Chapter 5 - Magnetostatics

Current = charges in motion,

$$I = \frac{dQ}{dt} = \frac{\text{Coulombs}}{\text{second}} = \text{Amperes}$$

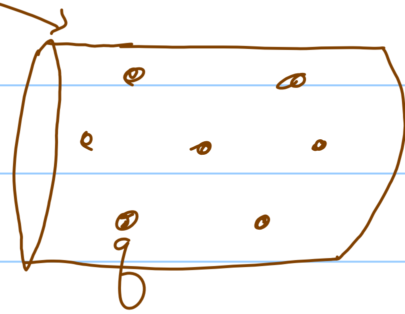
Current density If N charges q per unit volume move at velocity $\vec{v}$

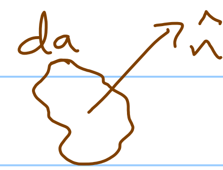
$\Rightarrow$ $\vec{J} = Nq\vec{v}$ is the current density

$\Rightarrow$ The total amount of $\frac{\text{Coulombs}}{\text{sec}}$ crossing a differential area element $\vec{da}$ is: $dI = \vec{J} \cdot \vec{da}$

$$= \vec{J} \cdot \hat{n}\, da$$

$N \frac{\text{Coul}}{m^3}$





Charge conservation

At every point in space, $\frac{\partial \rho(\vec{x},t)}{\partial t} + \nabla \cdot \vec{J}(\vec{x},t) = 0$

or in integral form,

$$\oint_S \vec{J} \cdot \vec{da} = -\frac{d}{dt}\int_V \rho(\vec{x},t)\, d^3x = -\frac{d}{dt} Q_{enclosed}$$

Here we focus on magnetostatics, where I and $\vec{J}$ are time-independent $\Rightarrow$ $\boxed{\frac{\partial \rho}{\partial t} = 0, \ \nabla \cdot \vec{J} = 0}$

Biot – Savart Law = magnetic "analogue" of Coulomb's Law

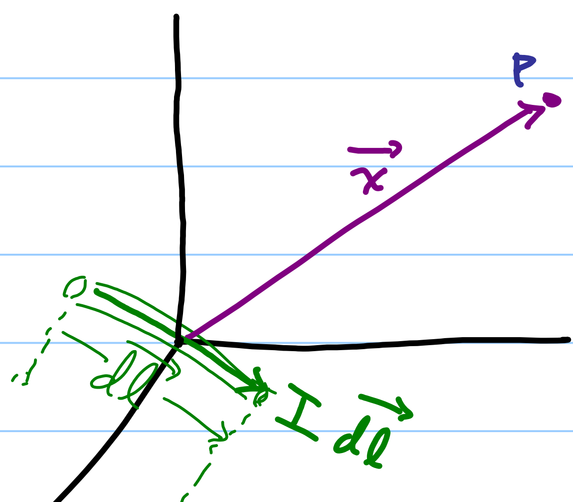
denote $\vec{B} = \begin{cases} \text{magnetic induction, or} \\ \text{magnetic flux density} \end{cases}$

Then consider an isolated, thin current element $I\vec{dl}$ of differential length $\vec{dl}$, located at the origin

Experiment: this $I\,\vec{dl}$ generates $\vec{dB}$ at $\vec{x}$, namely

$$\vec{dB} = k\,I\,\frac{\vec{dl} \times \vec{x}}{|\vec{x}|^3}\,,$$

in SI units, $k = \frac{\mu_0}{4\pi} \equiv 10^{-7}\,\frac{N}{A^2}$



$\vec{dB}$ at observation point P

and for a steady current that follows a complete closed circuit,

$$\boxed{\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi} \oint \frac{\vec{dl} \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}}$$

Units: $[B] = \frac{N}{A\cdot m}$ whereas $[E] = \frac{N}{C}$

$\Rightarrow [E] = [B]\cdot[\text{velocity}]$

You can also see this from the Lorentz force, $\vec{F} = q\vec{v}\times\vec{B} + q\vec{E}$

For some purposes it is desirable for $\vec{E}, \vec{B}$ to have the <u>same</u> dimensions, as in GAUSSIAN UNITS where

$$k = \frac{1}{c}\ , \quad c = \text{vacuum light speed}$$

e.g. the Lorentz force in Gaussian units is

$$\vec{F} = q\frac{\vec{v}}{c}\times\vec{B} + q\vec{E}$$

This is particularly helpful in relativity theory since the components of the field-strength tensor are made from $\vec{E}, \vec{B}$

