Next, Jackson Secs 5.8, 5.9 (temporarily skip Sec 5.7 concerning force, torque, energy)

Magnetostatics in Macroscopic Matter

We can visualize $\vec{B}$ as inducing magnetic dipoles $\vec{m}$ in matter, much as $\vec{E}$ induces electric dipoles $\vec{p}$. $\Rightarrow$ Thus, introduce the MAGNETIZATION,

$$\vec{M} \equiv \frac{\text{magnetic dipole moment}}{\text{unit volume}}$$

i.e. $\vec{M} = \sum_i N_i \langle \vec{m}_i \rangle$ where N_i = number density of dipoles of type i.

$\Rightarrow$ We can calculate the magnetic induction produced by a volume distribution of dipoles, starting with the vector potential,

$$d\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \vec{M}(\vec{x}\,') \times \frac{(\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3} d^3x'$$

(this is merely an application of $\vec{A}_{dipole} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$)

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int_V \vec{M}(\vec{x}\,') \times \nabla' \frac{1}{|\vec{x} - \vec{x}\,'|} d^3x'$$

$$= -\frac{\mu_0}{4\pi} \int_V \nabla' \times \left(\frac{\vec{M}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \right) d^3x' + \frac{\mu_0}{4\pi} \int_V \frac{\nabla' \times \vec{M}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x'$$

$$\vec{A} = \frac{\mu_0}{4\pi}\cancel{\oint_S \frac{\vec{M}(\vec{x}')\times\hat{n}'\,da'}{|\vec{x}-\vec{x}'|}} + \frac{\mu_0}{4\pi}\int_V \frac{\nabla'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$

0 surface integral vanishes if surface S is outside the medium.

So comparing this last term with $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$

it suggests that we might define

$$\boxed{\nabla\times\vec{M} \equiv \vec{J}_{bnd}} \equiv \text{"volume bound current density"}$$

And for some problems there could be a bound surface current density,

$$\boxed{\vec{M}\times\hat{n} \equiv \vec{K}_{bnd}}$$

Next, calculate $\vec{B} = \nabla\times\vec{A}$, returning to the differential equations, $\Rightarrow \nabla\times\vec{B} = \mu_0\vec{J}_{total} = \mu_0\vec{J}_{free} + \mu_0\nabla\times\vec{M}$

or if we define

$$\boxed{\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}} \equiv \text{MAGNETIC FIELD}$$

Then the macroscopic equations of magnetostatics are

$$\boxed{\begin{aligned}\nabla\times\vec{H} &= \vec{J}_{free}\\ \nabla\cdot\vec{B} &= 0\end{aligned}} \quad \text{analogous to} \longrightarrow \quad \boxed{\begin{aligned}\nabla\cdot\vec{D} &= \rho_{free}\\ \nabla\times\vec{E} &= 0\end{aligned}}$$

magnetostatics

electrostatics with $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

and as with $\vec{D}, \vec{E}$, we require a constitutive relation between $\vec{B}, \vec{H}$ in order to solve these equations.

For many substances, like isotropic paramagnetic or diamagnetic materials, the relationship is linear,

