To show that the aforementioned integral vanishes, use the identity $\vec{a}\cdot\nabla\psi = \nabla\cdot(\vec{a}\,\psi) - \psi\,\nabla\cdot\vec{a}$

$$\Rightarrow \int \vec{J}(\vec{x}\,')\cdot\nabla'\frac{1}{|\vec{x}-\vec{x}\,'|}\,d^3x'$$

$$= \int d^3x'\left\{\nabla'\cdot\left(\frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\right) - \frac{1}{|\vec{x}-\vec{x}\,'|}\nabla'\cdot\vec{J}(\vec{x}\,')\right\}$$

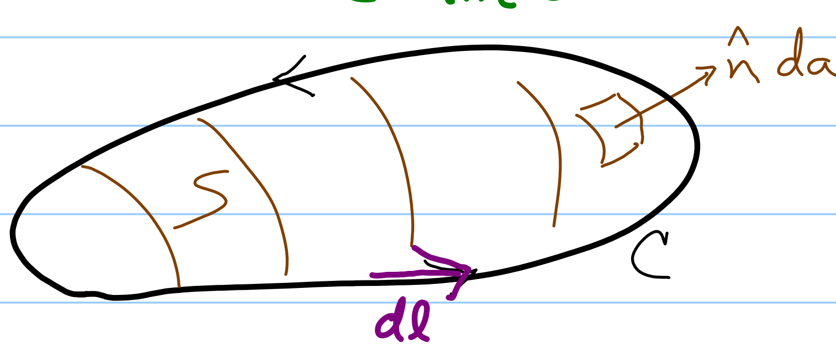
$$= \underbrace{\oint_S \frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\cdot d\vec{a}\,'}_{\to 0 \text{ if } S\to\infty \text{ for finite sources}} - \int_V \frac{\cancel{\nabla'\cdot\vec{J}(\vec{x}\,')}^{\;\to 0 \text{ in magnetostatics!}}}{|\vec{x}-\vec{x}\,'|}\,d^3x'$$

$= 0$, as we wished to prove.

$$\Rightarrow \boxed{\begin{aligned}\nabla\times\vec{B} &= \mu_0\vec{J}\\ \nabla\cdot\vec{B} &= 0\end{aligned}}$$

Ampere's Law

Consider an arbitrary open surface S with outward normal $\hat{n}$ and a closed loop C bounding it, traversed in the right-hand rule direction:



Now integrate $\nabla \times \vec{B} = \mu_0 \vec{J}$ over such an open surface S,

$$\Rightarrow \int \nabla \times \vec{B} \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

or by Stokes Theorem,

$$\boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{encircled}}$$

$\Rightarrow$ This is useful for problems of high symmetry, as with Gauss's Law in electrostatics

Solutions to differential laws of magnetostatics

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \nabla \cdot \vec{B} = 0$$

Case #1 – Source-Free regions

If $\vec{J} = 0$ in some region of space, then $\nabla \times \vec{B} = 0 \Rightarrow$ we can write

$$\vec{B} = -\nabla \Phi_M$$

where $\Phi_M \equiv$ magnetic scalar potential, which obeys

$$\nabla \cdot \vec{B} = 0 \Rightarrow \boxed{\nabla^2 \Phi_M = 0}$$

$\Rightarrow$ We can use all we know already about how to solve Laplace's equation!

However, the boundary conditions are different than electrostatics – discuss later.

Case #2 – General treatment based only on $\nabla \cdot \vec{B} = 0$

We can ALWAYS say that

$$\vec{B} = \nabla \times \vec{A}$$

for some vector field $\vec{A}(\vec{x}) \equiv$ vector potential

And, in fact we saw previously that

$$\vec{B} = \nabla \times \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')\, d^3x'}{|\vec{x} - \vec{x}\,'|}$$

So we expect that the vector potential should have the form:

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x'} + \cancel{\nabla \Psi(\vec{x})}$$

set to 0 for now, revisit in Chap. 6.

where $\Psi(\vec{x}) =$ an arbitrary scalar function

Note that $\vec{A}$ is simpler than $\vec{B}$ usually, because lines of $\vec{A}$ are mostly "parallel to $\vec{J}$" while the twists of $\vec{B}$ about $\vec{J}$ come from taking $\nabla \times \vec{A} = \vec{B}$.