Compare the SI expressions for $\vec{E}, \vec{B}$:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}\,')(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}\,d^3x'$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}\,')\times(\vec{x}-\vec{x}\,')\,d^3x'}{|\vec{x}-\vec{x}\,'|^3}$$

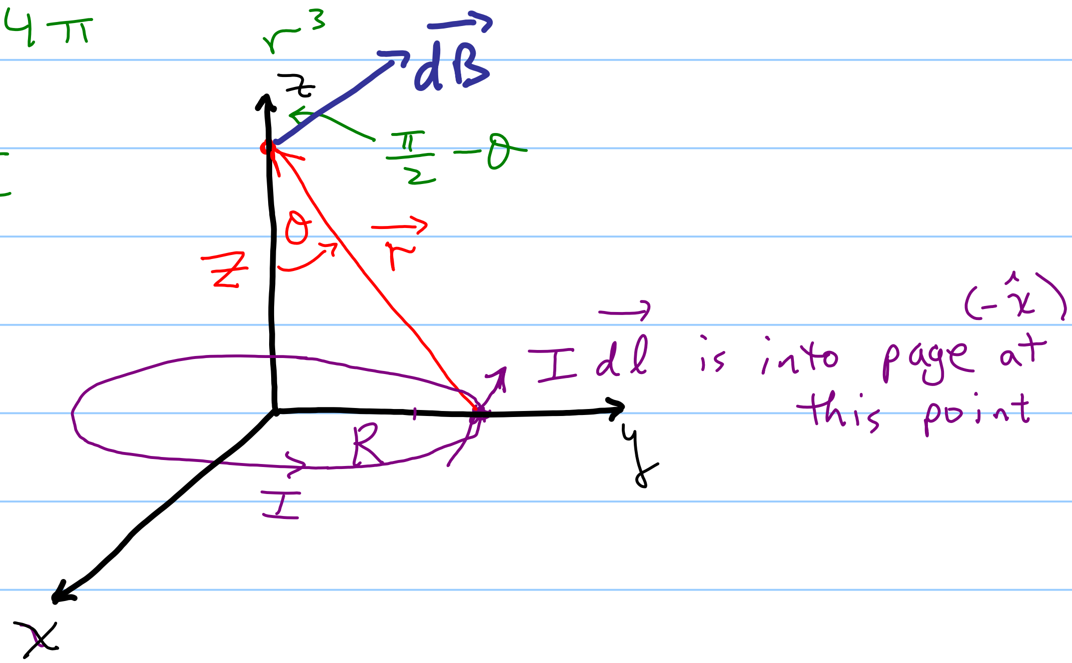
IMPORTANT: These two expressions have VERY different vectorial character!

Applications (a) $\vec{B}$ on axis of a circular current loop

We start from

$$\vec{dB} = \frac{\mu_0}{4\pi} I \frac{\vec{dl} \times \vec{r}}{r^3} \qquad \vec{r} = \vec{x} - \vec{x}'$$

$$= \frac{\mu_0 I}{4\pi} \frac{\vec{dl} \times \hat{r}}{z^2 + R^2}$$



$\Rightarrow$ By symmetry only B_z survives the integration

$$\Rightarrow \vec{B} = \hat{z} \frac{\mu_0 I}{4\pi} \frac{1}{z^2 + R^2} \int dl \cos\left(\frac{\pi}{2} - \theta\right) \qquad \sin\theta = \frac{R}{(z^2+R^2)^{1/2}}$$

$$= \hat{z} \frac{\mu_0 I}{4\pi} \frac{R}{(z^2+R^2)^{3/2}} (2\pi R)$$

or finally
$$\boxed{\vec{B} = \frac{\mu_0 I R^2}{2(z^2+R^2)^{3/2}} \hat{z}}$$

Parametric integral for thin wire loops

Suppose we describe a 3D current loop parametrically, i.e. as $\vec{l} = \vec{l}(s)$