$$\Rightarrow \quad \vec{B} = \mu \vec{H}$$

where μ = MAGNETIC PERMEABILITY

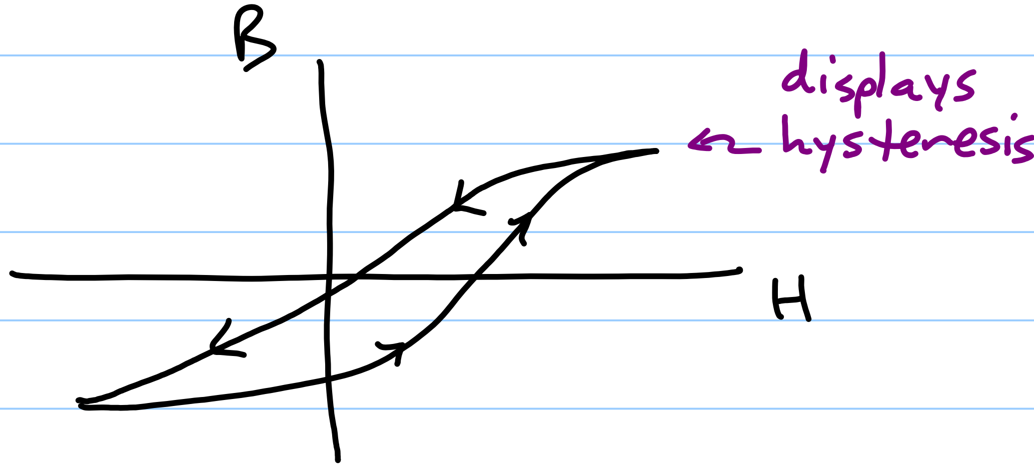
Typical values, $\left|\frac{\mu}{\mu_0} - 1\right| \simeq 10^{-5}$

and $\frac{\mu}{\mu_0} - 1 < 0$, diamagnetic substances $(\mu < \mu_0)$

$\frac{\mu}{\mu_0} - 1 > 0$, paramagnetic " $(\mu > \mu_0)$

In a ferromagnet the relationship is not linear, i.e. $\mu = \mu(H) \neq$ constant

<u>AND</u> μ can depend on the history of preparation



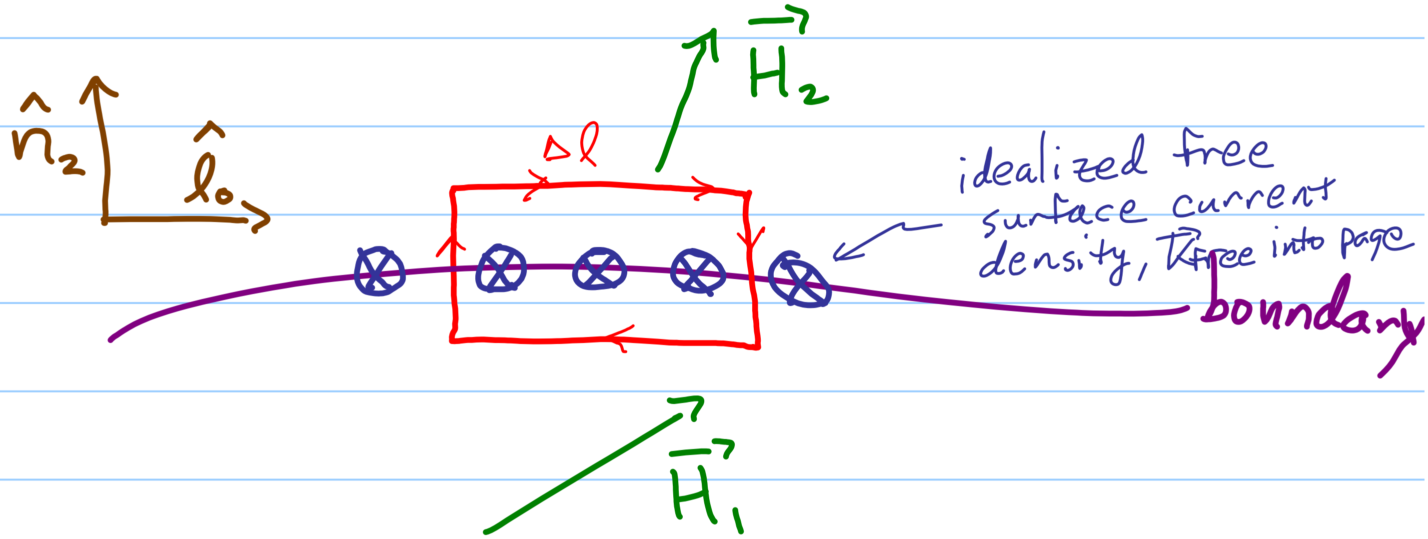
order of magnitude,

$$\frac{\mu}{\mu_0} \simeq 10 - 10^6 \text{ for ferromagnets}$$

Behavior of $\vec{B}$, $\vec{H}$ at boundaries

(1) $\nabla \cdot \vec{B} = 0 \implies B_\perp$ is continuous at any boundary, <u>always</u>

