Application Consider a uniform thin ring of total charge Q, of radius R, inside a <u>grounded</u>, <u>conducting</u> sphere of radius b. Find $\Phi(\vec{r})$ inside.

We can write this in terms of the Dirichlet GF that was just found, as

$$\Phi(\vec{x}) = \int_V \frac{G(\vec{x},\vec{x}\,')\rho(\vec{x}\,')}{4\pi\epsilon_0}\,d^3x'$$

$$-\frac{1}{4\pi}\oint_S \frac{\partial G(\vec{x},\vec{x}\,')}{\partial n'}\,\Phi(\vec{x}\,')\,da' \longrightarrow 0$$

since grounded

Observe that we could use $G(\vec{x},\vec{x}\,')$ from our preceding derivation, and set $a \to 0$, or else we could use G from our method of images derivation in Chapter 2. Their equality implies an identity,

$$G(\vec{x},\vec{x}\,') = \sum_{l,m}\left(\frac{-4\pi}{2l+1}\right)\frac{r_<^l}{b^{l+1}}\left[\left(\frac{r_>}{b}\right)^l - \left(\frac{b}{r_>}\right)^{l+1}\right] Y^*_{lm}(\hat{x}')\,Y_{lm}(\hat{x})$$

$$= \frac{1}{|\vec{x}-\vec{x}\,'|} - \frac{b}{r'\left|\vec{x}-\frac{b^2}{r'^2}\vec{x}\,'\right|}$$

The charged ring has charge density

$$\rho(\vec{x}\,') = \frac{Q}{2\pi R^2}\,\delta(r'-R)\,\delta(\cos\theta)$$

For the solution, let's write separate formulas for $r<R$ and for $r>R$

e.g., for $r<R$, $\Rightarrow r_< = r$, $r_> = R = r'$ (owing to $\delta(r'-R)$)

$$\Phi(r,\theta,\phi) = \sum_{lm} \frac{Q}{2\pi R^2}\frac{1}{4\pi\epsilon_0}\left(-\frac{4\pi}{2l+1}\right)\frac{r^l}{b^{l+1}} Y_{lm}(\theta,\phi)$$

$$\times \int_0^{2\pi} d\phi' \int_{-1}^{1} d(\cos\theta')\, Y^*_{lm}(\theta',\phi')\,\delta(\cos\theta')$$

$$\times \int_0^b r'^2 dr'\, \delta(r'-R)\left[\left(\frac{r'}{b}\right)^l - \left(\frac{b}{r'}\right)^{l+1}\right]$$

$$= 2\pi\,\delta_{m,0}\, Y_{lm}\left(\frac{\pi}{2},0\right) = \delta_{m,0}\left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(0)$$

$$R^2\left[\left(\frac{R}{b}\right)^l - \left(\frac{b}{R}\right)^{l+1}\right]$$

And the final answer, for $r<R$, is:

$$\Phi(r,\theta,\phi) = \frac{Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty} r^l\left(\frac{1}{R^{l+1}} - \frac{R^l}{b^{2l+1}}\right) P_l(0)\, P_l(\cos\theta)$$

and one could simplify this further if it is desired, using, e.g.

$$P_{2n+1}(0) = 0, \quad P_{2n}(0) = \frac{(-1)^n}{2^n n!}(2n-1)!!$$

Laplace's equation in cylindrical coordinates

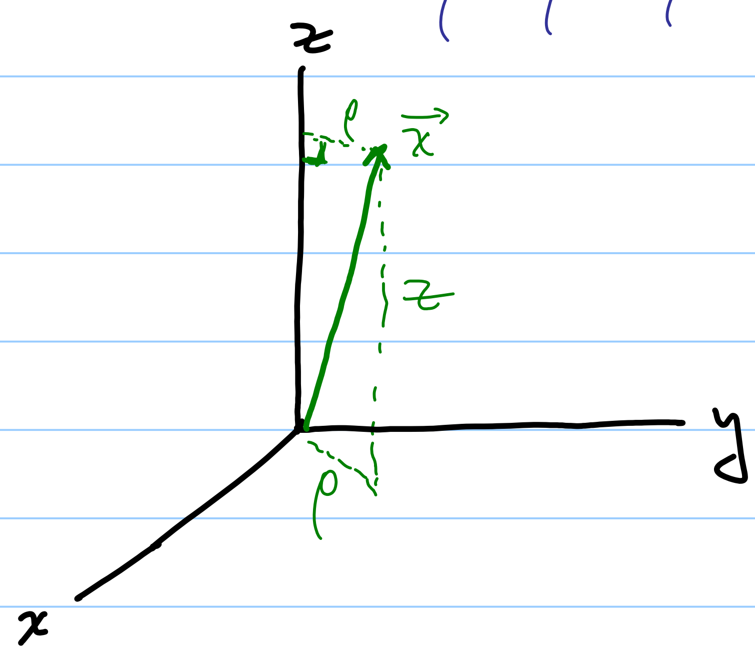
Now $(\rho, \phi, z) \equiv (q_1, q_2, q_3)$