for $s = s_0$ to s_f

$\Rightarrow$ Then $\quad \vec{dl} = \dfrac{\vec{dl}(s)}{ds}\, ds$

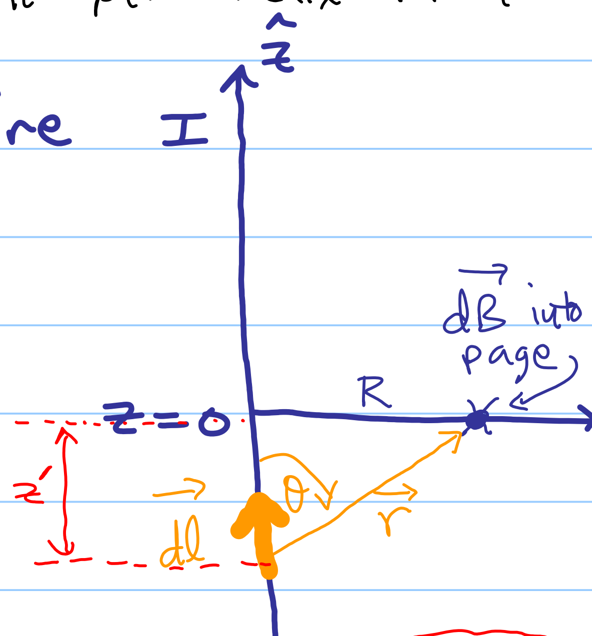
$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 I}{4\pi} \int_{s_0}^{s_f} ds \, \frac{\vec{dl}(s)}{ds} \times \frac{[\vec{x} - \vec{l}(s)]}{|\vec{x} - \vec{l}(s)|^3}}$$

e.g. a helical current might be described by

helix $\quad \vec{l}(s) = R\cos(2\pi s)\hat{x} + R\sin(2\pi s)\hat{y} + sh\hat{z}$

where h = rise in height per helix turn

(b) A long straight wire I



$$\Rightarrow \vec{dB} = \hat{\phi}\, \frac{\mu_0}{4\pi} \frac{dl}{z'^2 + R^2} \sin\theta$$

and $\sin\theta = \dfrac{R}{(z'^2 + R^2)^{1/2}}$

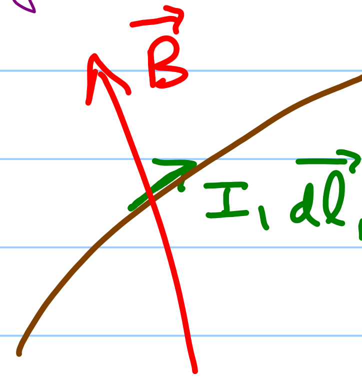
$$\Rightarrow B = \hat{\phi}\, \frac{\mu_0 I}{4\pi} \underbrace{\int_{-\infty}^{\infty} \frac{R\, dz'}{(z'^2 + R^2)^{3/2}}}_{= \frac{2}{R}}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 I}{2\pi R}\hat{\phi}}$$

Forces on currents

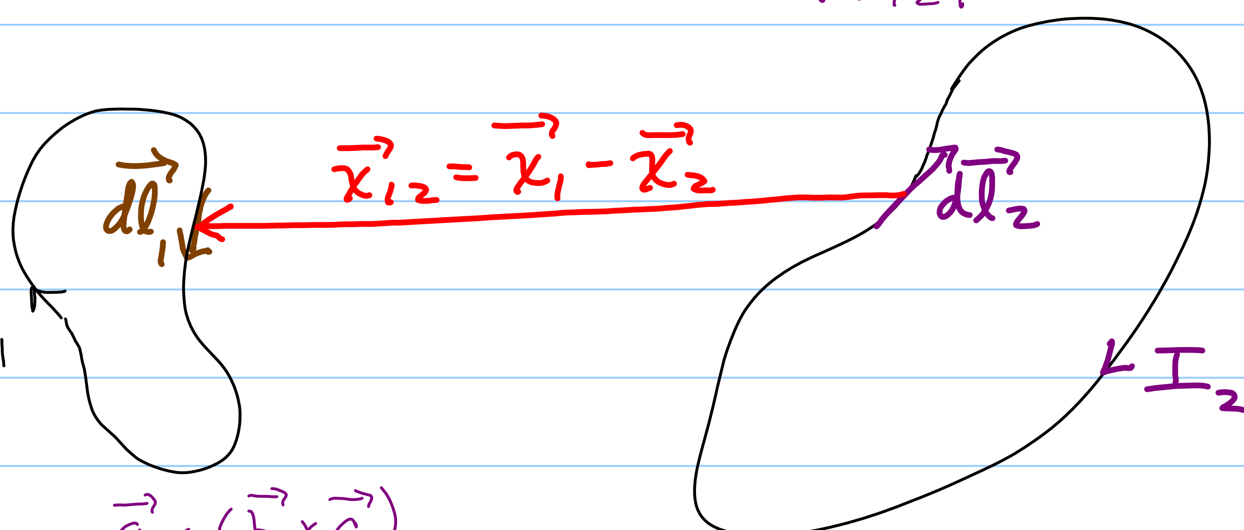
The force on a differential current element $I_1 \vec{dl}_1$ produced by another current's B-field is