$\vec{A}$ and $\vec{B}$ for a circular current loop, radius a

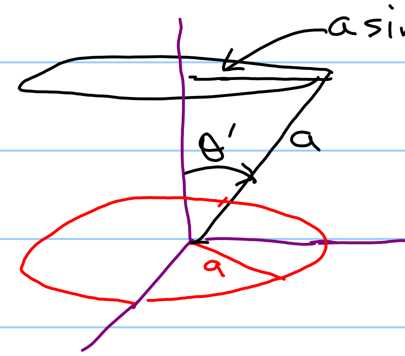
$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} \longrightarrow \frac{\mu_0 I}{4\pi}\oint \frac{\vec{dl}'}{|\vec{x}-\vec{x}'|}$$

used $\vec{J}(\vec{x}') = \hat{\phi}' I \dfrac{\sin\theta' \, \delta(\cos\theta')\, \delta(r'-a)}{a}$

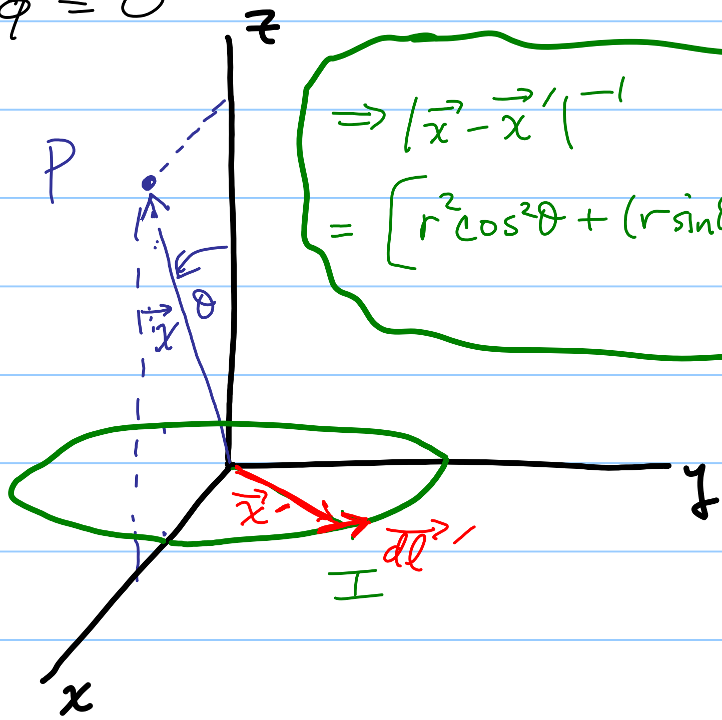
or in cylindrical coordinates $\vec{J}(\vec{x}') = \hat{\phi}' I \, \delta(z')\, \delta(\rho'-a)$

$\Rightarrow$ arranged such that

$$\underset{\substack{\text{area} \perp \\ \text{to } \vec{J}}}{\int} \vec{J}\, d^3x' = I\,\vec{dl}' = I a \sin\theta' d\phi' \, \hat{\phi}'$$



Because this problem has cylindrical symmetry, we may choose the field/observation point $\vec{x}$ to lie in the xz-plane, i.e. $\phi = 0$



$$\Rightarrow |\vec{x}-\vec{x}'|^{-1} = \left[r^2\cos^2\theta + (r\sin\theta - a\cos\phi')^2 + a^2\sin^2\phi'\right]^{-1/2}$$

One might treat this problem directly, as we will tackle it here, or by expanding

$\frac{1}{|\vec{x}-\vec{x}'|}$ in spherical harmonics (see Jackson pp 183–184) or as a cylindrical expansion, etc.