(2) $\nabla \times \vec{H} = \vec{J}_{free} \implies \int \nabla \times \vec{H} \cdot d\vec{a} = \oint \vec{H} \cdot d\vec{l} = I_{free}^{enclosed}$



$$\implies (\vec{H}_2 - \vec{H}_1) \cdot \hat{l}_0 \cancel{\Delta l} = (\vec{K}_f \cancel{\Delta l}) \cdot (\hat{n}_2 \times \hat{l}_0)$$

$$\implies (\vec{H}_2 - \vec{H}_1) \cdot \hat{l}_0 = \vec{K}_f \times \hat{n}_2 \cdot \vec{l}_0$$

or $(\vec{H}_2 - \vec{H}_1) = \vec{K}_f \times \hat{n}_2$

or $\boxed{\hat{n}_2 \times (\vec{H}_2 - \vec{H}_1) = \vec{K}_f}$

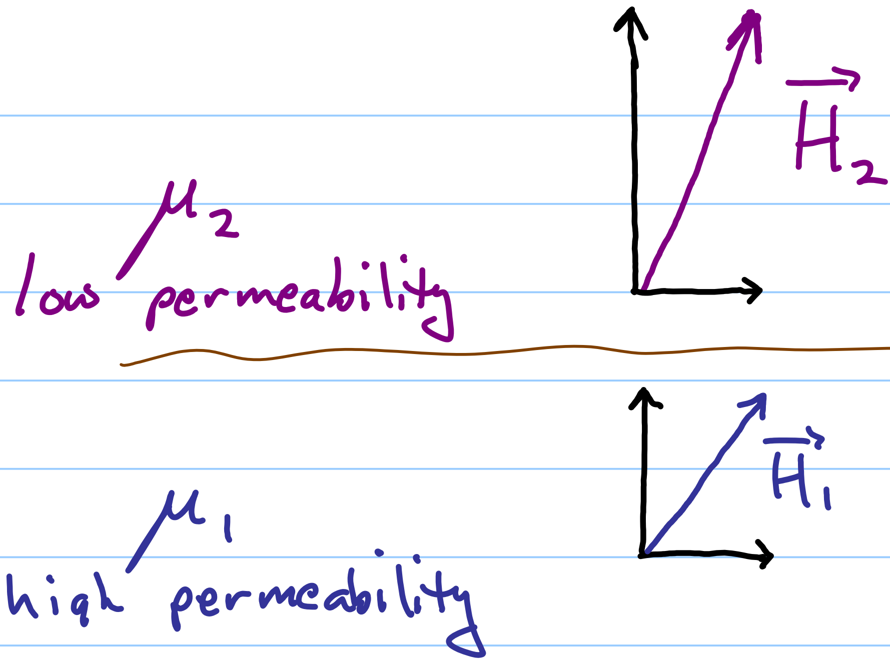
and in particular, if $\vec{K}_f = 0 \implies H_\parallel =$ continuous

For linear media, these BCs simplify to:

$$\vec{B}_2 \cdot \hat{n}_2 = \vec{B}_1 \cdot \hat{n}_2 \quad , \quad \vec{B}_2 \times \hat{n}_2 = \frac{\mu_2}{\mu_1} \vec{B}_1 \times \hat{n}_2$$

OR $$\vec{H}_2 \cdot \hat{n}_2 = \frac{\mu_1}{\mu_2} \vec{H}_1 \cdot \hat{n}_2 \quad , \quad \vec{H}_2 \times \hat{n}_2 = \vec{H}_1 \times \hat{n}_2$$

e.g.



