Read Jackson's Proof in Sec. 3.6 and/or test it, e.g using Mathematica.

$$\Rightarrow \boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\hat{x}') Y_{lm}(\hat{x})}$$

memorize this!

Next: Green's function construction

Again, for the Dirichlet problem, we pick $G \to 0$ on boundary surfaces, and then

$$\Phi(\vec{x}) = \int_V G(\vec{x},\vec{x}') \frac{\rho(\vec{x}')}{4\pi\epsilon_0} d^3x' - \frac{1}{4\pi}\oint_S \frac{\partial G}{\partial n'}(\vec{x},\vec{x}')\,\Phi(\vec{x}')\,da'$$

when $\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$

For many problems with spherical boundary surfaces, it is convenient to expand G into spherical harmonics, starting from:

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')$$

$$= \frac{1}{r^2}\delta(r-r') \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\hat{x}')\,Y_{lm}(\hat{x})$$

Strategies to construct multidimensional G:

(I) Eigenfunction in ALL coordinates

e.g., to solve

$$\nabla^2 G(\vec{x}, \vec{x}\,') = -4\pi\,\delta(\vec{x} - \vec{x}\,'), \qquad (1)$$

subject to some BCs, such as homogeneous Dirichlet BCs $\Rightarrow$ $G(\vec{x}, \vec{x}\,') = 0$ for $\vec{x}$ on S,

we can consider a companion problem, the homogeneous eigenvalue problem for eigenfunctions $\psi_n(\vec{x})$ obeying the SAME BCs as $G(\vec{x}, \vec{x}\,')$, i.e. consider

$$-\nabla^2 \psi_n(\vec{x}) = \lambda_n \psi_n(\vec{x}) \qquad (2)$$

Now, one can show that these $\psi_n(\vec{x})$ are eigenfunctions of a Hermitian operator and they can be chosen to be orthonormal and complete, i.e.

$$\int_V \psi_n^*(\vec{x})\,\psi_{n'}(\vec{x})\,d^3x = \delta_{nn'} \qquad (3)$$

and

$$\sum_n \psi_n^*(\vec{x}\,')\,\psi_n(\vec{x}) = \delta(\vec{x} - \vec{x}\,') \qquad (4)$$

So expanding G into this complete set looks like

$$G(\vec{x}, \vec{x}\,') = \sum_n c_n(\vec{x}\,')\,\psi_n(\vec{x}) \qquad (5)$$

and plug this into (1) to find the expansion coefficients $c_n(\vec{x}\,')$:

i.e. $-\nabla^2 \sum_n C_n(\vec{x}\,') \Psi_n(\vec{x}) = +4\pi\,\delta(\vec{x}-\vec{x}\,')$

$\hookrightarrow = \sum_n C_n(\vec{x}\,') \lambda_n \Psi_n(\vec{x}) = 4\pi \sum_n \Psi_n^*(\vec{x}\,') \Psi_n(\vec{x})$

$$\Rightarrow \quad \lambda_n C_n(\vec{x}\,') = 4\pi \Psi_n^*(\vec{x}\,')$$

Hence finally

$$\boxed{G(\vec{x},\vec{x}\,') = 4\pi \sum_n \frac{\Psi_n^*(\vec{x}\,')\Psi_n(\vec{x})}{\lambda_n}} \quad (6)$$

The case of a continuous eigenvalue spectrum

$\Rightarrow$ if λ_n are continuous, we must generalize the $\sum_n$ to an integral, e.g.

$$-\nabla^2 \Psi_{\vec{k}}(\vec{x}) = k^2 \Psi_{\vec{k}}(\vec{x})$$

If V = infinite free space, the solutions are

$$\Psi_{\vec{k}}(\vec{x}) = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{x}}$$

And when the eigenvalue spectrum is continuous, must generalize the normalization condition to

$$\delta(\vec{k}-\vec{k}\,') = \int_V \Psi_{\vec{k}\,'}^*(\vec{x}) \Psi_{\vec{k}}(\vec{x})\, d^3x$$

and the above derivation for G generalizes to

$$G(\vec{x},\vec{x}\,') = 4\pi \int d^3k \, \frac{\Psi_{\vec{k}}^*(\vec{x}\,')\Psi_{\vec{k}}(\vec{x})}{k^2}$$

