Chapter 3 Boundary value problems in electrostatics II

IMPORTANT: it is crucial whenever possible to solve a PDE in a coordinate system adapted to the shapes of the boundary surfaces.

Laplace's equation in spherical coordinates

$$\nabla^2 \Phi(r,\theta,\phi) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$
$$= 0$$

separable spherical solutions are found by setting $\Phi(r,\theta,\phi) = \frac{u(r)}{r} P(\theta) Q(\phi)$

$$\Rightarrow r^2\sin^2\theta\left\{\frac{u''(r)}{u(r)} + \frac{1}{r^2\sin\theta}\frac{(\sin\theta\, P'(\theta))'}{P(\theta)}\right\} + \frac{Q''(\phi)}{Q(\phi)} = 0 \qquad (*)$$

$\Rightarrow \frac{Q''(\phi)}{Q(\phi)} = -m^2 = \text{constant}$ since $(*)$ must hold for <u>all</u> r, θ, ϕ

$$\Rightarrow Q(\phi) = e^{\pm i m\phi}$$

and if the full angular range $0 \le \phi < 2\pi$ is allowed, $Q(\phi)$ must be single-valued,

$$\Rightarrow m = \text{any integer}$$

Returning to the r,θ equations,

$$\Rightarrow \underbrace{r^2\frac{u''(r)}{u(r)}}_{\text{all } r\text{-dependence}} + \underbrace{\frac{1}{\sin\theta}\frac{(\sin\theta\, P'(\theta))'}{P(\theta)} - \frac{m^2}{\sin^2\theta}}_{\text{all } \theta\text{-dependence}} = 0$$

⇒ The two circled terms must equal constants that sum to 0, call them $\pm \ell(\ell+1)$

$$\Rightarrow \quad -\frac{1}{\sin\theta}\left(\sin\theta\, P'(\theta)\right)' + \frac{m^2}{\sin^2\theta} P(\theta) = \ell(\ell+1) P(\theta) \qquad (**)$$

and then $\quad u''(r) - \frac{\ell(\ell+1)}{r^2} u(r) = 0 \qquad (***)$

For any given (ℓ, m) there are 2 linearly independent solutions called <u>Associated Legendre Functions</u>

$$P_{\ell m}(\cos\theta), \quad Q_{\ell m}(\cos\theta)$$

LegendreP[ℓ, m, Cos[θ]], in Mathematica
or Q

- If $m =$ integer, only the solutions $P_{\ell m}(\cos\theta)$ are regular at $\theta = 0$, and these will not be well-behaved at $\theta = \pi$ unless $\ell =$ integer $\geq |m|$
i.e. $\ell = |m|, |m|+1, \dots$ etc.

- If the electrostatic potential is known to be azimuthally symmetric about the z-axis, (meaning independent of ϕ), then only $m = 0$ can be present. The associated Legendre functions for $m=0$ are simpler + are called Legendre polynomials, $P_\ell(x)$ or LegendreP[ℓ, x] ($x = \cos\theta$ here)

Legendre Polynomial properties:

- Differential equation:

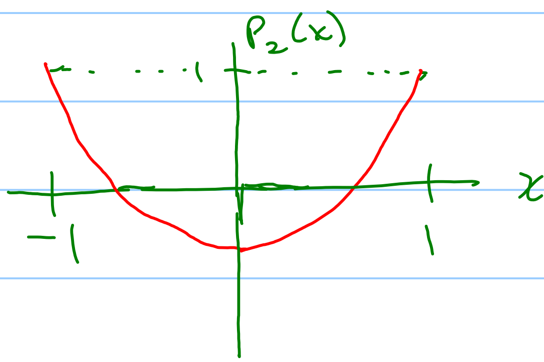
$$\frac{d}{dx}(1-x^2)\frac{dP_\ell}{dx} + \ell(\ell+1)P_\ell(x) = 0$$

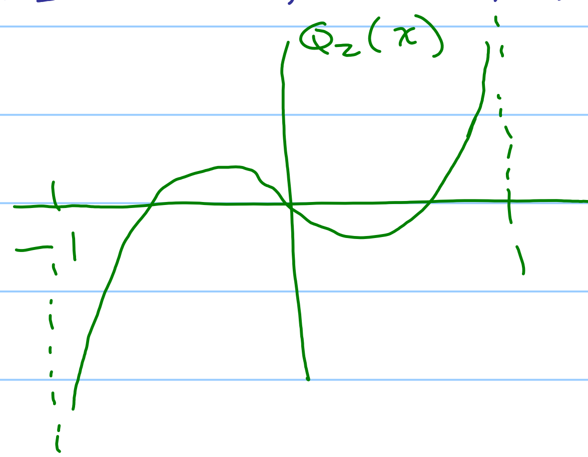
to solve it, this is simple via power series, or it can be coaxed out of Mathematica, e.g.

