Corollary to the preceding derivation

Consider a different limit now, where <u>all</u> charge $\rho(\vec{x}\,')$ lies OUTSIDE the sphere,

$\Rightarrow$ the same math applies <u>except</u> now we have $r_< = R$, $r_> = r'$

$$\Rightarrow \int_{r<R} \vec{E}(\vec{x})\, d^3x = -\frac{4\pi}{3} R^3 \left(\frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}\,')\, \hat{n}'\, d^3x'}{r'^2} \right)$$

but this is precisely $\vec{E}(0)$!

$$\Rightarrow \boxed{\int_{r<R} \vec{E}(\vec{x})\, d^3x = \frac{4}{3}\pi R^3\, \vec{E}(0)}$$

this holds provided NO CHARGE lies inside the sphere

Multipole expansion of the interaction energy of $\rho(\vec{x})$ with an external Field

The interaction energy with the external $\Phi(\vec{x})$ is:

$$W = \int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x$$

Suppose now that the potential $\Phi(\vec{x})$ is slowly varying throughout the region where $\rho(\vec{x}) \neq 0$, we can Taylor expand $\Phi(\vec{x})$ about a suitable origin inside ρ:

$$\Phi(\vec{x}) = \Phi(0) + \vec{x}\cdot\vec{\nabla}\Phi(0) + \frac{1}{2}\sum_{ij} x_i \frac{\partial^2 \Phi(0)}{\partial x_i \partial x_j} x_j + \ldots$$

$$= \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial E_j(0)}{\partial x_i} + \ldots$$

$$= \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{6}\sum_{i,j}\left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_j(0)}{\partial x_i} + \ldots$$

$\Rightarrow$ In that last term we have added a term

$$0 = \frac{r^2}{6}\nabla\cdot\vec{E},$$

in order to simplify the final expression.

Next, plugging this into the above energy expression gives the following:

$$W = q_{tot}\,\Phi(0) - \vec{E}(0)\cdot\int \vec{x}\,\rho(\vec{x})\,d^3x$$
$$-\frac{1}{6}\sum_{ij}\frac{\partial E_j(0)}{\partial x_i}\int\left(3x_i x_j - r^2\delta_{ij}\right)\rho(\vec{x})\,d^3x$$

or, more compactly,

$$W = q_{tot}\,\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\left[\nabla\cdot\left(\underline{Q}\,\vec{E}(\vec{x})\right)\right]_0 + \ldots$$

where $\underline{Q}$ = constant
= electric quadrupole Cartesian matrix.

e.g. 2 dipoles have:

$$W_{12} = \frac{\left[\vec{p}_1\cdot\vec{p}_2 - 3(\hat{n}\cdot\vec{p}_1)(\hat{n}\cdot\vec{p}_2)\right]}{4\pi\epsilon_0 r^3} + \frac{\vec{p}_1\cdot\vec{p}_2}{3\epsilon_0}\,\delta(\vec{x}_1-\vec{x}_2)$$

Electrostatics in macroscopic media:

- how matter affects or is affected by fields.

e.g. - why does light pass through a window, but not through wood?

some challenging aspects:

- Matter is mostly neutral, but it consists of charged particles

- Matter is rearranged, or polarized by external fields

- The polarization (usually expressed as induced dipole moments) produces "internal fields"

- The total field = sum of external + internal fields

- We must <u>average</u> over a "microsopically large but macroscopically small" volume, i.e. over a volume that might contain $\simeq 1000$ atoms

e.g., call $\vec{E}_{micro}(\vec{x})$ the primitive or microscopic E-field that obeys Maxwell's equations in free space and includes <u>ALL</u> charges in the universe

picture this as fluctuating wildly in space + time on the atomic scale, e.g. inside a solid.

$\Rightarrow$ We want to find equations obeyed by a coarse-grained or <u>averaged</u> <u>E-field</u>,

$$\Rightarrow \quad \boxed{\vec{E}(\vec{x}) = \frac{\displaystyle\int_{|\vec{x}-\vec{x}'|<R} d^3x'\, \vec{E}_{micro}(\vec{x}')}{\frac{4}{3}\pi R^3}}$$

Point (1) Since $\nabla \times \vec{E}_{micro} = 0$ everywhere in electrostatics, clearly $\underline{\nabla \times \vec{E} = 0}$ still after averaging

Hence $\vec{E}$ is still derivable from a <u>potential</u>

i.e. $\boxed{\vec{E} = -\nabla \Phi}$

Point (2) Next, determine the <u>internal</u> fields produced by electric dipole moments $\vec{p}$ induced in the medium. (Assume that higher multipoles only produce small corrections.)