Solution $\vec{l}'(\phi') = a\hat{x}\cos\phi' + a\hat{y}\sin\phi'$

$$\Rightarrow \vec{dl}' = -a\,\hat{x}\sin\phi'\,d\phi' + a\,\hat{y}\cos\phi'$$

$$= a\,\hat{\phi}'\,d\phi', \text{ since } \hat{\phi}' = -\hat{x}\sin\phi' + \hat{y}\cos\phi'$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} \frac{(-\hat{x}\sin\phi' + \hat{y}\cos\phi')\,d\phi'}{(r^2+a^2-2ar\sin\theta\cos\phi')^{1/2}}$$

$\Rightarrow$ the $\hat{y}$ integral survives but the $\hat{x}$ integrand is odd, so the integral vanishes

Thus, for $\vec{x} = (x, 0, z) = r(\sin\theta, 0, \cos\theta)$

$$\Rightarrow \vec{A}(\vec{x}) = \hat{y}\,\frac{\mu_0 I a}{4\pi}\int_0^{2\pi} \frac{\cos\phi'\,d\phi'}{(r^2+a^2-2ar\sin\theta\cos\phi')^{1/2}}$$

Do the integral, get =

$$\vec{A}(\vec{x}) = \cancel{\hat{y}}\,\hat{\phi}\;\mu_0 \frac{I a^2 \pi r\sin\theta}{4\pi(a^2+r^2)^{3/2}}\;{}_2F_1\!\left(\frac{3}{4}, \frac{5}{4}; 2; \frac{4a^2r^2\sin^2\theta}{(a^2+r^2)^2}\right)$$

Hypergeometric2F1[a,b,c,z] (Mathematica)

Or see Jackson Eq. 5.37

Note that $\vec{A}$ is in the $\hat{\phi}$ direction, which makes sense since the current is in the $\hat{\phi}$ direction.

Next, evaluate $\vec{B} = \nabla \times \vec{A}$

$$\Rightarrow \quad B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi)$$

$$B_\phi = 0$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)$$

Note: $_2F_1(a,b;c;z) = 1 + \frac{ab}{c\,1!}z + \frac{a(a+1)b(b+1)}{c(c+1)2!}z^2 + \ldots$

So the large-r behavior is:

$$A_\phi(r,\theta) \xrightarrow[r\to\infty]{} \frac{\mu_0 I a^2 r\sin\theta}{4(a^2+r^2)^{3/2}} + O(r^{-5})$$

$$\simeq \frac{\mu_0 I a^2 \sin\theta}{4r^2} \quad \text{as } r \to \infty$$

and after evaluating the curl,

$$B_r \xrightarrow[r\to\infty]{} \frac{\mu_0}{2\pi}(I\pi a^2)\frac{\cos\theta}{r^3}$$

$$B_\theta \xrightarrow[r\to\infty]{} \frac{\mu_0}{2\pi}(I\pi a^2)\frac{\sin\theta}{r^3}$$

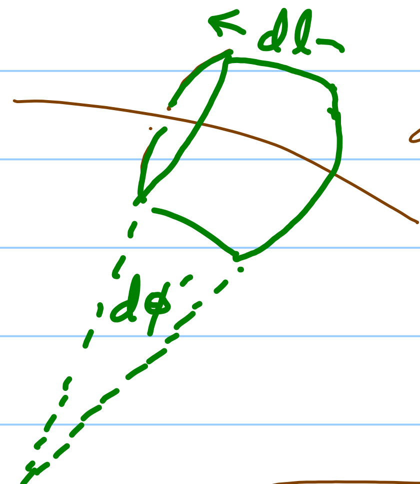
Now, these look very similar to the E-field produced by an electric dipole!

$\Rightarrow$ Call $\boxed{m = \pi a^2 I}$ the magnetic dipole moment

Aside on current δ-functions: previously we stated

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} \longrightarrow \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}'}{|\vec{x}-\vec{x}'|}$$

More careful discussion, integrate over a small volume δV oriented a differential length parallel to $\vec{J}$