DSolve[(1-x^2)P''[x] - 2x P'[x] + l(l+1)P[x] == 0, P[x], x]

returns

P[x] → C[1] LegendreP[l, x] + C[2] LegendreQ[l, x]





Things you should know about Legendre polynomials:

- First few:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$
$$\vdots$$

- Highest power of x in $P_\ell(x)$ is x^ℓ
- Parity under $x \rightarrow -x$ is $(-1)^\ell$, i.e. $P_\ell(-x) = (-1)^\ell P_\ell(x)$

- Rodriguez' formula $$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

- Orthogonality $$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\,dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

(can prove this using Rodriguez' formula, then integrate by parts)

- Completeness: over $-1 \le x \le 1$, can expand "any" function, as

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)$$

where $$A_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} P_\ell(x) f(x)\,dx$$

- Recurrence relations, e.g.

(a) $P'_{\ell+1} - P'_{\ell-1} = (2\ell+1)P_\ell$

(b) $(\ell+1)P_{\ell+1} - (2\ell+1)xP_\ell + \ell P_{\ell-1} = 0$

(c) $P'_{\ell+1} - xP'_\ell - (\ell+1)P_\ell = 0$

(d) $(x^2-1)P'_\ell - x\ell P_\ell + \ell P_{\ell-1} = 0$

for others, see Jackson Eq. 3.29

use such relations to evaluate integrals like

$$\int_{-1}^{1} x^k P_\ell(x) P_{\ell'}(x)\,dx,\ \text{etc.}$$

Finally, the radial equation is solved as:

$$r^2 \frac{u''(r)}{u(r)} = \ell(\ell+1) \Rightarrow u(r) = r^{\ell+1} \text{ or } r^{-\ell}$$

$$\Rightarrow \frac{u(r)}{r} = A_\ell r^\ell + B_\ell r^{-\ell-1}$$ is the general solution.

Azimuthally-symmetric boundary value problems

For problems having no ϕ-dependence, the general solution to $\nabla^2 \Phi = 0$ takes the form:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

where the constants A_ℓ, B_ℓ are found by imposing boundary conditions.

EXAMPLE Suppose a sphere of radius a has a specified surface potential equal to

$$V(\theta) = V_0 \cos^2\theta$$

BCs: Require $\Phi \xrightarrow[r\to 0]{}$ finite
and $\Phi \xrightarrow[r\to\infty]{} 0$ ← can demand this only for finite charge distributions

$\Rightarrow B_\ell \xrightarrow[\text{inside}]{} 0$, $A_\ell \xrightarrow[\text{outside}]{} 0$

i.e. $\Phi_{inside}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

$$\Phi_{outside}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Since there is no dipole layer, Φ = continuous,

$\Rightarrow$ $$\Phi_{inside}(a,\theta) = V(\theta) = \Phi_{outside}(a,\theta)$$

SOLUTION use these and orthonormality of $P_l(\cos\theta)$ to find A_l, B_l:

$$\text{e.g.} \sum_l A_l a^l P_l(\cos\theta) = V_0 \cos^2\theta$$

mutiply by $P_{l'}(\cos\theta) \equiv P_{l'}(x)$ and integrate:

$$\int_{-1}^{1} dx\, P_{l'}(x) \sum_l A_l a^l P_l(x) = V_0 \int_{-1}^{1} P_{l'}(x)\, x^2 dx$$

or $$\frac{2}{2l+1} A_l a^l = V_0 \int_{-1}^{1} P_l(x) x^2 dx$$

But notice that $x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)$

$$\Rightarrow \int_{-1}^{1} P_l(x)\, x^2 dx = \frac{2}{5}\left(\frac{2}{3}\right)\delta_{l,2} + \frac{1}{3}(2)\,\delta_{l,0}$$

$$\Rightarrow A_2 = \frac{2}{3}\frac{V_0}{a^2}, \quad A_0 = \frac{1}{3}V_0$$

and similarly $\frac{2}{5} B_2 a^{-3} = V_0 \frac{4}{15}$ or $B_2 = \frac{2}{3} a^3 V_0$

and $B_0 = \frac{1}{3} a V_0$

So finally $$\Phi_{inside}(r,\theta) = \frac{1}{3}V_0 + \frac{2}{3}V_0\left(\frac{r}{a}\right)^2 P_2(\cos\theta)$$

$$\Phi_{outside}(r,\theta) = \frac{1}{3}V_0 \frac{a}{r} + \frac{2V_0}{3}\frac{a^3}{r^3} P_2(\cos\theta)$$

