquad \Rightarrow \quad \vec{H} = -\nabla \Phi_M$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{and} \quad \nabla^2 \Phi_M = 0, \text{ inside \& out}$$

$$\Rightarrow \Phi_M^{\text{inside}} = \sum_{l=0}^{\infty} b_l r^l P_l(\cos\theta)$$

$$\Phi_M^{\text{outside}} = c_1 r P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{d_l}{r^{l+1}} P_l(\cos\theta)$$

at $r \to \infty$, we must have $-\nabla \Phi_M \to H_0 \hat{z} = \frac{B_0}{\mu_0} \hat{z}$

$$\Rightarrow \boxed{c_1 = -\frac{B_0}{\mu_0}}$$

Next demand continuity of Φ_M at $r=a$

$$\Rightarrow \quad b_l a^l = c_1 a \, \delta_{l,1} + \frac{d_l}{a^{l+1}}$$

Next demand continuity of B_r,

$$\Rightarrow \mu H_r^{in}\Big|_{r=a} = \mu_0 H_r^{out}\Big|_{r=a} \Rightarrow -\frac{\partial \Phi_M^{in}}{\partial r}\Big|_{r=a} = -\frac{\mu_0}{\mu} \frac{\partial \Phi_M^{out}}{\partial r}\Big|_{r=a}$$

$$\Rightarrow \ell\, b_\ell\, a^{\ell-1} = \frac{\mu_0}{\mu}\left[c_1 \delta_{\ell,1} - (\ell+1)\, d_\ell\, a^{-\ell-2}\right]$$

Note that this system of equations is linear, homogeneous for $\ell \neq 1 \Rightarrow b_\ell = 0 = d_\ell$ for $\ell \neq 1$

" inhomogeneous for $\ell = 1$

$\Rightarrow$ solution works out to be

$$b_1 = \frac{3\mu_0}{\mu + 2\mu_0} c_1 \,, \quad d_1 = -a^3 \frac{\mu - \mu_0}{\mu + 2\mu_0} c_1$$

So putting this together, we have

$$\Phi_M^{\text{inside}} = \frac{-3B_0}{\mu + 2\mu_0} r\cos\theta$$

$$\Phi_M^{\text{outside}} = -\frac{B_0}{\mu_0} r\cos\theta + \frac{B_0}{\mu_0} \frac{a^3\cos\theta}{r^2} \frac{\mu - \mu_0}{\mu + 2\mu_0}$$

and so $\boxed{\vec{H}^{\text{inside}} = -\nabla \Phi_M^{\text{in}} = \frac{3B_0}{\mu + 2\mu_0}\hat{z}}$

and $\boxed{\vec{B}^{\text{inside}} = \mu \vec{H}^{\text{inside}} = \frac{3\mu}{\mu + 2\mu_0} B_0 \hat{z}}$ = Constant inside!

while $\vec{H}^{\text{outside}} = -\nabla \Phi_M^{\text{outside}} = \frac{B_0 \hat{z}}{\mu_0} + \frac{\mu_0[3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}]}{\mu_0 4\pi r^3}$

where $\boxed{\vec{m} = 4\pi a^3 \vec{B}_0 \frac{\mu - \mu_0}{\mu + 2\mu_0}}$ = the induced magnetic dipole moment of the sphere

On your own, read Sec. 5.11, 5.12 and pp. 187 – 191

Relation between $\vec{m}$ and the orbital angular momentum $\vec{L}$

For a discrete set of moving particles,

$$\vec{J} = \sum_i q_i \vec{v}_i \, \delta(\vec{x} - \vec{x}_i)$$

If constant velocity $\vec{x}_i(t) = \vec{x}_i(0) + \vec{v}_i t$

since for 1-particle, $\vec{J} = \rho \vec{v}$ and $\rho = q\,\delta(\vec{x} - \vec{x}_i)$,

Now recall that $\vec{L} = M_i \vec{x}_i \times \vec{v}_i$

So $\vec{m} = \frac{1}{2} \int \vec{x} \times \vec{J} \, d^3x = \frac{1}{2} \sum_i q_i \vec{x}_i \times \vec{v}_i = \sum_i \frac{q_i \vec{L}_i}{2M_i}$

and if all particles are identical, e.g. $\frac{q_i}{M_i} = \frac{-e}{M}$

$\Rightarrow \boxed{\vec{m} = \frac{-e}{2M} \vec{L}}$ where $\vec{L} = \vec{L}_1 + \vec{L}_2 + \dots + \vec{L}_N$