$\Rightarrow$ Start from $\underline{\Phi}(\vec{x}) = \dfrac{\vec{p} \cdot (\vec{x} - \vec{x}')}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|^3}$

for a dipole at $\vec{x}'$. We can rewrite this in differential form, for the potential $d\underline{\Phi}$ produced by a dipole $d\vec{p}$ at $\vec{x}'$:

$$d\Phi(\vec{x}) = \frac{d\vec{p}}{4\pi\epsilon_0} \cdot \nabla' \frac{1}{|\vec{x} - \vec{x}'|} = \frac{-d\vec{p}}{4\pi\epsilon_0} \cdot \nabla \frac{1}{|\vec{x} - \vec{x}'|}$$

So if we have a <u>continuous</u> average distribution of dipole moment, we write $d\vec{p}(\vec{x}') = \vec{P}(\vec{x}') \, d^3x'$ where we have defined

$$\vec{P}(\vec{x}') \equiv \text{polarization} = \frac{\text{average dipole moment}}{\text{unit volume}}$$
$$= \sum_i N_i \langle \vec{p}_i \rangle$$

where n_i = number density of the ith type of molecule, having an average electric dipole moment, $\langle\vec{p}_i\rangle$

Of course, in addition to the dipoles that represent "bound charge", we could also have some "additional" or "free" charge present, ρ_{free}

But for now, consider only the dipoles.

$$\Rightarrow \quad \Phi(\vec{x}) = \int_V \frac{\vec{P}(\vec{x}')}{4\pi\epsilon_0} \cdot \nabla' \frac{1}{|\vec{x}-\vec{x}'|} d^3x'$$

or recalling the identity:

$$\nabla' \cdot \left(\frac{\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|} \right) = \frac{1}{|\vec{x}-\vec{x}'|} \nabla' \cdot \vec{P}(\vec{x}') + \vec{P}(\vec{x}') \cdot \nabla' \frac{1}{|\vec{x}-\vec{x}'|}$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \nabla' \cdot \left(\frac{\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|} \right) d^3x' - \int_V \frac{d^3x'}{4\pi\epsilon_0} \frac{\nabla' \cdot \vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

and from the divergence theorem, of course, this becomes

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\left(-\nabla' \cdot \vec{P}(\vec{x}')\right) d^3x'}{(\vec{x}-\vec{x}')} + \frac{1}{4\pi\epsilon_0} \oint_S \frac{\vec{P}(\vec{x}') \cdot \hat{n}' da'}{|\vec{x}-\vec{x}'|}}$$

Interpretation These 2 integrals now have the SAME FORM as for the potential produced by a volume "bound charge" $\rho_b \equiv -\nabla' \cdot \vec{P}(\vec{x}')$ and/or a surface bound charge $\sigma_b \equiv \vec{P}(\vec{x}') \cdot \hat{n}'$

where "b" stands for "bound charges"

Thus, interpreted this way, our old formulas for $\Phi(\vec{x})$ still work in the presence of media, namely

$$\Phi(\vec{x}) = \int_V \frac{\rho(\vec{x}\,')\,d^3x'}{4\pi\epsilon_0|\vec{x}-\vec{x}\,'|} + \int_S \frac{\sigma(\vec{x}\,')\,da'}{4\pi\epsilon_0|\vec{x}-\vec{x}\,'|}$$

provided we use:

$$\rho(\vec{x}\,') = \rho_{free}(\vec{x}\,') + \rho_b(\vec{x}\,') \quad \leftarrow -\nabla'\cdot\vec{P}(\vec{x}\,')$$

and

$$\sigma(\vec{x}\,') = \sigma_{free}(\vec{x}\,') + \sigma_b(\vec{x}\,') \quad \leftarrow \vec{P}(\vec{x}\,')\cdot\hat{n}'$$

And since this equation has the same form for $\Phi(\vec{x})$ as before (without media), we know:

$$-\nabla^2\Phi(\vec{x}) = \nabla\cdot\vec{E} = \frac{4\pi}{4\pi\epsilon_0}(\rho_{free}+\rho_b)$$

or $\nabla\cdot\vec{E} + \frac{1}{\epsilon_0}\nabla\cdot\vec{P} = \frac{\rho_{free}}{\epsilon_0}$

Or defining the ELECTRIC DISPLACEMENT as

$$\vec{D} \equiv \epsilon_0\vec{E} + \vec{P},$$

we obtain the appropriate Maxwell equation in the presence of macrosopic media,

$$\boxed{\nabla\cdot\vec{D} = \rho_{free}} \Rightarrow \boxed{\oint_S \vec{D}\cdot d\vec{a} = Q_{enclosed}^{free}}$$

This, in addition to $\boxed{\nabla\times\vec{E}=0,}$ <u>defines</u> the electrostatic field.

Generally, we cannot solve the Maxwell equations

$$\nabla\cdot\vec{D}=\rho$$

$$\nabla\times\vec{E}=0$$

unless we know how $\vec{D}$ and $\vec{E}$ are related.

<u>Often</u>, it is a good approximation to assume, especially for <u>weak</u> fields, that the induced polarization is proportional to $\vec{E}$, i.e. $\vec{P}=\chi_e\epsilon_0\vec{E}$

where χ_e is the electric susceptibility of the medium = CONSTANT for an isotropic medium.

$$\Longrightarrow \quad \vec{D}=\epsilon\vec{E} \Rightarrow \epsilon=\epsilon_0(1+\chi_e) \text{ is called the DIELECTRIC CONSTANT}$$

<u>An Important Special Case</u>

Often ϵ = constant within some region. When this is true, we simply have

$$\nabla\cdot\vec{E}=\rho/\epsilon \quad \text{WITHIN THAT REGION,}$$

while different regions are connected by BOUNDARY CONDITIONS, namely

$$\boxed{(\vec{D}_2-\vec{D}_1)\cdot\hat{n}_{2\leftarrow 1}=\sigma_F}\;\boxed{(\vec{E}_2-\vec{E}_1)\times\hat{n}_{2\leftarrow 1}=0}$$