EXAMPLE Construct the potential everywhere in space for an azimuthally-symmetric problem, starting from the solution known only on the symmetry axis.

e.g. since we know that in a charge-free region of space, the general ϕ-independent potential is

$$\Phi(r,\theta) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

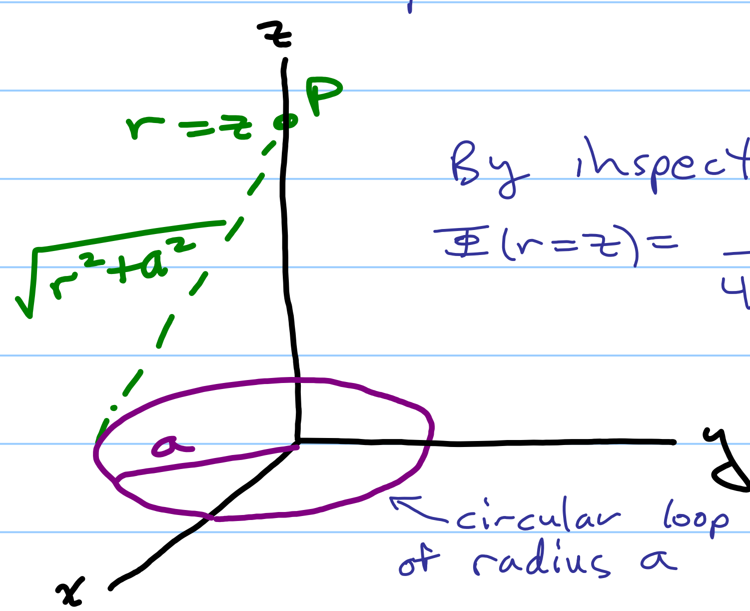
we can evaluate this on axis where $\theta = 0$, $P_\ell(1) = 1$,

$$\Rightarrow \quad \Phi(r,0) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)$$

We can turn this logic around, as follows:

Suppose we know $\Phi(r=z)$ along the z-axis for an azimuthally-symm. problem. If we expand $\Phi(r=z)$ in a series, we can deduce $\Phi(r,\theta)$ everwhere.

To be concrete, consider a ring of total charge Q in the xy-plane,



By inspection

$$\Phi(r=z) = \frac{Q}{4\pi\epsilon_0\sqrt{r^2+a^2}}$$

Next, expand Φ at $r > a$ as a binomial expansion:

$$\Phi(r=z) = \frac{Q}{4\pi\epsilon_0 r}\left(1+\frac{a^2}{r^2}\right)^{-1/2}$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left\{1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{a^2}{r^2}\right)^2 + \dots\right\}$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left\{1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{3}{8}\frac{a^4}{r^4} - \frac{15}{48}\frac{a^6}{r^6} + \dots\right\}$$

$$= \sum_\ell \frac{B_\ell}{r^{\ell+1}} \quad \text{with } B_0 = \frac{Q}{4\pi\epsilon_0},\ B_1 = 0$$

$$B_2 = \frac{-Qa^2}{8\pi\epsilon_0},\ \text{etc...}$$

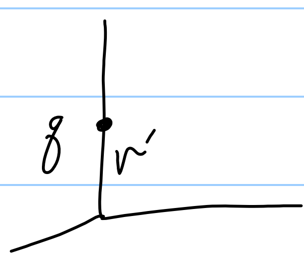
Then by the above logic, the potential off-axis at $r > a$ must be

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{r} - \frac{a^2}{2r^3}P_2(\cos\theta) + \frac{3}{8}\frac{a^4}{r^5}P_4(\cos\theta) + \dots\right\}$$

Another application of this logic:

The potential at $\vec{x}$ due to a point charge q at $\vec{x}'$ is $\Phi(\vec{x}, \vec{x}') = \dfrac{q}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|}$

If we choose $\vec{x}'$ to lie on the z-axis then we have axial symmetry, i.e. $r' = z'$