$= $ total orbital angular momentum

orbital magnetic moment

whereas in quantum physics one sees that the spin angular momentum of an e^- has a magnetic moment too:

$$\vec{m}_s = 2(1.00116\ldots)\left(\frac{-e}{2M_e}\right)\vec{S}$$

(this is derived from the Dirac equation + quantum electrodynamics)

Force $\vec{F}$ and torque $\vec{N}$ on a localized current distribution:

Start from our earlier force expression,

$$\vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x$$

and insert a Taylor series for $\vec{B}$, produced by <u>other</u> currents outside of $\vec{J}$

$$\Longrightarrow \quad \vec{B}(\vec{x}) = \vec{B}(0) + (\vec{x}\cdot\nabla)\,\vec{B}\Big|_{\vec{x}=0} + \ldots$$

i.e. $\quad B_i(\vec{x}) = B_i(0) + \underbrace{(\vec{x}\cdot\nabla)B_i}$

derivatives of B_i are evaluated at $\vec{x} \to 0$, which we write as a shorthand as $(\vec{x}\cdot\nabla)\vec{B}(0)$

Then $\vec{F} = -\vec{B}(0) \times \int \vec{J}(\vec{x})\, d^3x$

$$+ \int \vec{J}(\vec{x}) \times [(\vec{x}\cdot\nabla)\vec{B}(0)]\, d^3x$$

Moreover, $\int \vec{J}(\vec{x})\, d^3x = 0$ in magnetostatics (proved on p. 154 of these notes)

Or using the Levi-Civita symbol, we have

Cartesian unit vectors, $\hat{e}_1 = \hat{x}, \hat{e}_2 = \hat{y}, \hat{e}_3 = \hat{z}$

$$\vec{F} = \sum_{ijk} \epsilon_{ijk}\, \hat{e}_k \int J_i\, (\vec{x}\cdot\nabla) B_j(0)\, d^3x$$

but notice that

$$\int J_i\, (\vec{x}\cdot\nabla) B_j(0)\, d^3x = \sum_\ell (\nabla B_j)^{(0)}_\ell \int x_\ell J_i\, d^3x$$

$$= \frac{1}{2}\sum_\ell (\nabla B_j)^{(0)}_\ell \int \underbrace{(x_\ell J_i - x_i J_\ell)}_{= \sum_{k'} (\vec{x}\times\vec{J})_{k'}\,\epsilon_{\ell i k'}}\, d^3x,$$

(using Eq. 5.52 and below)

Hence $\vec{F} = \frac{1}{2} \sum_{\substack{ijk\\ lk'}} \hat{e}_k \epsilon_{ijk} (\nabla B_j^{(0)})_l \epsilon_{lik'} \left[\int \vec{x} \times \vec{J} \, d^3x \right]_{k'}$

Now we have a key identity we can use:

$$\boxed{\sum_i \epsilon_{ijk} \epsilon_{lik'} = \sum_i \epsilon_{ijk} \epsilon_{ik'l} = \delta_{jk'} \delta_{kl} - \delta_{jl} \delta_{kk'}}$$

also note: $\sum_{ij} \epsilon_{ijk} \epsilon_{ijn} = 2\delta_{kn}$ and $\sum_{ijk} \epsilon_{ijk} \epsilon_{ijk} = 6$

So $\vec{F} = \frac{1}{2} \sum_{jklk'} \hat{e}_k (\nabla B_j^{(0)})_l \left(\int \vec{x} \times \vec{J} \, d^3x \right)_{k'} (\delta_{jk'} \delta_{kl} - \delta_{jl} \delta_{kk'})$

and so, after evaluating these sums over l, k', we get

$$\vec{F} = \frac{1}{2} \sum_{jk} \hat{e}_k \left[(\nabla B_j^{(0)})_k \left(\int \vec{x} \times \vec{J} \, d^3x \right)_j - \cancel{(\nabla B_j^{(0)})_j} \left(\int \vec{x} \times \vec{J} \, d^3x \right)_k \right]$$