It will be instructive, especially later in the course, to evaluate this integral. We can use a "trick", namely to choose the z-axis in k-space to lie along the direction of $\vec{x}-\vec{x}'$. We may do this <u>because</u> we're integrating over all k-space.

$$G(\vec{x},\vec{x}') = \frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{k^2 dk}{k^2}\int_{-1}^{1} d(\cos\theta_k)\int_0^{2\pi} d\phi_k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

where θ_k = angle between $\vec{k}$ and $(\vec{x}-\vec{x}')$

$$= \frac{1}{2\pi^2}(2\pi)\int_0^\infty dk\int_{-1}^{1} d(\cos\theta_k)\, e^{ikR\cos\theta_k}$$

where $R = |\vec{x}-\vec{x}'|$

$$\Rightarrow G(\vec{x},\vec{x}') = \frac{1}{\pi}\int_0^\infty dk\,\frac{e^{ikR}-e^{-ikR}}{ikR}$$

$$= \frac{2}{\pi}\int_0^\infty dk\,\frac{\sin kR}{kR} = \frac{2}{\pi R}\overbrace{\int_0^\infty du\,\frac{\sin u}{u}}^{"\frac{\pi}{2}"}$$

So finally $\boxed{G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}}$ = an alternative way to find this!

Method II Eigenfunction expansion in all coordinates EXCEPT one.

e.g. for a spherical problem, write an <u>ansatz</u> that is motivated by the spherical harmonic completeness relation, namely

$$G(\vec{x}, \vec{x}\,') = \sum_{l,m} g_l(r,r')\, Y^*_{lm}(\vartheta',\phi')\, Y_{lm}(\vartheta,\phi)$$

Now use ∇^2 in the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\vec{L}^2(\theta,\phi)}{r^2}, \quad \text{where}$$

$$\vec{L}^2(\theta,\phi) = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

and recall that $\vec{L}^2 Y_{lm}(\vartheta,\phi) = l(l+1)\, Y_{lm}(\vartheta,\phi)$

Thus we can find the equation $g_l(r,r')$ obeys,

$$\nabla^2 G(\vec{x},\vec{x}\,') = \sum_{lm} Y^*_{lm}(\hat{x}')\, Y_{lm}(\hat{x})\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] g_l(r,r')$$

$$= -4\pi\, \frac{\delta(r-r')}{r^2}\,\delta(\cos\vartheta - \cos\vartheta')\,\delta(\phi-\phi')$$

and this equation will be satisfied if

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] g_l(r,r') = -4\pi\,\frac{\delta(r-r')}{r^2}$$

$\Rightarrow$ We must solve this equation, subject to the appropriate BCs, over $a \le r \le b$, and then we will have $G(\vec{x},\vec{x}\,')$

Solution of the radial equation = 2^{nd}-order linear diff. eqn.

Think of $g_l(r,r')$ = function of r, for fixed r'.

$\Rightarrow$ At any $r \neq r'$, the most general solution in r must be a linear combination of 2 independent solutions of the HOMOGENEOUS DIFF. EQN.

i.e. $$g_\ell(r,r') = \begin{cases} A r^\ell + B r^{-\ell-1}, & r<r' \\ C r^\ell + D r^{-\ell-1}, & r>r' \end{cases}$$

where A, B, C, D all depend on ℓ and r' but not r.

