Also, in some problems we will the values of $k$ at which $J_\nu(k\rho_0)$ vanishes, i.e. the roots of

$$J_\nu(x_{\nu n}) = 0$$

with $x_{\nu n} = n^{th}$ zero of $J_\nu(x)$

Tables of these $x_{\nu n}$ can be found, e.g., in the NIST Tables, the old Abramowitz + Stegun, but also Mathematica can provide them with the statement BesselJZeros[$\nu$, n]

First n zeroes of $J_\nu(x_{\nu n}) = 0$.

**Example application** Find the potential $\Phi$ inside a right-circular cylinder of radius $a$, height $L$, all of whose walls are held at $\Phi = 0$ (think of a grounded conductor), Except for the top cap at $z = L$, which is instead held at $V(\rho,\phi) = V_0 \frac{\rho}{a} w(\phi)$ and $w(\phi)$ is some known function of $\phi$.

**Solution** Start from a sum over the separable Bessel solutions, regular at $\rho = 0$, which vanish at $\rho = a$:

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left( \frac{A_{m,n} e^{im\phi} + A_{-m,n} e^{-im\phi}}{1+\delta_{m0}} \right)$$
$$\times J_m\left(x_{mn}\frac{\rho}{a}\right) \sinh\left(x_{mn}\frac{z}{a}\right)$$

i.e. write $k = \frac{x_{mn}}{a}$ in our earlier separable solution.

This has been designed to satisfy the PDE

$$\nabla^2 \Phi = 0$$

and all BCs in the problem except one, the one at $z=L$. To find the $A_{mn}$ such that this last BC is obeyed, use the orthogonality relation obeyed by

$$e^{im\phi} J_m\left(x_{mn}\frac{\rho}{a}\right)$$

For a derivation, see Jackson pp. 114-115, namely,

$$\int_0^a J_m\left(x_{mn'}\frac{\rho}{a}\right) J_m\left(x_{mn}\frac{\rho}{a}\right)\rho\, d\rho = \delta_{nn'}\frac{a^2}{2}\left[J_{m+1}(x_{mn})\right]^2$$

Another useful aside: from the last result, notice that the functions

$$u_n^{(m)}(\rho) = \frac{\sqrt{2}}{a J_{m+1}(x_{mn})} J_m\left(x_{mn}\frac{\rho}{a}\right)$$

form a complete, orthonormal set, with weight function $\rho$, over $0 \le \rho \le a$, for each $m = 0, 1, 2, \ldots$

i.e. $\int_0^a u_n^{(m)}(\rho)\, u_{n'}^{(m)}(\rho)\,\rho\, d\rho = \delta_{nn'}$

and $\sum_{n=1}^{\infty} u_n^{(m)}(\rho)\, u_n^{(m)}(\rho') = \frac{\delta(\rho-\rho')}{\rho}$

for $0 \le \rho \le a$ and $0 \le \rho' \le a$

To complete this solution, use "Fourier's Trick" again, using orthogonality in $\rho, \phi$, i.e.

$$A_{\pm m,n} = \frac{1}{2\pi} \frac{1}{\sinh\left(\frac{x_{mn} L}{a}\right)} \frac{2}{a^2 [J_{m+1}(x_{mn})]^2}$$

$$\times \int_0^{2\pi} e^{\mp i m\phi} d\phi \int_0^a \rho\, d\rho\, J_m\left(x_{mn}\frac{\rho}{a}\right) V(\rho,\phi)$$

$$= \frac{V_0}{\pi a^2 (J_{m+1}(x_{mn}))^2} \frac{\int_0^{2\pi} e^{\mp i m\phi} w(\phi)\, d\phi}{\sinh\left(x_{mn}\frac{L}{a}\right)}$$

$$\times \left(\int_0^a \frac{\rho^2}{a} J_m\left(x_{mn}\frac{\rho}{a}\right) d\rho\right)$$

## An aside on orthogonality + completeness

## ⟹ Read this on your own!

We have already commented that eigenfunctions $u_n(x)$ of the Sturm–Liouville operator are orthogonal in general, if the $u_n(x)$ all obey the same BCs on $\frac{u'}{u}$ at the boundaries.