Now, the potential at any point $\vec{x} = r\hat{z}$ on the z-axis is of course

$$\Phi(r=z, r'=z') = \frac{q}{4\pi\epsilon_0 |r-r'|} = \frac{q}{4\pi\epsilon_0}\begin{cases} \frac{1}{r}\left(1-\frac{r'}{r}\right)^{-1}, & r'<r \\ \frac{1}{r'}\left(1-\frac{r}{r'}\right), & r'>r \end{cases}$$

or in a more succinct notation, <u>define</u>

$$r_< = \min(r, r')$$

$$r_> = \max(r, r')$$

So in general, on axis,

$$\Phi(r=z, r'=z') = \frac{q}{4\pi\epsilon_0}\frac{1}{r_>}\left(1-\frac{r_<}{r_>}\right)^{-1}$$

and performing a binomial expansion gives

$$\Phi(r=z, r'=z') = \frac{q}{4\pi\epsilon_0}\frac{1}{r_>}\left\{1 + \frac{r_<}{r_>} + \frac{(-1)(-2)}{2!}\left(\frac{r_<}{r_>}\right)^2 + \dots\right\}$$

$$= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}$$

and as before, this result generalizes off-axis to

$$\Phi(r,\theta; r', 0) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

Finally, noting that there was nothing special about choosing $\vec{x}'$ on the z-axis, we have generally

$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)}$$

where γ = angle between $\vec{x}$ and $\vec{x}'$

This last result is also equal to $G(\vec{x}, \vec{x}')$, the free space Green's function with no finite boundaries.

Note that we can write $\cos\gamma$ in terms of the separate spherical coordinates of $\vec{x}$ and $\vec{x}'$, as follows:

Start from $\hat{x} \cdot \hat{x}' = \cos\gamma$

but $\vec{x} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$\vec{x}' = r'(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$

and $\hat{x} \cdot \hat{x}' = \sin\theta\sin\theta'(\cos\phi\cos\phi' + \sin\phi\sin\phi') + \cos\theta\cos\theta'$

or $$\boxed{\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')}$$

useful!

Read sec. 3.4 on your own.

Next, suppose the system has <u>no azimuthal symmetry</u> $\Rightarrow$ NEED SPHERICAL HARMONICS!

The angular diff. equation is now

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)P = 0$$

& solutions are associated Legendre functions,

$$P_l^m(x), \quad x = \cos\theta, \quad -1 \le x \le 1$$

and for $l, m =$ integers, an explicit formula is

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

Other relations: $P_\ell^m(1) = \delta_{m,0}$

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell, \text{ for } -\ell \le m \le \ell$$

and also,
$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

Important: The $P_\ell^m(x)$ with the SAME m but different ℓ are orthogonal and complete, i.e.

$$\int_{-1}^{1} P_\ell^m(x)\, P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell,\ell'}$$

Recalling that the ϕ-dependence for the Laplace equation solutions must be $e^{im\phi}$, the full solution in any charge-free region can be written (assuming no angular boundaries) as:

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right) P_\ell^m(\cos\theta)\, e^{im\phi}$$

But it is usually more convenient if we work instead with spherical harmonics, having unit normalization, $Y_{\ell m}(\theta,\phi)$, obeying orthonormality,

$$\int d\Omega\, Y^*_{\ell' m'}(\theta,\phi)\, Y_{\ell m}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\Rightarrow Y_{lm}(\vartheta, \phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\vartheta)\, e^{im\phi}$$

whose completeness relation reads:

$$\sum_{lm} Y^*_{lm}(\vartheta', \phi')\, Y_{lm}(\vartheta, \phi) = \delta(\phi - \phi')\,\frac{\delta(\vartheta - \vartheta')}{\sin\vartheta}$$

$$= \delta(\phi - \phi')\,\delta(\cos\vartheta - \cos\vartheta')$$

or with the useful abbreviations in spherical coordinates, $\hat{x} \equiv (\vartheta, \phi)$, $\hat{x}' \equiv (\vartheta', \phi')$

$$\Rightarrow \sum_{l,m} Y^*_{lm}(\hat{x}')\, Y_{lm}(\hat{x}) = \delta^{(2)}(\hat{x} - \hat{x}')$$

Miscellaneous useful relations:

$$Y^*_{lm}(\vartheta, \phi) = (-1)^m\, Y_{l,-m}(\vartheta, \phi)$$

and $$Y_{l0}(\vartheta, \phi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\vartheta)$$

and in Mathematica,

SphericalHarmonicY[L, M, θ, φ]

Spherical Harmonic Addition Theorem:

important! $$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\vartheta', \phi')\, Y_{lm}(\vartheta, \phi)$$

where $\cos\gamma = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\phi - \phi')$,

i.e. γ = angle between $\hat{x}$ and $\hat{x}'$

or $$P_l(\hat{x}\cdot\hat{x}') = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\hat{x}')\, Y_{lm}(\hat{x})$$