<u>Dirichlet case</u> Must satisfy the following 4 equations

① $g_\ell(a,r') = 0 = A a^\ell + B a^{-\ell-1}$

② $g_\ell(b,r') = 0 = C b^\ell + D b^{-\ell-1}$

③ Continuity at $r=r'$

$\Rightarrow \quad A r'^\ell + B r'^{-\ell-1} = C r'^\ell + D r'^{-\ell-1}$

④ Derivative <u>discontinuity</u> at $r=r'$, caused by $\delta(r-r')$

To derive the precise form of this deriv. discontinuity integrate the diff. eqn. for $g_\ell(r,r')$ over r from $r=r'-\epsilon$ to $r=r'+\epsilon$, then take $\lim_{\epsilon\to 0^+}$

i.e.

$$\int_{r=r'-\epsilon}^{r=r'+\epsilon} r^2\,dr \left\{ \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial g(r,r')}{\partial r} - \cancel{\frac{\ell(\ell+1)}{r^2} g_\ell(r,r')} \right\} = -4\pi \int_{r'-\epsilon}^{r'+\epsilon} r^2 dr\, \frac{\delta(r-r')}{r^2}$$

0 since continuous

$$\Rightarrow r^2 \frac{\partial g_\ell(r,r')}{\partial r}\Bigg|_{r=r'-\epsilon}^{r=r'+\epsilon} = -4\pi$$

or $$\boxed{\frac{\partial g_\ell(r,r')}{\partial r}\Bigg|_{r=r'+\epsilon} - \frac{\partial g_\ell(r,r')}{\partial r}\Bigg|_{r=r'-\epsilon} = \frac{-4\pi}{r'^2}}$$

This gives the 4^{th} equation needed to find the 4 constants A, B, C, D, namely:

$$l C r'^{\,l-1} + (-l-1) D r'^{\,-l-2} - [\, l A r'^{\,l-1} - (l+1) B r'^{\,-l-2} = -\frac{4\pi}{r'^2}$$

=> Solution of these 4 equations and 4 unknowns gives the solution in the form written as Jackson Eq. 3.125, namely for a Dirichlet BC problem with spherical conducting boundaries at $r=a$ and $r=b$, the GF at $a \le r \le b$ is

$$G(\vec{x},\vec{x}\,') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\hat{x}')Y_{lm}(\hat{x})\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}$$

and it is readily seen that the free space GF is recovered upon setting $a \to 0$, $b \to \infty$.

1-dimensional Green Functions – general treatment for the Sturm–Liouville operator.

(see, e.g. Arfken + Weber, Chap. 9)

This 2nd-order differential operator looks in general like

$$\hat{\mathcal{L}}_x = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

Let's find the Dirichlet GF obeying

$$\hat{\mathcal{L}}_x\, g(x,x') = -\frac{\delta(x-x')}{\beta(x')}$$

over $a \le x \le b$, subject to $g(a,x') = 0 = g(b,x')$

Again, there are two linearly-independent solutions to the homogeneous differential equation, $\boxed{\hat{\mathcal{L}}_x u(x) = 0}$ (*)

Call $u_1(x)$ = a solution of (*) obeying the LEFT boundary condition, $u_1(a) = 0$

and $u_2(x)$ = a solution obeying $u_2(b) = 0$

Then the GF must have the form

$$g(x,x') = \begin{cases} C_1 u_1(x), & a \le x \le x' \\ C_2 u_2(x), & x' \le x \le b \end{cases}$$

where c_1, c_2 depend on x'.

Next impose continuity at $x = x'$

$$\Rightarrow \boxed{c_1 u_1(x') = c_2 u_2(x')} \quad (**)$$

Next find the derivative discontinuity at $x = x'$, by integrating:

$$\lim_{\epsilon \to 0^+} \int_{x=x'-\epsilon}^{x=x'+\epsilon} \hat{\mathcal{L}}_x g(x,x')\,dx = \int_{x'-\epsilon}^{x'+\epsilon} \left(-\frac{\delta(x-x')}{\beta(x')}\right) dx$$

$$\Rightarrow \boxed{p(x') \frac{dg(x,x')}{dx}\Bigg|_{x=x'+\epsilon} - p(x') \frac{dg(x,x')}{dx}\Bigg|_{x=x'-\epsilon} = \frac{-1}{\beta(x')}} \quad (***)$$