$dl = a\,d\phi$

$$\Rightarrow \int_{\delta V} \vec{J}\,d^3x' =$$

$$\int \hat{\phi}'\frac{I}{a}\delta(r'-a)\frac{\delta(\theta'-\frac{\pi}{2})}{\sin\theta'}\, r'^2\sin\theta'\,d\theta'\,d\phi'\,dr'$$

$$= I\,a\,d\phi'\,\hat{\phi}'$$

$= I\,d\vec{l}'$ as desired if we set $\boxed{\vec{J}(\vec{x}') = I\hat{\phi}'\frac{\delta(r'-a)}{a}\delta(\theta'-\frac{\pi}{2})}$

Back to magnetic dipoles:

Observe that $\hat{k}\times\frac{\vec{x}}{|\vec{x}|} = \hat{\phi}\sin\theta$

$$\Rightarrow \boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}}$$

where $\vec{m} = \pi a^2 I\,\hat{z}$ for a circular current loop in the xy-plane

and

$$\nabla\times\vec{A} = \frac{\mu_0}{4\pi}\left(\vec{m}\left(\vec{\nabla}\cdot\frac{\vec{x}}{|\vec{x}|^3}\right) - (\vec{m}\cdot\vec{\nabla})\frac{\vec{x}}{|\vec{x}|^3}\right)$$

$$\nabla\cdot\frac{\vec{x}}{|\vec{x}|^3} = \frac{1}{|\vec{x}|^3}(\nabla\cdot\vec{x}) + \vec{x}\cdot\nabla\frac{1}{|\vec{x}|^3} = \frac{3}{|\vec{x}|^3} - \frac{3}{2}\vec{x}\cdot\frac{(2x\hat{x}+2y\hat{y}+2z\hat{z})}{|\vec{x}|^5} = 0$$

and $(\vec{m}\cdot\nabla)\dfrac{\vec{x}}{|\vec{x}|^3} = m\dfrac{\partial}{\partial z}\dfrac{\vec{x}}{|\vec{x}|^3} = m\dfrac{\hat{z}}{|\vec{x}|^3} - m\dfrac{\vec{x}\,z}{|\vec{x}|^5}3$

$$= -\frac{3(\vec{m}\cdot\vec{x})\vec{x}}{|\vec{x}|^5} + \frac{\vec{m}}{|\vec{x}|^3}$$

define $\hat{n} \equiv \dfrac{\vec{x}}{|\vec{x}|}$

derive this term later!

So $\vec{B} = \nabla\times\vec{A} = \dfrac{\mu_0}{4\pi}\left(\dfrac{3\hat{n}(\vec{m}\cdot\hat{n}) - \vec{m}}{|\vec{x}|^3} + \dfrac{8\pi}{3}\vec{m}\,\delta(\vec{x})\right)$

Multipole expansion of the $\vec{B}$- field

As in Jackson, only expand up to the dipole term

Assume all currents $\vec{J}$ are localized,

and use $\dfrac{1}{|\vec{x}-\vec{x}'|} \xrightarrow[r\gg r']{} \dfrac{1}{|\vec{x}|} + \dfrac{\vec{x}'\cdot\vec{x}}{|\vec{x}|^3} + \ldots$

(this is from the Taylor expansion, $\psi(\vec{x}+\vec{h}) \simeq \psi(\vec{x}) + \vec{h}\cdot\nabla\psi(\vec{x}) + \ldots$

Consider the i^{th} Cartesian component of $\vec{A}$:

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left(\frac{1}{|\vec{x}|}\int_V J_i(\vec{x}')\,d^3x' + \frac{\vec{x}}{|\vec{x}|^3}\cdot\int_V J_i(\vec{x}')\vec{x}'\,d^3x' + \ldots\right)$$

use the identity: $\displaystyle\int_V J_i(\vec{x}')d^3x' = \int_V (\nabla' x_i')\cdot\vec{J}(\vec{x}')d^3x'$

$$= \int_V \left[\nabla'\cdot(x_i'\vec{J}(\vec{x}')) - x_i'\,\underbrace{\nabla'\cdot\vec{J}(\vec{x}')}_{\to 0 \text{ in magnetostatics}}\right]d^3x'$$

$$= \oint_S x_i'\,\vec{J}(\vec{x}')\cdot d\vec{a}' = 0$$

since the surface S is beyond the range of $\vec{J}$

Conclusion: There is no MONOPOLE contribution to $\vec{A}$ $\Longrightarrow$ no magnetic charge!

