Note $\vec{P}$ is often NOT proportional to $\vec{E}$, notably:

(i) in ferroelectric materials, where $\vec{P}$ can be nonzero even when $\vec{E} = 0$

or

(ii) for strong E-field where $\vec{P}$ can be a more complicated, nonlinear function of E, e.g. in nonlinear optics

There is another important class of problems where the relationship between $\vec{P}$ and $\vec{E}$ is LINEAR but ANISOTROPIC, as in a birefringent crystal like calcite. For such a case, we can write:

$$\vec{P} = \epsilon_0 \underline{\chi_e} \vec{E}$$

where $\underline{\chi_e}$ is a 3x3 matrix rather than a scalar.

Boundary Value Problems with Dielectrics

Category I Problems solvable using image charge methods: e.g. a point charge q near a planar interface between 2 dielectrics.

$z > 0$

$\epsilon = \epsilon_1$

Φ_1

q at $(0,0,d) \equiv \vec{x}_0$

$z = 0$ plane

$z < 0$

$\epsilon = \epsilon_2$

Φ_2

try image q' at $(0,0,-d)$ $'' \ \vec{x}_0'$

when calculating Φ_1 at $z > 0$

Then a plausible hypothesis is that $\Phi(\vec{x})$ can be written as follows, for some q',

$$\Phi_1 = \frac{q}{4\pi\epsilon_1 |\vec{x} - \vec{x}_0|} + \frac{q'}{4\pi\epsilon_1 |\vec{x} - \vec{x}_0'|}$$

and another plausible guess/ansatz is that Φ_2 can be written in the form:

$$\Phi_2 = \frac{1}{4\pi\epsilon_2} \frac{q''}{|\vec{x} - \vec{x}_0|}$$

Recall that it is not allowed to put an image charge into the volume of interest!

Next, attempt to match the B.C.s for some choice of q', q'', using $\frac{1}{|\vec{x}-\vec{x}_0|} = [x^2+y^2+(z-d)^2]^{-1/2}$

$$\frac{1}{|\vec{x}-\vec{x}_0'|} = [x^2+y^2+(z+d)^2]^{-1/2}$$

Since there are no free surface charges, the BCs are

$$\epsilon_1 E_z\Big|_{z=0^+} = \epsilon_2 E_z\Big|_{z=0^-}$$

and

$$E_x\Big|_{z=0^+} = E_x\Big|_{z=0^-}$$

i.e.,

$$-\epsilon_1 \frac{\partial \Phi_1}{\partial z}(x,y,0) = -\epsilon_2 \frac{\partial \Phi_2}{\partial z}(x,y,0)$$

$$\Rightarrow \frac{-\epsilon_1}{4\pi\epsilon_1}\left\{\frac{+qd}{(\rho^2+d^2)^{3/2}} - \frac{q'd}{(\rho^2+d^2)^{3/2}}\right\} = \frac{-\epsilon_2}{4\pi\epsilon_2}\frac{q''d}{(\rho^2+d^2)^{3/2}}$$

and

$$-\frac{\partial \Phi_1}{\partial x}(x,y,0) = -\frac{\partial \Phi_2}{\partial x}(x,y,0)$$

or

$$-\frac{1}{4\pi\epsilon_1}\left[\frac{-qx}{(\rho^2+d^2)^{3/2}} - \frac{q'x}{(\rho^2+d^2)^{3/2}}\right] = -\frac{1}{4\pi\epsilon_2}\frac{(-q''x)}{(\rho^2+d^2)^{3/2}}$$

and simplifying gives:

$$\left\{\begin{aligned} q - q' &= q'' \\ \frac{q+q'}{\epsilon_1} &= \frac{q''}{\epsilon_2}\end{aligned}\right.$$

2 equations & 2 unknowns to solve

and the solution is $q' = -\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\right)q$, $q'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}q$

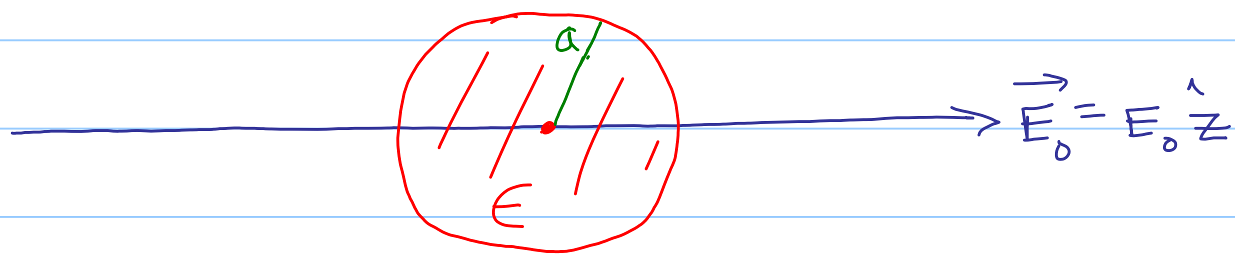
On your own, verify the following:

(1) The potential is continuous at the boundary $\Rightarrow$ we could have imposed this as a requirement