<u>Careful statement</u>: Let $\{u_n(x)\}$ be the set of <u>all</u> eigenfunctions of

can prove that the $\lambda_n$ are <u>real</u>

$$\hat{\mathcal{L}}\, u_n(x) + \lambda_n w(x)\, u_n(x) = 0,$$

over $a \le x \le b$, where $w(x)$ is a nonnegative weight function, and where each $u_n(x)$ obeys the BCs $\frac{u_n'(a)}{u_n(a)} = \alpha$, $\frac{u_n'(b)}{u_n(b)} = \beta$

Then one can show (see, e.g. Arfken + Weber, Chap. 9):

ORTHOGONALITY $\int_a^b u_n^*(x)\, u_m(x)\, w(x)\, dx = 0$

if $\lambda_n \neq \lambda_m$. Moreover, if the eigenvalue spectrum is discrete (countable), then it is possible to normalize the eigenfunctions to unity, as

$$\int_a^b |u_n(x)|^2 w(x)\, dx = 1$$

and we assume hereafter that this has been done $\Rightarrow \int_a^b u_n^*(x)\, u_m(x)\, w(x)\, dx = \delta_{nm}$

**Completeness** One can also show that the $\{u_n(x)\}_{n=1,2,\ldots\infty}$ are complete. To derive the actual completeness relation, start from the following expansion of an arbitrary function $F(x)$, over $a \le x \le b$: $F(x) = \sum_n c_n u_n(x)$ (I)

The $c_n$ obtained from "Fourier's Trick" are found the usual way (multiply by $u_m^*(x)\, w(x)$ and integrate)

$$\Rightarrow c_m = \int_a^b u_m^*(x')\, F(x')\, w(x')\, dx'$$

use $x'$ as dummy integration variable.

Now plug this back into (I)

$$\Rightarrow F(x) = \int_a^b \Big(\sum_n u_n^*(x')\, u_n(x)\Big) F(x')\, w(x')\, dx'$$

$\Rightarrow$ For this to hold at all $x$, we MUST have:

$$\sum_n u_n^*(x')\, u_n(x) = \delta(x-x')/w(x')$$

# Back to cylindrical electrostatics problems

For some problems we will need the <u>other</u> family of Laplace equation solutions, which are oscillatory in $z$ rather than $\rho$.

i.e. where we chose $\frac{Z''(z)}{Z(z)} = k^2$, we <u>could</u> have chosen $-K^2$

$\Rightarrow$ This new family of solutions can be obtained by analytical continuation, which means setting $k \to iK$ with $K$ = real, $\geq 0$ now.

$\Rightarrow$ solutions for $Z(z)$ become $\sinh(iKz) = i \sin Kz$

$\cosh(iKz) = \cos Kz$

whereas the Bessel solutions become

$$J_\nu(iK\rho) \;,\; N_\nu(iK\rho),$$

and both of these <u>diverge</u> <u>exponentially</u> at $K\rho \to \infty$ analogous to $\sinh K\rho$, $\cosh K\rho$.

Because of this divergence, it is often convenient to work with a different $2^{nd}$ solution called $K_\nu(K\rho)$ instead of with $N_\nu(iK\rho)$, where $K_\nu(K\rho)$ is defined to be regular at $K\rho \to \infty$. (<u>Analogy</u>: this is similar to the fact that <u>one</u> linear combination of $\sinh Kz$, $\cosh Kz$, namely $e^{-Kz}$, is regular at $\infty$.)

Similarly, it is conventional to introduce a new solution $I_\nu(\kappa\rho)$ proportional to $J_\nu(i\kappa\rho)$ such that $I_\nu(\kappa\rho)$ is real for real $\kappa\rho>0$, and still regular at $\kappa\rho\to 0$

**Definitions of non-oscillatory Bessel functions:**

$$I_\nu(x) \equiv i^{-\nu} J_\nu(ix) = \text{real for real } x$$

$$K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1}\left[J_\nu(ix) + i N_\nu(ix)\right]$$

$$= \frac{\pi}{2} i^{\nu+1} \underbrace{H_\nu^{(1)}(ix)}$$

↖ Hankel function of the $1^{st}$ kind.

