Example problem 3.16

(a) δ-Function orthogonality proof

Consider the 2-dim^l Fourier transform of a function $f(x,y)$:

$$F(s,\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\, f(x,y)\, e^{i(sx+\sigma y)}$$

and now specialize to a particular $f(x,y)$ written in polar coordinates,

$f(x,y) \to f(r)\,e^{im\theta}$, where $x = r\cos\theta$, $y = r\sin\theta$

and call $s = \rho\cos\alpha$, $\sigma = \rho\sin\alpha$ $\quad(*)$

$$\Rightarrow F(\rho,\alpha) = \frac{1}{2\pi}\int_0^\infty r\,dr\int_0^{2\pi} d\theta\; f(r)\exp[ir\rho\underbrace{(\cos\theta\cos\alpha+\sin\theta\sin\alpha)}_{=\cos(\theta-\alpha)} + im\theta]$$

and the inverse of this Fourier transform is

$$f(r)\,e^{im\theta} = \frac{1}{2\pi}\int_0^\infty \rho\,d\rho\int_0^{2\pi} d\alpha\; F(\rho,\alpha)\, e^{-ir\rho\cos(\theta-\alpha)} \quad (**)$$

Refer now to Gradsteyn + Ryzhik 8.411.1,

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta + iz\sin\theta}\, d\theta$$

and set $\beta \equiv \theta - \frac{\pi}{2}$ here

$$\Rightarrow \quad J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos\beta - in\beta - in\frac{\pi}{2}}\, d\beta$$

comparing this with our original transform $(*)$ gives

$$F(\rho,\alpha) = e^{im(\alpha+\frac{\pi}{2})}\int_0^\infty r\,f(r)\,J_m(\rho r)\,dr$$

and similarly $(**)$ gives

$$f(r)\,e^{im\theta} = \frac{1}{2\pi}\int_0^\infty \rho\,d\rho\int_0^{2\pi} d\alpha\left\{\frac{1}{2\pi}\int_0^\infty r'\,dr'\int_0^{2\pi} d\theta'\right.$$

$$\left.\left[f(r')\,e^{i(r'\rho\cos(\theta'-\alpha)+m\theta')}\, e^{-ir\rho\cos(\theta-\alpha)}\right]\right\}$$

$$= \frac{1}{2\pi}\int_0^\infty \rho\, d\rho \int_0^\infty r'\, dr' \int_0^{2\pi} d\alpha\, e^{-ir\rho\cos(\theta-\alpha)}$$

$$\times \left\{ \frac{1}{2\pi}\int_0^{2\pi} d\theta'\, e^{i(r'\rho\cos(\theta'-\alpha)+m\theta')} \right\}$$

and $\hookrightarrow \{\ \} = J_m(r'\rho)\, e^{im\alpha}$

and $\frac{1}{2\pi}\int_0^{2\pi} d\alpha\, e^{-ir\rho\cos(\theta-\alpha)+im\alpha}$

$= J_m(-r\rho)\, e^{im\theta}$ by this logic

So putting it all together we see that

$$f(r)e^{im\theta} = e^{im\theta}\int_0^\infty r'\, dr' \Big(\underbrace{\int_0^\infty \rho\, d\rho\, J_m(\rho r') J_m(\rho r)}_{\text{"}\frac{\delta(r-r')}{r'}\text{"}} \Big)$$

So we're done!

3.16 (b)

The Green's function with no conducting surfaces obeys

$$\nabla_x^2 G(\vec{x}, \vec{x}\,') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')$$

To satisfy the δ-functions in ρ, ϕ, take G in the form (an ansatz):

$$G(\vec{x}, \vec{x}\,') = \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \int_0^\infty k\, dk\, J_m(k\rho) J_m(k\rho') Z_{km}(z,z')$$

Plugging this in gives an equation in z:

$$-k^2 Z(z,z') + Z''(z,z') = -4\pi\delta(z-z')$$

and the appropriate homogeneous solutions are

$u_1(z) = e^{kz}$ = regular @ $z \to -\infty$, if $k \geq 0$

and $u_2(z) = e^{-kz}$ = regular at $z \to +\infty$ if $k \geq 0$

so $$Z(z,z') = -4\pi \frac{u_1(z_<)\,u_2(z_>)}{W(u_1,u_2)}$$

$$\Rightarrow \quad Z(z,z') = \frac{4\pi}{2k} e^{k(z_< - z_>)} = \frac{2\pi}{k} e^{-|z-z'|}$$

and plugging this back in to our ansatz gives another identity:

$$G(\vec{x},\vec{x}\,') = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') e^{-k|z-z'|}$$

$$= \frac{1}{|\vec{x}-\vec{x}\,'|}$$

3.16 (c) As a special case of this formula, set $\rho'=0 \Rightarrow$ all $J_m(k\rho') \longrightarrow 0$ unless $m=0$ and also set $z' \to 0$, which gives

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty e^{-k|z|} J_0(k\rho)\, dk \qquad (1)$$

Next let $z' \to 0$ and $\phi' \to 0$ to see another special case:

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}\,'|} = (\rho^2+\rho'^2-2\rho\rho'\cos\phi)^{-1/2}$$
$$= \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im\phi} J_m(k\rho) J_m(k\rho') e^{-k|z|}$$

and one more identity is found by replacing ρ in Eq. (1) above by $(\rho^2+\rho'^2-2\rho\rho'\cos\phi)^{1/2}$, namely

$$\int_0^\infty e^{-k|z|} J_0[k(\rho^2+\rho'^2-2\rho\rho'\cos\phi)^{1/2}]\, dk$$
$$= \int_0^\infty dk\, e^{-k|z|} \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho) J_m(k\rho')$$

and since this holds for ALL ρ and z, the integrands must be equal

$$\Rightarrow J_0\left[k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi}\right] = \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho) J_m(k\rho')$$

Next, prove that $e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho)$

Here I will use the generating function for Bessel functions:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{m=-\infty}^{\infty} t^m J_m(x)$$

In this identity let $t \to e^{i(\phi+\frac{\pi}{2})}$

$$\Rightarrow e^{\frac{x}{2}[2i\sin(\phi+\frac{\pi}{2})]} = e^{ix\cos\phi} = \sum_{m=-\infty}^{\infty} e^{im\phi} \underbrace{e^{i\frac{m\pi}{2}}}_{i^m} J_m(x)$$

or setting $x = k\rho \Rightarrow \boxed{e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho)}$

Alternative method to show orthonormality of continuous eigenfunctions more generally

[i.e. alternative solution of Jackson 3.16(a)]
consider p. 99–100 as an "appendix", not covered in class

Start from the Sturm–Liouville operator

$$\boxed{\hat{\mathcal{L}} \equiv \frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{\nu^2}{\rho^2}}$$

whose eigenfunctions are Bessel functions:

$$\hat{\mathcal{L}}\, J_\nu(k\rho) = -k^2 J_\nu(k\rho)$$

and $\hat{\mathcal{L}}\, J_\nu(k'\rho) = -k'^2 J_\nu(k'\rho)$

Then consider $(k^2 - k'^2)\int_0^\infty \rho\, d\rho\, J_\nu(k\rho)\, J_\nu(k'\rho) \equiv I$

so

$$I = \int_0^\infty \rho\, d\rho \left\{ [\hat{\mathcal{L}} J_\nu(k'\rho)]\, J_\nu(k\rho) - J_\nu(k'\rho)\,[\hat{\mathcal{L}} J_\nu(k\rho)] \right\}$$

$$= \int_0^\infty d\rho \left\{ \left[\frac{d}{d\rho}\left(\rho \frac{dJ_\nu(k'\rho)}{d\rho}\right)\right] J_\nu(k\rho) - J_\nu(k'\rho)\left[\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(k\rho)}{d\rho}\right)\right] \right.$$
$$\left. + \cancel{\rho\, J_\nu(k'\rho)\, J_\nu(k\rho)\left(\frac{\nu^2 - \nu^2}{\rho^2}\right)} \xrightarrow{} 0 \right\}$$

but observe that we can rewrite the nonvanishing part of the integrand as:

$$I = \int_0^\infty d\rho\, \frac{d}{d\rho}\left\{ \rho\frac{dJ_\nu(k'\rho)}{d\rho} J_\nu(k\rho) - \rho\, J_\nu(k'\rho)\frac{dJ_\nu(k\rho)}{d\rho} \right\}$$

or $\displaystyle I = \lim_{R\to\infty} \Big\{ \rho\, W[J_\nu(k\rho), J_\nu(k'\rho)] \Big\}\Big|_0^R$

Now, this vanishes at the lower limit, while the upper limit can be evaluated using the asymptotic expansion, namely

$$J_\nu(k\rho) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi k\rho}\right)^{1/2} \cos\Big(k\rho \underbrace{- \frac{\nu\pi}{2} - \frac{\pi}{4}}_{\equiv \beta}\Big)$$

So $$W[J_\nu(kR), J_\nu(k'R)] \xrightarrow[R\to\infty]{} \left(\frac{2}{\pi R}\right)(kk')^{-1/2}$$

$$\times \left\{ \cos(kR+\beta)(-k')\sin(k'R+\beta) + k\sin(kR+\beta)\cos(k'R+\beta) \right\}$$

now set $k'-k \equiv \Delta$, i.e. replace $k' \to k+\Delta$ everywhere expand in powers of small Δ:

$$\Rightarrow \lim_{R\to\infty} W = \frac{2}{\pi R}(-1)\underbrace{[k(k+\Delta)]^{-1/2}}_{\simeq \frac{1}{k}}\left\{ \underbrace{k\sin(\Delta R)} - \Delta\cos(kR+\beta)\sin((k+\Delta)R+\beta) \right\}$$

only term relevant to the δ-function!

oscillates infinitely fast and vanishes at $R\to\infty$

$$\Rightarrow \int_0^\infty \rho J_\nu(k\rho) J_\nu(k'\rho)\,d\rho = \lim_{R\to\infty} \frac{2}{\pi}\frac{\sin R\Delta}{k^2-k'^2}, \text{ and } k+k' \simeq 2k$$

$$= \frac{1}{2k}\lim_{R\to\infty}\frac{2}{\pi}\frac{\sin R\Delta}{\Delta} = \frac{\delta(k-k')}{k} \checkmark$$

and we're done!