(2) Compute the polarization surface charge density, $\sigma_b = \vec{P} \cdot \hat{n}$ $\leftarrow$ Jackson calls this σ_{pol}

Category II – Boundary value problem solved by separation of variables

e.g. a dielectric sphere in a uniform E-field



$\Rightarrow$ Here we have AZIMUTHAL SYMMETRY and NO FREE CHARGES, $\rho_{free} = 0$

$$\Rightarrow \Phi_{in} = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$$

$$\Phi_{out} = \sum_\ell \left(B_\ell r^\ell + \frac{C_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Boundary Conditions

(1) $\lim_{r\to\infty} \Phi_{out} = -E_0 z = -E_0 r P_1(\cos\theta)$

i.e. $\boxed{B_1 = -E_0}$ and all other $B_\ell = 0$ for $\ell \neq 1$

(2) $E_{\parallel}$ is continuous $\Rightarrow$ $-\frac{1}{r}\frac{\partial \Phi_{in}}{\partial\theta}\Big|_{r=a} = -\frac{1}{r}\frac{\partial \Phi_{out}}{\partial\theta}\Big|_{r=a}$

$\Rightarrow A_\ell a^\ell = \frac{C_\ell}{a^{\ell+1}}$ if $\ell \neq 1$, and $A_1 a = B_1 a + \frac{C_1}{a^2}$

(3) $\epsilon E_\perp(in) = \epsilon_0 E_\perp(out)$

$\Rightarrow -\epsilon \frac{\partial \Phi_{in}}{\partial r}\Big|_{r=a} = -\epsilon_0 \frac{\partial \Phi_{out}}{\partial r}\Big|_{r=a}$

$\Rightarrow -\epsilon \ell A_\ell a^{\ell-1} = -\epsilon_0(-\ell-1) C_\ell a^{-\ell-2}$, $\ell \neq 1$

and $-\epsilon A_1 = -\epsilon_0\left(-E_0\right) + \frac{2C_1}{a^2}$, $\ell = 1$

These are 2 equations, 2 unknowns for each ℓ, whose solutions are:

$$A_\ell = C_\ell = 0, \quad \ell \neq 1$$

$$A_1 = \frac{-3E_0}{2+\frac{\epsilon}{\epsilon_0}}, \quad C_1 = \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) a^3 E_0$$

$\Rightarrow$ the final solution is

$$\Phi_{in}(r,\theta) = -\frac{3}{2+\frac{\epsilon}{\epsilon_0}} E_0 r\cos\theta, \quad r<a$$

$$\Phi_{out}(r,\theta) = -E_0 r\cos\theta + \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) E_0 \frac{a^3}{r^2}\cos\theta, \quad r>a$$

Note that here again, Φ is continuous at a.

$$\Rightarrow \vec{E}_{inside} = \frac{3}{\frac{\epsilon}{\epsilon_0}+2} E_0 \hat{z}$$

(observe that E_{inside} is smaller than E_0, if $\epsilon > \epsilon_0$)

and $\vec{E}_{outside} = E_0\hat{z} + \vec{E}_{dipole}$

where

$\vec{E}_{dipole}$ is the field due to a dipole of magnitude $\vec{p} = \hat{z}\, 4\pi\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{\frac{\epsilon}{\epsilon_0}+2}\right) a^3 E_0$

i.e. $\vec{p} = \int d^3x\, \vec{P}(x)$ where

$$\vec{P}(\vec{x}) = (\epsilon-\epsilon_0)\vec{E} = 3\epsilon_0\left(\frac{\frac{\epsilon}{\epsilon_0}-1}{\frac{\epsilon}{\epsilon_0}+2}\right)\vec{E}_0$$

or more directly using $\vec{E}_{out} = -\vec{\nabla}\Phi_{out}$,

$$\vec{E}_{out} = E_0\hat{z} - \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) E_0 a^3 \nabla\left(\frac{\cos\theta}{r^2}\right)$$

$$= E_0\hat{z} - \left(\frac{\frac{\epsilon}{\epsilon_0}-1}{\frac{\epsilon}{\epsilon_0}+2}\right) E_0 a^3 \left(-\frac{\cos\theta\,\hat{r}}{r^3}(2) - \frac{\sin\theta\,\hat{\theta}}{r^3}\right)$$

Compare with Eq. 4.12

Observe that the bound (polarization) charge density is 0 throughout the volume $r<a$, since there $\vec{P}(\vec{x}) = \text{constant}$,

$$\Rightarrow -\vec{\nabla}\cdot\vec{P}(\vec{x}) = 0$$

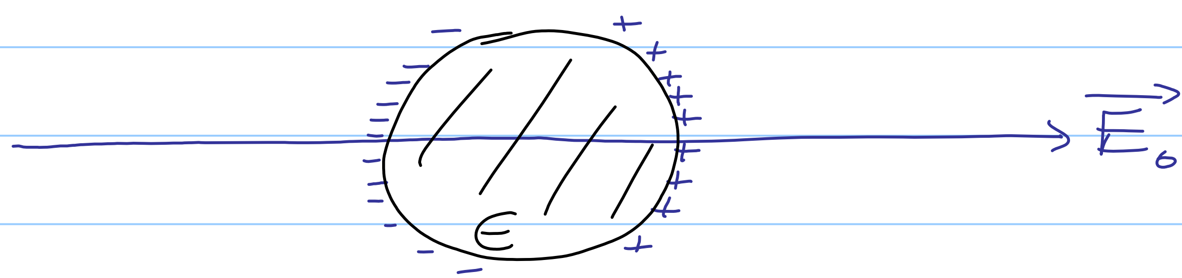
However, a surface bound charge remains,