**Terminology and limiting forms**

$I_\nu(x)$ and $K_\nu(x)$ are often called

MODIFIED BESSEL FUNCTIONS

Small $x$ $\quad I_\nu(x) \xrightarrow[|x|\to 0]{} \frac{1}{\nu!}\left(\frac{x}{2}\right)^\nu$ = regular at 0 if $\nu \geq 0$

$$K_\nu(x) \xrightarrow[|x|\to 0]{} \begin{cases} -\ln x - \gamma + \ln 2 + \dots, & \nu = 0 \\ 2^{-\nu-1}(\nu-1)!\, x^{-\nu}, & \nu > 0 \end{cases}$$

= IRREGULAR at 0

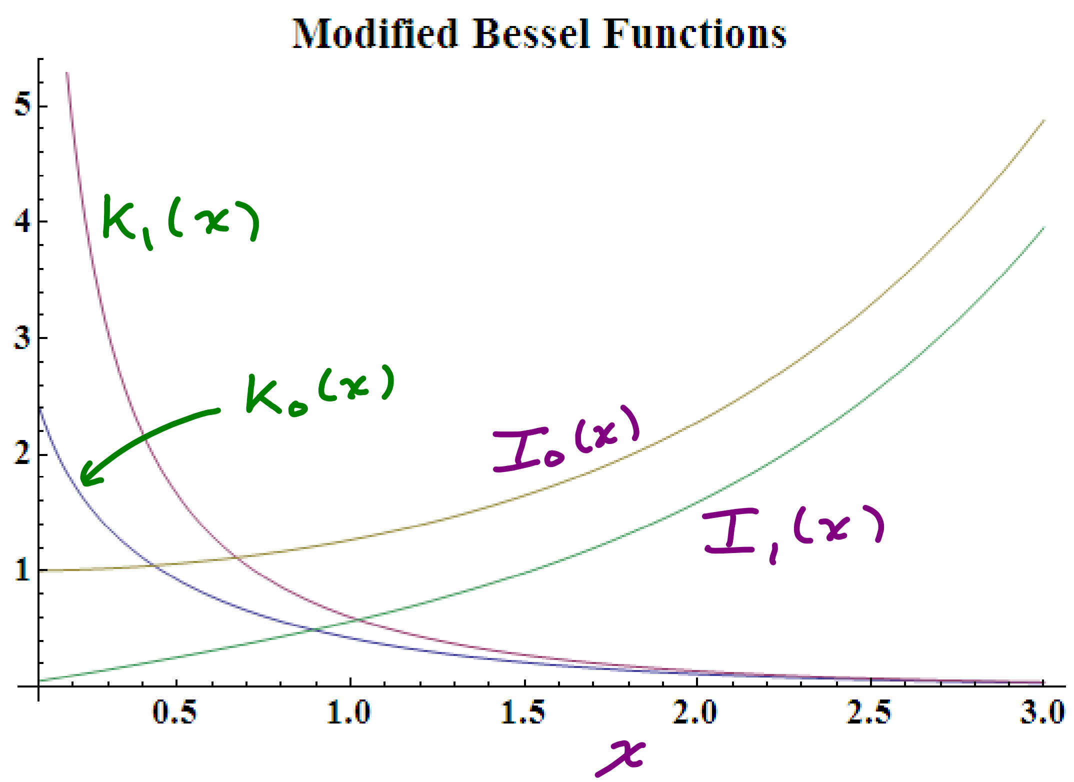
Large $|x|$ $\quad I_\nu(x) \xrightarrow[|x|\to\infty]{} \text{const} \times \frac{e^x}{\sqrt{x}}$, $\Rightarrow$ IRREGULAR at $\infty$

$$K_\nu(x) \xrightarrow[|x|\to\infty]{} \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \Rightarrow \text{REGULAR at } \infty$$

```
In[3]:= $DefaultFont = {"HelveticaBold", 16};

In[6]:= Plot[{BesselK[0, x], BesselK[1, x], BesselI[0, x], BesselI[1, x]}, {x, 0.1, 3},
         BaseStyle → {FontWeight → "Bold", FontSize → 18},
         PlotLabel → "Modified Bessel Functions"]
```

Out[6]=




**Example** This class of solutions is useful for problems, especially if the top ($z=L$) and bottom ($z=0$) caps of a cylinder are held at $\Phi = 0$. Then we can utilize a complete, orthonormal set of solutions in $z$ that vanish at $z=0, L$, with wavenumbers $k = \frac{n\pi}{L}$ : $\left(\frac{2}{L}\right)^{1/2} \sin\frac{n\pi z}{L}$,

$n = 1, 2, 3, \ldots$

The solutions *inside* this cylinder are then

$$\Phi(\rho, \phi, z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{im\phi} \left(\frac{2}{L}\right)^{1/2} \sin\frac{n\pi z}{L}\, I_m\!\left(\frac{n\pi\rho}{L}\right)$$

- $K_m$ cannot be present if $\rho = 0$ is within the volume $V$ since $K_m$ = irregular at $\rho = 0$
- Note that $I_{-m}(x) = I_m(x)$ for integer $m$.