Observe that if $\mu_1 >> \mu_2$, then

$$\boxed{|H_{2,\perp}| >> |H_{1,\perp}|}$$

$\Rightarrow \vec{H}_2$ is oriented NORMAL to the surface in the limit $\frac{\mu_1}{\mu_2} \to \infty$

Exception: UNLESS $\vec{H}_1$ is accurately parallel to the surface

ALSO, since $B_{2,\parallel} = \frac{\mu_2}{\mu_1} B_{1,\parallel} \Rightarrow B_{2,\parallel} \simeq 0$

$\Rightarrow H_\parallel$ and $B_\parallel$ vanish just outside an infinitely permeable material, just as $E_\parallel, D_\parallel$ vanish just outside a conductor surface.

Boundary Value Problems in Magnetostatics

We can ALWAYS write $\vec{B} = \nabla \times \vec{A}$

A Most general case has $\vec{H} = \vec{H}(\vec{B}) \Rightarrow \nabla \times \vec{H}(\nabla \times \vec{A}) = \vec{J}_f$

That last equation = hard to solve!

But for a linear medium, $\vec{B} = \mu \vec{H}$

$$\Rightarrow \nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) = \vec{J}_f$$

or throughout any region where μ = constant,

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J}_f$$

or $\boxed{\nabla^2 \vec{A} = -\mu \vec{J}}$ ← in the Coulomb gauge where $\nabla \cdot \vec{A} = 0$

Next solve the PDE, match solutions at boundaries using the BCs derived above.

(B) Suppose $\vec{J}_f = 0$ in some region of space

$$\Rightarrow \nabla \times \vec{H} = 0$$

⇒ In any such region, we can introduce a MAGNETIC SCALAR POTENTIAL Φ_M

i.e. $\vec{H} = -\nabla \Phi_M$ $\Rightarrow \nabla \cdot \vec{B} = 0$ or $\nabla \cdot (\mu \vec{H}) = 0$

$$\Rightarrow \boxed{\nabla^2 \Phi_M = 0}$$

in regions where μ = constant, so we can use all the standard methods, and the BCs above.

(C) Hard Ferromagnets

– Treatment when $\vec{J}_f = 0$, $\vec{M}(\vec{x}) = $ given

$\Longrightarrow$ Here of course, $\vec{B} \neq \mu \vec{H}$

$\Rightarrow \nabla \cdot \vec{B} = 0 = \mu_0 \nabla \cdot (\vec{H} + \vec{M})$

$\Longrightarrow \nabla \cdot (-\nabla \Phi_M) = -\nabla \cdot \vec{M}$

or introducing $\rho_M = -\nabla \cdot \vec{M} =$ "effective magnetic charge density"

$\Rightarrow$ we have $\boxed{\nabla^2 \Phi_M = -\rho_M}$

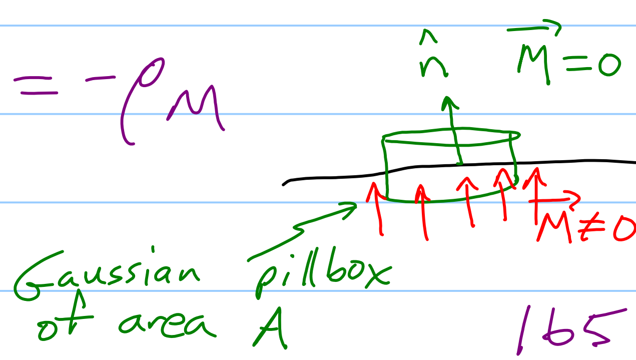
and, by analogy with electrostatics the solution is

$$\Phi_M = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

Next, observe that we can sometimes usefully treat a discontinuity of $\vec{M}$ at a surface like a "surface magnetization,"

i.e. consider "Gauss's Law for $\vec{M}$" in a problem with no free charge or current

$\Longrightarrow$ Start from $\nabla \cdot \vec{M} = -\rho_M$



Then $\oint \vec{M} \cdot d\vec{a} = -\int d^3x \, \rho_M(\vec{x})$

or $-(\vec{M} \cdot \hat{n}) A = -A \, \sigma_M(\vec{x})$

i.e. $\boxed{\sigma_M = \vec{M} \cdot \hat{n}}$ acts like an EFFECTIVE magnetic surface-charge density

Hence we can write separate volume and surface terms:

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi} \int_V \frac{\nabla' \cdot \vec{M}'(\vec{x}') \, d^3x'}{|\vec{x} - \vec{x}'|} + \frac{1}{4\pi} \int_S \frac{\hat{n}' \cdot \vec{M}(\vec{x}') \, da'}{|\vec{x} - \vec{x}'|}$$

And of course in many problems, $\vec{M}$ = constant inside, so Φ_M will then be determined ONLY by the surface term.