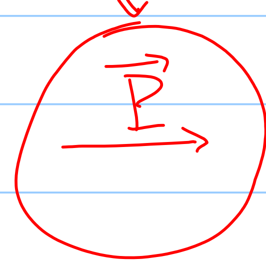
$$\sigma_b = \vec{P} \cdot \hat{r} = 3\epsilon_0 \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2} \right) E_0 \cos\theta$$

Comment: The same math would apply to a spherical CAVITY in a dielectric medium, except that we must interchange $\epsilon \longleftrightarrow \epsilon_0$ everywhere!

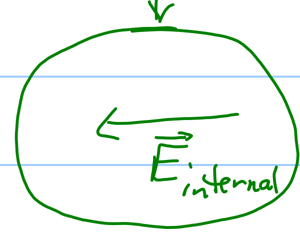
e.g. $\vec{E}_{in} = \dfrac{3\epsilon}{2\epsilon + \epsilon_0} \vec{E}_0$, which is $> E_0$ if $\epsilon > \epsilon_0$

Qualitative Picture for a dielectric sphere



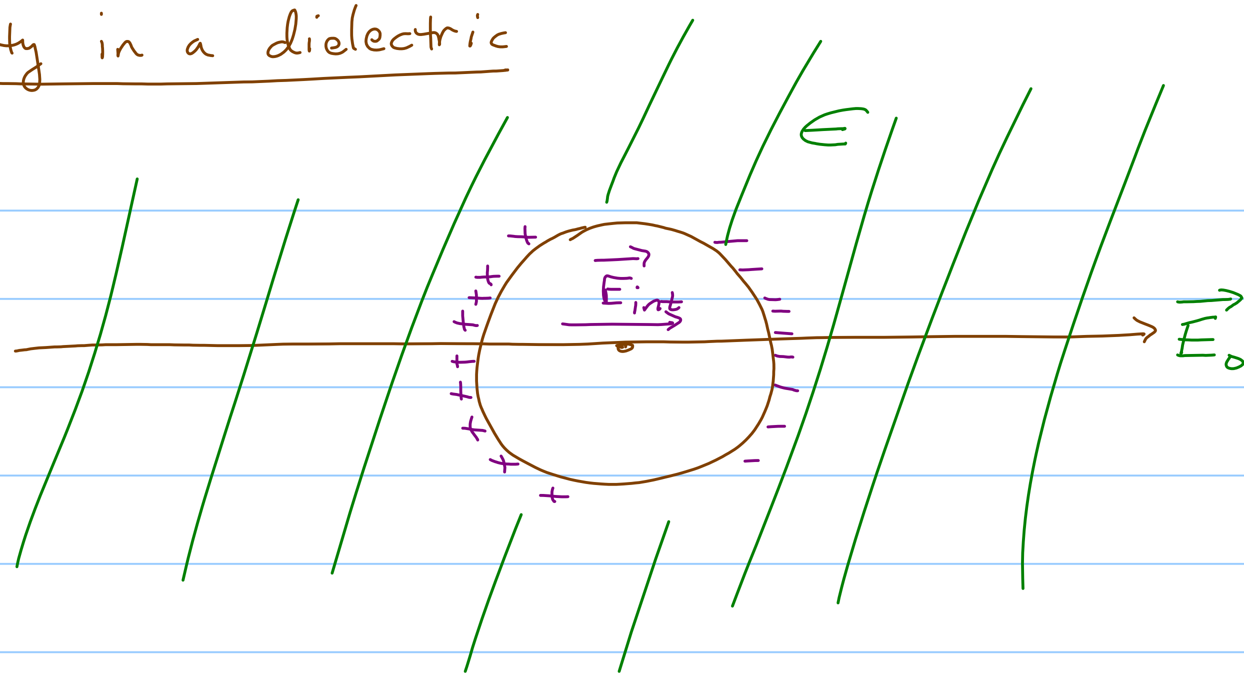


polarization induced by $\vec{E}_0$



$\vec{E}$-field induced by bound polarization charges σ_b REDUCES $\vec{E}$ when added to $\vec{E}_0$

Cavity in a dielectric



Now, for a cavity, the $\vec{E}$-field due to σ_b INCREASES the field higher than what it is in the dielectric far from the cavity.

IMPORTANT: When there are no free charges in a medium, we expect $\rho_b = -\nabla \cdot \vec{P} = 0$ since $\nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon} \to 0$ and $\vec{P} = \epsilon_0 \chi_e \vec{E}$