$$d\vec{F}_1 = I_1 \vec{dl}_1 \times \vec{B}$$



If $\vec{B}$ is produced by a second current loop I_2, then the total force that I_2 causes on I_1 is:

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{\vec{dl}_1 \times (\vec{dl}_2 \times \vec{x}_{12})}{|\vec{x}_{12}|^3}$$



or using $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$$\Rightarrow d\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \left[\frac{(\vec{dl}_1 \cdot \vec{dl}_2)\vec{x}_{12}}{|\vec{x}_{12}|^3} + \frac{\vec{dl}_2 (\vec{dl}_1 \cdot \vec{x}_{12})}{|\vec{x}_{12}|^3} \right]$$

perfect differential gives 0 when integrated over closed loop! $\rightarrow$ $\left(-\vec{dl}_1 \cdot \nabla_1 \frac{1}{|\vec{x}_{12}|}\right)$

Hence $$\boxed{\vec{F}_{12} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint\oint (d\vec{l}_1 \cdot d\vec{l}_2) \frac{\vec{x}_{12}}{|\vec{x}_{12}|^3}}$$

$= -\vec{F}_{21}$ by inspection

$\Rightarrow$ this obeys Newton's 3^{rd} law

More generally the force on a current is

$$\vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x$$

and the torque is

$$\vec{N} = \int \vec{x} \times (\vec{J} \times \vec{B})\, d^3x$$

Derivation the differential equations of magnetostatics

Begin from $\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} d^3x'$

or

$$\vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \nabla_x \frac{1}{|\vec{x} - \vec{x}'|} d^3x'$$

Aside: a vector identity is $\vec{\nabla}\psi \times \vec{a} = \nabla \times (\psi \vec{a}) - \psi \vec{\nabla} \times \vec{a}$

where $\vec{a} = \vec{J}(\vec{x}')$ $\quad \psi = 1/|\vec{x} - \vec{x}'|$

$$\Rightarrow \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \left[\int \vec{\nabla}_x \times \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' - \int \frac{\overbrace{(\vec{\nabla}_x \times \vec{J}(\vec{x}'))}^{0}}{|\vec{x} - \vec{x}'|} d^3x' \right.$$

since $\vec{J}(\vec{x}')$ is indep of $\vec{x}$

$$\text{So } \boxed{\vec{B}(\vec{x}) = \vec{\nabla} \times \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')\, d^3x'}{|\vec{x} - \vec{x}\,'|}}$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

<u>Next, derive $\vec{\nabla} \times \vec{B}$:</u>

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \left(\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x' \right)$$

and use $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Hence

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla}_x \left[\int \vec{\nabla}_x \cdot \left(\frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \right) d^3x' \right]$$

$$- \frac{\mu_0}{4\pi} \int \nabla_x^2 \left(\frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \right) d^3x'$$

$$= \frac{\mu_0}{4\pi} \nabla_x \int \vec{J}(\vec{x}\,') \cdot \nabla_x \frac{1}{|\vec{x} - \vec{x}\,'|} d^3x' - \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}\,') \underbrace{\nabla_x^2 \frac{1}{|\vec{x} - \vec{x}\,'|}}_{-4\pi\delta(\vec{x} - \vec{x}\,')} d^3x'$$

$$= -\frac{\mu_0}{4\pi} \nabla_x \int \vec{J}(\vec{x}\,') \cdot \nabla' \frac{1}{|\vec{x} - \vec{x}\,'|} d^3x'$$

we will now demonstrate that this integral vanishes!

$$+ \mu_0 \int \vec{J}(\vec{x}\,') \, \delta(\vec{x} - \vec{x}\,') d^3x'$$

$$= \boxed{\mu_0 \vec{J}(\vec{x}) = \nabla \times \vec{B}(\vec{x})}$$