and $h_\rho = 1, \; h_\phi = \rho, \; h_z = 1$

So recalling that

$$\nabla^2 \Phi(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\right) + \text{cyclic permutations} \ldots$$

$$\Rightarrow \nabla^2 \Phi(\rho, \phi, z) = 0 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$



Again solve this PDE by separation of variables. That is, we initially look for separable solutions, of the form

$$\Phi(\rho, \phi, z) = R(\rho)\, Q(\phi)\, Z(z)$$

which gives the separated equations

$$\frac{d^2 Q(\phi)}{d\phi^2} + \nu^2 Q(\phi) = 0 \Rightarrow e^{\pm i\nu\phi} \text{ are the 2 solns when } \nu \neq 0$$

and $$e^{i\nu\phi}\Big|_{\nu\to 0} \longrightarrow A + B\phi$$

Next,

$$\frac{d^2 Z(z)}{dz^2} - k^2 Z(z) = 0$$

$$\Rightarrow Z(z) = e^{\pm kz}$$

or $\sinh kz$, $\cosh kz$...

and finally in ρ,

$$\Rightarrow \frac{d^2 R(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dR(\rho)}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R(\rho) = 0$$

or dividing this last equation by k^2 and calling $x = k\rho$, gives

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$$

This can be solved by power series (Fröbenius method), be sure you know how to do this! The solution is

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j \, (x/2)^{2j}}{j!\,\Gamma(\nu+j+1)}$$

Observation, since the differential equation depends only on ν^2, we know immediately that

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(-\nu+j+1)}\left(\frac{x}{2}\right)^{2j}$$

is ALSO a solution to the diff. equation!

Theory of the Neumann Bessel Function.

These two functions $J_\nu(x)$, $J_{-\nu}(x)$ could serve as our two linearly-independent solutions, provided they are in fact independent.

We can assess their independence by looking at their asymptotic forms (see Arfken + Weber, or Morse + Feshbach):

$$J_\nu(x) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

and

$$J_{-\nu}(x) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

This is proportional to $J_\nu(x)$ when ν = integer

i.e. this shows that $J_{-m}(x) = (-1)^m J_m(x)$ when m = integer.

However, if we could define a solution $N_\nu(x)$ that behaves asymptotically like

$$N_\nu(x) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right),$$

then it would be linearly-independent of $J_\nu(x)$ at all values of ν, x! Actually, we can find such a solution by starting from the identity,

$$\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\cos\pi\nu - \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$
$$= \cos\left(x + \frac{\nu\pi}{2} - \pi/4\right)$$

i.e. $J_\nu(x)\cos\pi\nu - N_\nu(x)\sin\pi\nu = J_{-\nu}(x)$

and this in turn suggests that we should <u>define</u>

$$\boxed{N_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}}$$

which is called the "Neumann function", or Bessel function of the 2^{nd} kind, sometimes written instead as $Y_\nu(x)$

Through this construction $J_\nu(x)$ and $N_\nu(x)$ are now linearly-independent for <u>ALL</u> values of ν, including $\nu = m =$ integer!

For electrostatics problems, these Bessel functions are most useful when $\Phi \to 0$ boundary conditions are imposed on a surface in ρ, because they are <u>oscillatory</u> in ρ for k=real, i.e. when $k^2 > 0$. It is also <u>crucial</u> to recognize that for small distances,

$$J_\nu(k\rho) \xrightarrow[k\rho\to 0]{} \text{regular} \quad \text{for} \quad \nu \geq 0$$

$$N_\nu(k\rho) \xrightarrow[k\rho\to 0]{} \text{IRREGULAR at all } \nu \geq 0$$

See, e.g., Jackson Eqs. 3.89, 3.90