Now consider the 2^{nd} term: $\int \vec{x}\cdot\vec{x}\,'\,\vec{J}(\vec{x}\,')\,d^3x'$

First, we must derive a useful identity:

$$\int_V \nabla\cdot(f\vec{J}g)\,d^3x = \oint_S fg\,\vec{J}\cdot d\vec{a} = 0$$

since $\vec{J}$ = localized,

assuming f, g are arbitrary scalar functions

$$\Rightarrow 0 = \int [\nabla(fg)\cdot\vec{J} + fg\,\underbrace{\nabla\cdot\vec{J}}_{\to 0}]\,d^3x$$

$\to 0$ in magnetostatics

$$\Rightarrow \boxed{\int_V (\nabla f\cdot\vec{J}g + f\vec{J}\cdot\nabla g)\,d^3x = 0}$$

Now apply this identity, set $f = x_i'$, $g = x_j'$

$$\Longrightarrow \int (x_j' J_i + x_i' J_j)\,d^3x' = 0$$

$$\Rightarrow \vec{x}\cdot\int \vec{x}\,' J_i\,d^3x' = \sum_j x_j \int x_j' J_i\,d^3x'$$

which we can rewrite as

$$= -\frac{1}{2}\sum_j x_j \int (x_i' J_j - x_j' J_i)\,d^3x'$$

which now looks like a cross product, $(\vec{x}\,'\times\vec{J})_k\,\epsilon_{ijk}$

$$= -\frac{1}{2}\sum_{j,k}\epsilon_{ijk}\,x_j \int (\vec{x}\,'\times\vec{J})_k\,d^3x' \quad (*)$$

$$= -\frac{1}{2}\left[\vec{x}\times\int (\vec{x}\,'\times\vec{J}(\vec{x}\,'))\,d^3x'\right]_i$$

e.g., suppose 'i' = 'x-component'

$$\Rightarrow (\text{above expression})_i = \underbrace{-\tfrac{1}{2}\, x \int \cancel{(x' J_x - x' J_x)\, d^3x'}}_{\to\, 0}$$

$$- \tfrac{1}{2}\, y \int (x' J_y - y' J_x)\, d^3x'$$

$$- \tfrac{1}{2}\, z \int (x' J_z - z' J_x)\, d^3x'$$

$$= -\tfrac{1}{2}\, y \int (\vec{x}\,' \times \vec{J})_z\, d^3x'$$

$$+ \tfrac{1}{2}\, z \int (\vec{x}\,' \times \vec{J})_y\, d^3x'$$

$$= -\tfrac{1}{2} \left(\vec{x} \times \int (\vec{x}\,' \times \vec{J})\, d^3x' \right)_x$$

Note that Eq. (*) above has used the Levi-Civita antisymmetric symbol ϵ_{ijk}, also called the "completely antisymmetric unit tensor"

<u>Definition</u>

$$\epsilon_{ijk} \equiv \begin{cases} 1, & \text{if } (i,j,k) = (1,2,3) \text{ or any cyclic permutation} \\ -1, & \text{if an anticyclic permutation of } (1,2,3) \\ 0, & \text{if 2 or more indices are equal.} \end{cases}$$

The bottom line is that we have proved above that

$$\int (\vec{x} \cdot \vec{x}\,')\, \vec{J}(\vec{x}\,')\, d^3x' = -\tfrac{1}{2}\, \vec{x} \times \int (\vec{x}\,' \times \vec{J}(\vec{x}\,'))\, d^3x'$$

whereby we now have for an ARBITRARY current loop,

$$\boxed{\vec{A}_{dipole}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}}$$

, where in general the magnetic dipole moment of the current distribution is

$$\boxed{\vec{m} = \tfrac{1}{2} \int \vec{x}\,' \times \vec{J}(\vec{x}\,')\, d^3x'}$$

This generalizes our earlier result for a circular current loop, and we see that, as before, we have

$$\vec{B}_{dipole} = \nabla \times \vec{A}_{dipole} = \frac{3(\vec{m}\cdot\hat{n})\hat{n} - \vec{m}}{r^3}, \quad \text{at } r \neq 0$$