Solve this along with (**) to get $c_1(x')$, $c_2(x')$

$$\Rightarrow \text{Find } c_1(x') = \frac{-u_2(x')}{\beta(x')p(x')[u_1(x')u_2'(x') - u_1'(x')u_2(x')]}$$

$$c_2(x') = \frac{-u_1(x')}{\beta(x')p(x')[u_1(x')u_2'(x') - u_1'(x')u_2(x')]}$$

Notational aside: an important quantity in the theory of differential equations is the Wronskian: $W(u_1, u_2) \equiv u_1'(x) u_2(x) - u_1(x) u_2'(x)$

For our Sturm–Liouville operator one can prove that, for any 2 solutions $u_1(x)$, $u_2(x)$ of the homogeneous diff. equation, $p\,W(u_1,u_2) = \text{constant}$

Thus in concise form, we have

$$\boxed{g(x,x') = \frac{-1}{\beta(x')\,p\,W(u_1,u_2)} u_1(x_<)\, u_2(x_>) \quad \text{is the GF for} \quad \frac{d}{dx} p(x) \frac{dg}{dx} + q(x)\, g = -\frac{\delta(x-x')}{\beta(x')}}$$

extremely useful!

Application to the radial GF for Laplace's equation in spherical coordinates

The diff. equation is:

$$\frac{d}{dr} r^2 \frac{d}{dr} g_\ell(r,r') - \ell(\ell+1)\, g_\ell(r,r') = -4\pi\,\delta(r-r')$$

which matches our Sturm–Liouville eqn if we identify

$$p(r) = r^2$$
$$q = -\ell(\ell+1)$$
$$\beta(r) = \frac{1}{4\pi}$$

$a \le r \le b$

$$\Rightarrow \quad g_\ell(r,r') = \frac{-4\pi\; u_1(r_<)\, u_2(r_>)}{r'^2\, W[u(r'), u_2(r')]}$$

where $u_i(r)$ obey the homogeneous diff eqn,

$$\frac{d}{dr} r^2 \frac{du_i(r)}{dr} - \ell(\ell+1) u_i(r) = 0$$

and $u_1(a) = 0$, $u_2(b) = 0$

Aside: if you forget what these solutions are, you can always TRY a power law, e.g. try $u(r) = r^\gamma$ and see whether it works, e.g. here

$$\frac{d}{dr} r^2 (\gamma r^{\gamma-1}) - \ell(\ell+1) r^\gamma = 0 = [\gamma(\gamma+1) - \ell(\ell+1)] r^\gamma = 0$$

$\Rightarrow$ this solution works provided $\gamma(\gamma+1) - \ell(\ell+1) = 0$, i.e. $\gamma = \ell$ or $-\ell-1$

$\Rightarrow$ the general solution is $u(r) = A r^\ell + B r^{-\ell-1}$

So the linear combinations that obey homogeneous Dirichlet BCs are

$$u_1(r) = \left(\frac{r}{a}\right)^\ell - \left(\frac{a}{r}\right)^{\ell+1}, \quad u_2(r) = \left(\frac{r}{b}\right)^\ell - \left(\frac{b}{r}\right)^{\ell+1}$$

whose p×Wronskian works out to be

$$r'^2 W(u_1, u_2) = (2\ell+1)\left(\frac{b^{\ell+1}}{a^\ell} - \frac{a^{\ell+1}}{b^\ell}\right)$$

giving the radial GF,

$$g_\ell(r, r') = \frac{-4\pi}{(2\ell+1)\left(\frac{b^{\ell+1}}{a^\ell} - \frac{a^{\ell+1}}{b^\ell}\right)} \left[\left(\frac{r_<}{a}\right)^\ell - \left(\frac{a}{r_<}\right)^{\ell+1}\right] \left[\left(\frac{r_>}{b}\right)^\ell - \left(\frac{b}{r_>}\right)^{\ell+1}\right]$$

and $G(\vec{x}, \vec{x}') = \sum_{\ell m} g_\ell(r, r') Y^*_{\ell m}(\hat{x}') Y_{\ell m}(\hat{x})$