Example - the uniformly magnetized sphere

$$\vec{M} = M_0 \, \hat{z}$$

(i) Scalar potential method:

$$\nabla \times \vec{H} = 0 \implies \vec{H} = -\nabla \Phi_M$$

volume term vanishes since $\nabla \cdot \vec{M} = 0 \implies \Phi_M(\vec{x}) = \oint_S \frac{\hat{n}' \cdot \vec{M}(\vec{x}') \, d^3x'}{4\pi |\vec{x} - \vec{x}'|}$

So from our previous work, we can write

$$\Phi_M(\vec{x}) = \frac{M_0 a^2}{4\pi} \int d\Omega' \frac{\cos\theta'}{|\vec{x} - \vec{x}'|},$$

integrated over all $|\vec{x}'| = a$ on S

$\Rightarrow$ plug in our expansion into Y_{lm}'s:

$$\Phi_M = \frac{M_0 a^2}{4\pi} \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \left(\frac{4\pi}{3}\right)^{1/2} \underbrace{\int d\Omega' Y_{10}(\hat{x}') Y^*_{lm}(\hat{x}')}_{= \delta_{l,1}\delta_{m,0}} Y_{lm}(\hat{x})$$

giving finally $\Phi_M = \frac{M_0 a^2}{3} \frac{r_<}{r_>^2} \left(\frac{4\pi}{3}\right)^{1/2} Y_{10}(\hat{x})$

$$\Phi_M = \frac{M_0 a^2}{3} \frac{r_<}{r_>^2} \cos\theta$$

with $r_< = \min(r, a)$, $r_> = \max(r, a)$

$$\Rightarrow \Phi_M(r<a) = \frac{M_0 a^2}{3a^2} z = \frac{M_0 z}{3}$$

whereby $\boxed{\vec{H}_{in} = -\frac{M_0 \hat{z}}{3} = -\frac{\vec{M}}{3} \text{ at } r<a.}$

and $\boxed{\vec{B}_{in} = \mu_0(\vec{H} + \vec{M}) = \frac{2\mu_0}{3}\vec{M}}$

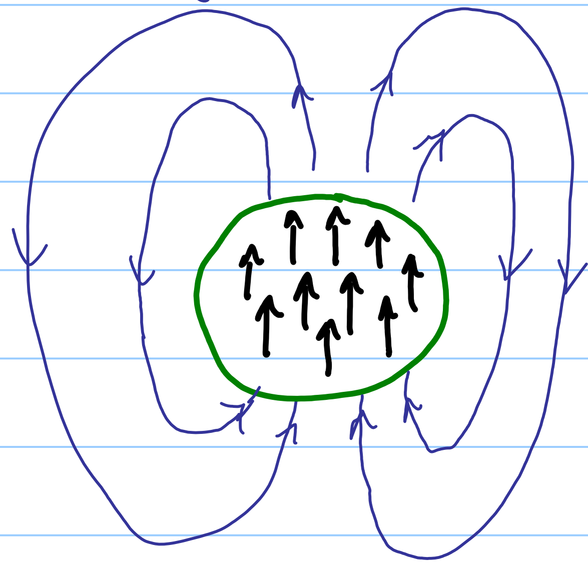
Whereas $\boxed{\Phi_M^{outside} = \frac{M_0 a^3}{3r^2}\cos\theta = \frac{\vec{m}\cdot\vec{x}}{4\pi r^3},}$

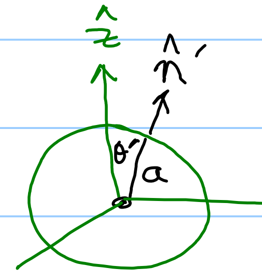
after setting $\vec{m} = \frac{4}{3}\pi a^3 \vec{M}$

= magnetic dipole

moment of entire sphere.