# Green's Function in Cylindrical Coordinates

**Problem** Find $G(\vec{x}, \vec{x}')$ in free space, using oscillatory solutions in $\phi$ and $z$, i.e. $e^{ikz}$

In this infinite space volume $V$, we want to solve

$$\nabla_x^2 G(\vec{x}, \vec{x}') = -4\pi \frac{\delta(\rho-\rho')}{\rho} \delta(\phi-\phi') \delta(z-z')$$

**IDEA** Start from the completeness relations in $\phi$ and $z$, namely

$$\text{(i)} \quad \delta(z-z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(z-z')} dk$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \cos k(z-z') \, dk$$

$$\text{(ii)} \quad \delta(\phi-\phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

Then building the Green's function out of these forms, with a still-unknown $\rho, \rho'$-dependence,

$$\Rightarrow G(\vec{x}, \vec{x}') = \frac{1}{2\pi^2} \sum_m \int_0^{\infty} dk \cos k(z-z') \, e^{im(\phi-\phi')} g_m(\rho, \rho'; k)$$

Plug this into the PDE to find the equation obeyed by $g_m(\rho, \rho'; k)$:

i.e.

$$\nabla^2 G(\vec{x},\vec{x}') = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right) G(\vec{x},\vec{x}')$$

$$= \frac{1}{2\pi^2}\sum_m \int_0^\infty dk \cos k(z-z') e^{im(\phi-\phi')} \left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - k^2 - \frac{m^2}{\rho^2}\right\} g_m(\rho,\rho';k)$$

$$= -4\pi\frac{\delta(\rho-\rho')}{\rho}\left\{\frac{1}{2\pi^2}\sum_m e^{im(\phi-\phi')}\int_0^\infty \cos k(z-z')dk\right\}$$

Clearly, this last equation will be satisfied iff $g_m$ obeys:

$$\cancel{\frac{1}{\rho}}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} g_m(\rho,\rho';k) - \left(k^2+\frac{m^2}{\rho^2}\right) g_m(\rho,\rho';k) = -4\pi\frac{\delta(\rho-\rho')}{\cancel{\rho}}$$

and observe that this fits the Sturm-Liouville equation discussed previously, namely

$$\frac{d}{dx}p(x)\frac{d}{dx}g(x,x') + q(x)\,g(x,x') = -\frac{\delta(x-x')}{\beta(x')}$$

where we derived $g(x,x') = \dfrac{-u_1(x_<)u_2(x_>)}{\beta(x')p(x')W[u_1,u_2]\big|_{x'}}$

So to apply this here, we must let

$x \equiv \rho \qquad p \longrightarrow \rho\,, \quad \beta \longrightarrow \frac{1}{4\pi}$

And the appropriate solutions of the homogeneous differential equation are:

$u_1(\rho) = I_m(k\rho)$, obeys the B.C. at $\rho\to 0$

and $u_2(\rho) = K_m(k\rho)$, obeys the B.C. at $\rho\to\infty$

-and $\rho' W(I_m,K_m)\big|_{\rho'}$ can be evaluated at ANY $\rho'$ since it is constant.

I will evaluate it at $\rho \to 0$ where

$$I_m(k\rho) \xrightarrow[\rho\to 0]{} \frac{1}{|m|!}\left(\frac{k\rho}{2}\right)^{|m|}$$

$$K_m(k\rho) \xrightarrow[\rho\to 0]{} 2^{|m|-1}(|m|-1)!\,(k\rho)^{-|m|}, \quad \text{for } m\neq 0$$

and thus $\quad \rho' W(I_m, K_m)_{\rho'} = \lim\limits_{\rho'\to 0} \frac{\rho'}{2|m|}\left(-\frac{|m|}{\rho'} - \frac{|m|}{\rho'}\right) = -1$

and this result TURNS OUT to hold <u>also</u> for $m=0$. (Prove this on your own!)

Therefore, finally, $g_m(\rho,\rho';k) = 4\pi I_m(k\rho_<)K_m(k\rho_>)$ and another expression for the Green's function with NO BOUNDARIES PRESENT is:

$$G(\vec{x},\vec{x}') = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\,\cos k(z-z')\,e^{im(\phi-\phi')}\,I_m(k\rho_<)\,K_m(k\rho_>)$$

$$= \frac{1}{|\vec{x}-\vec{x}'|}$$