Next, we consider small $r = |\vec{x}|$ carefully

Consider $\displaystyle\int_{r<R} \vec{B}(\vec{x})\, d^3x = \int_{r<R} \nabla\times\vec{A}\, d^3x = R^2 \oint d\Omega\, \hat{n}\times\vec{A}$

$$= -\frac{R^2\mu_0}{4\pi}\int d^3x'\, \vec{J}(\vec{x}')\times \underbrace{\int \frac{\hat{n}\, d\Omega}{|\vec{x}-\vec{x}'|}}_{= \frac{4\pi}{3}\hat{n}'\frac{r_<}{R^2}}$$

r' here

$$= -\frac{R^2\mu_0}{4\pi}\int d^3x'\, \vec{J}(\vec{x}')\times\hat{n}'\left(\frac{4\pi r'}{3R^2}\right)$$

(refer to pp. 109–110)

but of course $\hat{n}'r' = \vec{x}'$, hence:

$$\int_{r<R} \vec{B}(\vec{x})\, d^3x = -\frac{\mu_0}{3}\underbrace{\int d^3x'\, \vec{J}(\vec{x}')\times\vec{x}'}_{= -2\vec{m}}$$

$$= \frac{2\mu_0}{3}\vec{m}$$

And so finally we have for a magnetic dipole,

$$\boxed{\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left(\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3} + \frac{8\pi}{3}\vec{m}\,\delta(\vec{x})\right)}$$

this term is important in atomic hyperfine interaction!

Other implications

That last derivation implies also that

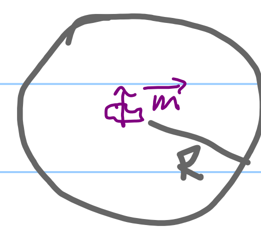
$$\int_{r<R} \vec{B}(\vec{x})\, d^3x = \frac{4}{3}\pi R^3 \vec{B}(0) \quad \text{if all current } \vec{J}(\vec{x})$$

lies OUTSIDE the sphere, just like we found for $\vec{E}$

i.e. there are two separate cases of interest:

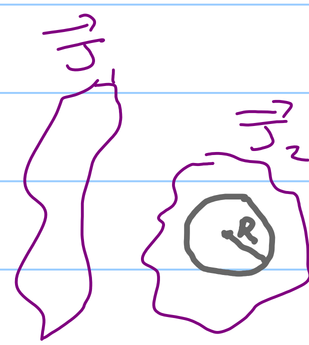
<u>Case (1)</u> – sphere <u>outside</u> a tiny dipole:

$$\int_{r<R} \vec{B}(\vec{x})\, d^3x = \frac{2\mu_0}{3}\vec{m}$$



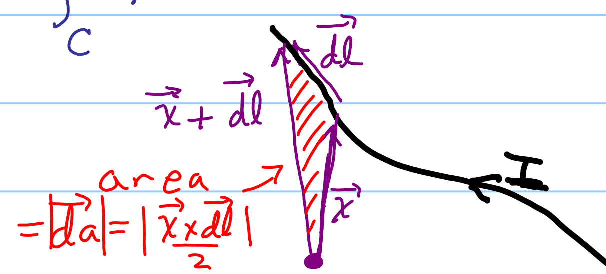
<u>Case (2)</u> Sphere <u>inside</u> of ALL Currents:

$$\int_{r<R} \vec{B}(\vec{x})\, d^3x = \frac{4}{3}\pi R^3 \vec{B}(0)$$



<u>Quick</u> <u>check</u> of our formula for $\vec{m}$, for a <u>planar</u> loop:

$$\vec{m} = \frac{1}{2}\int \vec{x} \times \vec{J}(\vec{x})\, d^3x = \frac{1}{2} I \oint_C \vec{x} \times \vec{d\ell}$$



So from the construction → we find for a <u>planar</u> loop,

$\oint \frac{\vec{x} \times \vec{d\ell}}{2} = a\hat{n}$, where a = loop area

and $\vec{m} = I\vec{a}$