Conclusion: there is a constant $\vec{B}, \vec{H}$ inside a uniformly magnetized sphere, while outside the sphere you get simply a pure dipole field.





(ii) Alternative solution using vector potential

Start from

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \cancelto{0}{\int_V \frac{\nabla' \times \vec{M}(\vec{x}')\, d^3x'}{|\vec{x} - \vec{x}'|}} + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{M}(\vec{x}') \times \hat{n}'\, da'}{|\vec{x} - \vec{x}'|}$$

and use $\vec{M}(\vec{x}') \times \hat{n}' = M_0 \hat{z} \times \hat{n}' = M_0 \sin\theta' \hat{\phi}'$

$$= M_0 \sin\theta' (-\sin\phi' \hat{x} + \cos\phi' \hat{y})$$

Recalling that $\sin\theta' \sin\phi' = -\left(\frac{8\pi}{3}\right)^{1/2} \left(\frac{Y_{11} + Y_{1,-1}}{2i}\right)$

and

$$\sin\theta' \cos\phi' = -\left(\frac{8\pi}{3}\right)^{1/2} \left(\frac{Y_{11} - Y_{1,-1}}{2}\right)$$

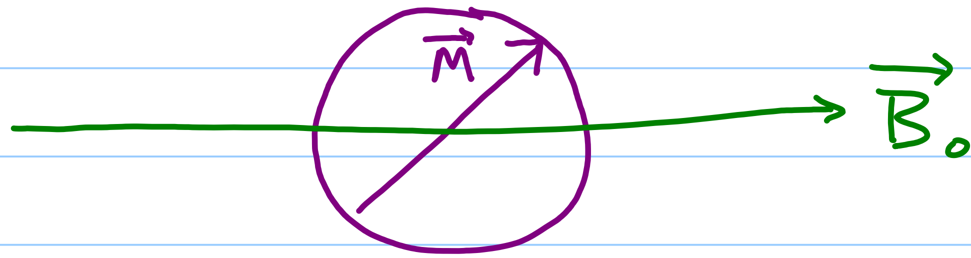
$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0 M_0 a^2}{4\pi} \int d\Omega' \sum_{l,m} \frac{4\pi}{2l+1} Y^*_{lm}(\hat{x}') Y_{lm}(\hat{x}) \frac{r_<^l}{r_>^{l+1}}$$

$$\times \left(\frac{8\pi}{3}\right)^{1/2} \left\{ -\hat{x}(-1) \frac{Y_{11}(\hat{x}') + Y_{1-1}(\hat{x}')}{2i} + \hat{y}(-1) \frac{Y_{11}(\hat{x}') - Y_{1-1}(\hat{x}')}{2} \right\}$$

$\Rightarrow$ In the end, we see simply that

$$\boxed{\vec{A} = \frac{\mu_0 M_0 a^2}{3} \frac{r_< \sin\theta}{r_>^2} \hat{\phi} \quad, \text{etc.}}$$

Uniformly magnetized sphere in an external $\vec{B}$-field



Because we have linear equations, we can simply add a constant $\vec{B}_0$ everywhere, i.e.

$$\vec{B}_{in} = \vec{B}_0 + \frac{2\mu_0}{3}\vec{M} \qquad (1)$$

$$\vec{B}_{outside} = \vec{B}_0 + \frac{\mu_0}{4\pi}\left(\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{r^3}\right) \qquad (2)$$

where $\vec{m} = \frac{4}{3}\pi a^3 \vec{M}$ (3)

and $\vec{H}_{in} = \frac{\vec{B}_{in}}{\mu_0} - \vec{M} = \frac{\vec{B}_0}{\mu_0} - \frac{1}{3}\vec{M}$

and $\vec{H}_{out} = \frac{\vec{B}_{out}}{\mu_0}$ (since $\vec{M} = 0$ outside the sphere)