$\rightarrow 0$, since $\nabla \cdot \vec{B} = 0$

Thus $\vec{F} = \sum_{jk} \hat{e}_k \frac{\partial}{\partial x_k} B_j^{(0)} m_j = \boxed{\nabla (\vec{m} \cdot \vec{B}) = \vec{F}}$

and this result is analogous to $\vec{F} = \nabla (\vec{p} \cdot \vec{E})$ for an electric dipole

Also, since we expect $\vec{F} = -\nabla W_{dipole}$,

we can interpret $W_{dipole} = -\vec{m} \cdot \vec{B}$ as the energy of a magnetic dipole $\vec{m}$ in an external $\vec{B}$-field

NOTE: This is not actually the TOTAL ENERGY because it neglects the energy needed to maintain steady currents as the dipole is brought in from ∞.

Similarly, relevant equations for torque on a localized current:

$\vec{N} = \int \vec{x}' \times (\vec{J} \times \vec{B}(0)) \, d^3x'$ + higher order terms

or $\vec{N} = \int [(\vec{x}' \cdot \vec{B}(0)) \vec{J} - (\vec{x}' \cdot \vec{J}) \vec{B}(0)] \, d^3x'$

Faraday's Law of Induction

Now allow time-dependence, i.e. $\frac{\partial \rho}{\partial t} \neq 0$, $\frac{\partial \vec{J}}{\partial t} \neq 0$

Experiment: Start by defining

(1) $\mathcal{E} = \oint_C \vec{E} \cdot \vec{dl}$ = "electromotive force" or emf

(2) $\Phi_B = \int_S \vec{B} \cdot \vec{da}$ = magnetic flux, Jackson's F

recall — The sense of direction of $\vec{da}$ is in accordance with the right-hand rule

Now, Faraday's experiments showed that

$$\mathcal{E} = -k \frac{d\Phi}{dt}, \quad \begin{cases} k = 1, & \text{SI units} \\ k = \frac{1}{c}, & \text{Gaussian units} \end{cases}$$

Let's work in SI units,

$$\Rightarrow \mathcal{E} = \frac{-d\Phi_B}{dt}$$

volts

$\frac{\text{Tesla}-m^2}{\text{sec}}$

On your own, read Jackson's Sec 5.15 on Galilean invariance concept.

Observations (a) In $\oint_C \vec{E} \cdot \vec{dl}$, C can be any closed path in space, i.e it need not be a wire, though it could be one.

(b) The magnetic flux can change by changing $\vec{B}$ or the shape, size, or orientation of C

$\Rightarrow$ of course physics does not depend on the reference frame.

(c) It is understood that $\vec{E}$ is the electric field at $\vec{dl}$ in the coordinate system in which $\vec{dl}$ is at rest

Galilean Transformation

Let $\vec{E}, \vec{B}$ be the fields in the laboratory frame and let $\vec{E}'$ be the E-field in a frame moving at velocity $\vec{v}$ relative to the laboratory, i.e. along with the charge q whose velocity is $\vec{v}$ in the laboratory frame.

$\Rightarrow$ Then we must have $\boxed{\vec{E}' = \vec{E} + \vec{v} \times \vec{B}}$,

in order to be consistent with the Lorentz force,

$$\vec{F}' = q\vec{E}' \Longleftrightarrow \vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Differential Form of Faraday's Law

Consider an arbitrary time-independent loop C

$$\Rightarrow -\frac{d}{dt}\Phi_B = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{da} = \oint_C \vec{E} \cdot \vec{dl} = \int_S \nabla \times \vec{E} \cdot \vec{dl}$$

and since the loop C is arbitrary, the integrands are equal

$$\Rightarrow \boxed{\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0}$$

(which generalizes our result $\nabla \times \vec{E} = 0$ from before in electrostatics)

Energy in the magnetic field

Let's now discuss what happens to the energy as we increase $\vec{J}$ up from 0, and use Faraday's Law to account for the fact that we perform work against the induced emf as the current is increased.

$\Rightarrow$ We interpret this work as the "ENERGY STORED IN THE MAGNETIC FIELD"

Then the rate at which work is done by the battery or power supply, in the course of changing the flux and inducing currents & emf; is

$$P = \frac{dW}{dt} = -I\mathcal{E}, \text{ but } \mathcal{E} = -\frac{d\Phi_B}{\partial t}$$

Hence $\frac{dW}{dt} = I \frac{d\Phi_B}{dt}$